LECTURE 10
MATH 261A

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1. Representations of $\mathfrak{sl}(2,\mathbb{C})$

1.1. **Motivation.** We might wonder if we have a presentation of an algebra as matrices, why we care about additional representations? For example, we have a standard representation of $\mathfrak{sl}(2,\mathbb{C})$, so why do we care about anything else?

The point is, this isn't all about $\mathfrak{sl}(2,\mathbb{C})$. Of course we do understand $\mathfrak{sl}(2,\mathbb{C})$, but what we really want to understand is geometric representations coming from actions of $\mathfrak{sl}(2,\mathbb{C})$. So we really want to develop Lie groups as a tool rather than something to be studied.

Example 1. Let $X \subseteq \mathbb{CP}^n$ be a smooth projective variety over $\mathbb{C}$. One of the most important invariants we can associate to X is the cohomology $H^*(X,\mathbb{C})$, which is a vector space which has something to do with X. Then we have the following theorem:

Theorem 1 (Hard Lefschetz)**.** *$H^*(X,\mathbb{C})$ is naturally an $\mathfrak{sl}(2,\mathbb{C})$ representation.*

1.2. **Classification.** Recall we were about to prove the following last time:

Theorem 2. *$\mathbf{Rep}_{fd}(\mathfrak{sl}(2,\mathbb{C}))$ is semisimple, and the irreducibles are*

$$V_n = \mathrm{Sym}^n(V_1)$$

where V_1 is the standard representation.

Proof. Inside $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{g}$, consider the subalgebra $\mathfrak{h} = \mathbb{C}\langle H\rangle \subseteq \mathfrak{g}$. This is a one-dimensional abelian subalgebra.[1] Finite dimensional representations of $\mathfrak{h}$ are the same as finite dimensional vector spaces with an endomorphism:

$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{h}) = \langle H \circlearrowright V \mid H \in \mathrm{Ext}(V)\rangle \simeq \mathbb{C}[H]\text{-}\mathbf{Mod}$$

Every time you see a module over a polynomial algebra, you should think of the eigenline. So think of $\mathbb{C}$ as the eigenline of H. Then

$$V = \bigoplus V_{\lambda_i}$$

Now what can we say about representations of $\mathfrak{h}$ that come from $\mathfrak{g}$? This is not sort of mathematically canonical, but our strategy will be to consider the real direction as special. So project the eigenline to $\mathbb{R}$, which is of course ordered, which will allow us to analyze this picture from right to left.

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[1] In particular it is maximal, which makes it a Cartan subalgebra by definition. We will be seeing these later.

Definition 1. We will call the eigenvalues of an $\mathfrak{h}$ representation the *weights*. The *highest weight* will be the weight with real part $\geq$ the others. Call any vector in the eigenspace V_{λ_i} of the highest weight λ_i a *highest weight vector*. A general $v \in V_\lambda$ is said to be of weight λ.

Now bring X and Y into the picture. We now make a fundamental observation. Suppose $v \in V$ is of weight λ. Let's apply X and Y to v. The weight of $X \cdot v$ is just the eigenvalue of $X \cdot v$ under the action of H. But we can write:

$$HXv = XHv + [H, X]\, v = XHv + 2Xv$$

If $Hv = \lambda v$, i.e. v is an eigenvector, then

$$HXv = X\lambda v + 2Xv = (2 + \lambda)\, Xv$$

so it just shifted the eigenvalue by 2. Similarly, Y shifts the eigenvalue by -2.

Now we have to make sure nothing goes wrong when we act X and Y on the generalized eigenvectors.

Exercise 1. Show that if $(H - \lambda I)^n\, V = 0$, then

$$(H - (\lambda + 2)\, I)^n\, Xv = 0$$

and similarly for Y.

Solution. Proceed by induction. So suppose $(H - \lambda I)^{n-1}\, Xv = 0$. Then we can write:

$$\begin{aligned}(H - \lambda I)^n\, Xv &= (H - \lambda I)^{n-1}\, (HXv - \lambda IXv) \\ &= (H - \lambda I)^{n-1}\, ((2 + \lambda)\, Xv - \lambda Xv) \\ &= 2\, (H - \lambda I)^{n-1}\, Xv = 0\end{aligned}$$

as desired.

