LECTURE 11
MATH 261 A

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1. Continued proof from last time

Recall we were in