So in conclusion, $X : V_\lambda \to V_{\lambda+2}$ and $Y : V_\lambda \to V_{\lambda-2}$.

Now we want to use this to find the irreducibles. Suppose V is a finite dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ representation. The first step is to find a highest weight λ_{hw}, and choose some eigenvector $v_{\text{hw}} \in V_{\lambda_{\text{hw}}}$.

Remark 1. This exists, because of the following. When you look at a Jordan block, the first vector is an eigenvector. So it doesn't matter if λ_{hw} yields an eigenspace or a generalized eigenspace, since there will still be an eigenvector either way.

If H was the only operator, this would be irreducible since $Hv_{\text{hw}} = \lambda_{\text{hw}} v_{\text{hw}}$. But now we have X and Y as well. Since λ_{hw} is the highest weight[2], $Xv_{\text{hw}} = 0$. Now start applying Yv_{hw} to get something in $V_{\lambda_{\text{hw}}-2}$, and continue applying Y. Of course since V is finite dimensional, this will eventually terminate.

Claim 1. The vectors

$$\mathbb{C}\left\langle v_{\text{hw}}, Yv_{\text{hw}}, Y^2 v_{\text{hw}}, \cdots \right\rangle \subseteq V$$

comprise an irreducible representation. In particular, if V is irreducible, then this is an equality.

[2] Professor Nadler says you should be yelling highest weight in your sleep.

Proof. The first thing this is saying is that these vectors span a subspace of the representation which is invariant under the operators. It is clear that H and Y preserve this so we need to show X preserves it. Of course $Xv_{\mathrm{hw}} = 0$, and

$$XYv_{\mathrm{hw}} = \cancel{YXv_{\mathrm{hw}}} + [X, Y]\, v_{\mathrm{hw}} = Hv_{\mathrm{hw}} = \lambda_{\mathrm{hw}} v_{\mathrm{hw}}$$

which is in this subspace.

Exercise 2. Iterate this process.

Solution. Proceed by induction. So suppose $XY^{n-1}v_{\mathrm{hw}} \in \left\langle Y^i v_{\mathrm{hw}} \right\rangle$. Then we can write:

$$\begin{aligned} XY^n v_{\mathrm{hw}} &= YXY^{n-1}v_{\mathrm{hw}} + [X, Y]\, Y^{n-1}v_{\mathrm{hw}} \\ &= YXY^{n-1}v_{\mathrm{hw}} + HY^{n-1}v_{\mathrm{hw}} \in \mathbb{C}\left\langle Y^i v_{\mathrm{hw}} \right\rangle \end{aligned}$$

as desired.

It is clear that this is irreducible, because if you defined any sort of proper nontrivial subspace it would not be closed under the action of Y. □

Next we will analyse the possible weight spaces. To do this, we will introduce universal highest weight modules.[3]

Definition 2. A *Verma* module I_λ of highest weight λ is

$$I_\lambda = \mathcal{U}\left(\mathfrak{sl}\left(2, \mathbb{C}\right)\right) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$$

Note that this is infinite dimensional.

Recall we had an adjunction where $\mathcal{U}$ was adjoint to Forget : **Lie-Alg** → **Alg**. This means

$$\mathrm{Hom}_{\mathbf{Alg}}\left(\mathcal{U}\mathfrak{g}, A\right) = \mathrm{Hom}_{\mathbf{Lie\text{-}Alg}}\left(\mathfrak{g}, \mathrm{Forget}\left(A\right)\right)$$

As a special case, for $\mathfrak{g} = \mathfrak{gl}\left(n, \mathbb{C}\right)$, this means n-dimensional $\mathfrak{g}$ representations are just n-dimensional $\mathcal{U}\mathfrak{g}$-modules.

Recall that we can explicitly write the enveloping algebra as:

$$\mathcal{U}\mathfrak{g} = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} / \left(XY - YX = [X, Y]\right)$$

The idea here is that $\mathcal{U}\mathfrak{g}$ allows us to work with products of operators of $\mathfrak{g}$.

The subalgebra $\mathfrak{b} \subseteq \mathfrak{sl}\left(2, \mathbb{C}\right)$ is a Borel subalgebra:

$$\mathfrak{b} = \mathbb{C}\left\langle H, X\right\rangle = \left\langle \begin{pmatrix} a & u \\ 0 & -a \end{pmatrix} \mid a, u \in \mathbb{C} \subseteq \mathfrak{sl}\left(2, \mathbb{C}\right) \right\rangle$$

Note that this is a maximal solvable subalgebra.

Finally $\mathbb{C}_\lambda$ is the one dimensional complex vector space with one vector v such that H acts by multiplication by λ and X acts as 0:

$$Hv = \lambda v \qquad\qquad Xv = 0$$

In other words, the $\mathbb{C}_\lambda$ comprise the irreducible representations of $\mathfrak{b}$. In particular, these are all one-dimensional.

Exercise 3. Show that the $\mathbb{C}_\lambda$ comprise the irreducible representations of $\mathfrak{b}$.

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[3] This might be against better judgement, but Professor Nadler says he just can't help himself.

Solution. First notice that $[\mathfrak{b},\mathfrak{b}]$ must be inside $\ker\rho$. Since $[\mathfrak{b},\mathfrak{b}] = \mathbb{C}\langle X\rangle$, this means X must act trivially. Then we know there must be some λ eigenvalue of H, so we can decompose this space if it is not of a single dimension.

The point here, is that we take the enveloping algebra, and then every time we see an H or an X, we can act by these rules and cancel.

Claim 2. I_λ has basis v, Yv, ...

Remark 2. This is a special case of the Poincaré-Birkhoff-Witt (PBW) theorem.

Proof. Look at some monomial. Using the bracket, we can rewrite this as a sum of monomials of the form $Y^a H^b X^c$.

So I_λ is spanned by vectors of the form $Y^a H^b X^c \otimes v$. For $c \neq 0$, we can use that we are tensoring over $\mathcal{U}(\mathfrak{b})$ to move X to the other side:

$$Y^a H^b X^c \otimes v = Y^a H^b X^{c-1} \otimes Xv = 0$$

So we may as well assume $c = 0$, and if $b \neq 0$,

$$Y^a H^b \otimes v = Y^a H^{b-1} \otimes Hv = \lambda\left(Y^a H^{b-1} \otimes v\right)$$

□

We do a sample calculation in V_λ to see the flavor of this:

$$\begin{aligned} XY^2 v &= XYYv = (YX + [X,Y])\,Yv = YXYv + HYv \\ &= Y\,(YX + [X,Y])\,v + (YH + [H,Y])\,v \\ &= \cancel{Y^2 X v} + \lambda Yv + \lambda Yv - 2Yv = \boxed{2\,(\lambda - 1)\,Yv} \end{aligned}$$

Lemma 1. *The action of H on the basis is given by:*

$$HY^j v = (\lambda - 2j)\,Y^j v$$

Proof. Proceed by induction. So assume

$$HY^{j-1} v = (\lambda - 2\,(j-1))\,Y^{j-1} v$$

and then this allows us to write:

$$\begin{aligned} HY^j v &= (YH + [H,Y])\,Y^{j-1} v \\ &= YHY^{j-1} v - 2Y^j v \\ &= Y\left(\left(\lambda - 2\,(j-1)\,Y^{j-1}\right)\right) - 2Y^j v \\ &= (\lambda - 2\,(j-1))\,Y^j - 2Y^j \\ &= (\lambda - 2j)\,Y^j \end{aligned}$$

as desired. □

Lemma 2. *The action of X on the basis is given by:*

$$XY^j v = j\,(\lambda - (j-1))\,Y^{j-1} v$$

Proof. Proceed by induction. So assume

$$XY^{j-1}v = (j-1)(\lambda - (j-2))Y^{j-2}v$$

then this lets us write:

$$\begin{aligned} XY^{j}v &= (YX + [X,Y])Y^{j-1}v \\ &= YXY^{j-1}v + HY^{j-1}v \end{aligned}$$

now we can rewrite each of these terms. First, using the induction hypothesis we can write:

$$\begin{aligned} YXY^{j-1}v &= Y(j-1)(\lambda - (j-2)) \\ &= ((j-1)\lambda - (j-2)(j-1))Y^{j-1}v \end{aligned}$$

and using lemma 1 we can write:

$$HY^{j-1}v = (\lambda - 2(j-1))Y^{j-1}v$$

which means

$$\begin{aligned} XY^{j}v &= ((j-1)\lambda - (j-2)(j-1) - \lambda + 2(j-1))Y^{j-1}v \\ &= j(\lambda - (j-1))Y^{j-1}v \end{aligned}$$

as desired. □

Remark 3. The basic idea of Verma modules is to somehow get a universally nonterminating object.

So for each λ there is this Verma module as defined above, but now this in fact has the universal property:

Exercise 4. Check that if λ is the highest weight:

$$\mathrm{Hom}_{\mathbf{Rep}(\mathfrak{g})}(I_\lambda, V) = \lambda \text{ eigenspace}$$

Solution. Let $f \in \mathrm{Hom}_{\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})}(W_\lambda, V)$. This is completely determined by where it takes the basis of W_λ, and in particular, since this must respect the action of $\mathfrak{g}$, it is completely specified by where it takes the vector v. It must take this to some element of V_λ in order to preserve the action of H, and therefore we can associate f to the image $f(v) \in V_\lambda$.

This is a kind of standard adjunction, where we ask for the Hom of V_λ to any V, be the same as a Hom from $\mathbb{C}_\lambda$ as a $\mathcal{U}(\mathfrak{b})$ module. So we need to find vectors which are killed by X, and for which H acts as λ.

Now we return to representations of $\mathfrak{sl}(2,\mathbb{C})$. We have a canonical map $V_{\lambda_{\mathrm{hw}}} \to V$ which simply sends $v \to v_{\lambda_{\mathrm{hw}}}$. There must be some kernel, since this is a map from an infinite dimensional thing to a finite dimensional thing.

Claim 3. If we similarly generate from some different $v'_{\mathrm{hw}} \in V'_{\lambda'_{\mathrm{hw}}}$ in some representation V', we obtain isomorphic irreducible subspaces.

I.e. there is somehow no ambiguity. The isomorphism is the obvious one.

To be continued next time...

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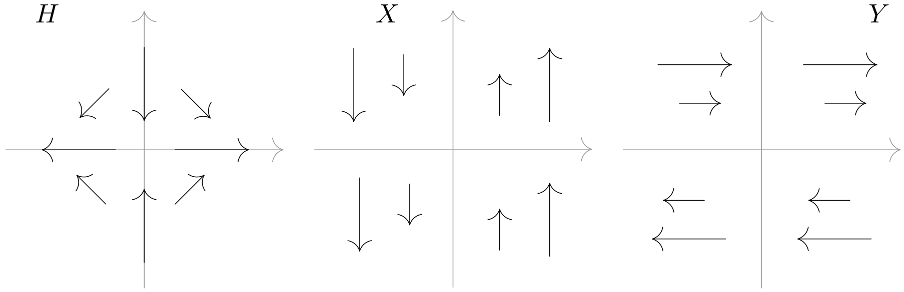

FIGURE 1. The vector fields which H, X, and Y are mapped to. The first can be thought of as being sort of hyperbolic, and the second two are shears.

1.3. Where does this come from.

Recall

$$\mathrm{Sym}^n(W) \subset W^{\otimes n}$$

consists of the Σ_n-symmetric tensors.

How would someone come up with theorem 2? Imagine we start with $\mathrm{SL}(2,\mathbb{C}) \circlearrowright V_1 = \mathbb{C}^2 = \mathbb{C}\langle u, v\rangle$. Then we differentiate this to give us: $\mathfrak{g} \to \mathbf{Vect}(\mathbb{C}^2)$. In particular, calculate

$$\frac{d}{dt}\left(e^{tH}\cdot w\right)|_{t=0} = \frac{d}{dt}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}|_{t=0} = \begin{pmatrix} w_1 \\ -w_2\end{pmatrix}$$

so

$$H \mapsto u\partial_u - v\partial_v \qquad X \mapsto u\partial_v \qquad Y \mapsto v\partial_u$$

These vector fields can be visualized in fig. 1. Now consider the polynomial functions on $\mathbb{C}^2$, $\mathbb{C}[u,v]$, then these vector fields act on this, to get

$$\mathbb{C}[u,v] = \mathbb{C} \oplus \mathbb{C}\langle u, v\rangle \oplus \mathbb{C}\left\langle u^2, uv, v^2\right\rangle \oplus \cdots$$

which is effectively a decomposition in the irreducibles Sym^n.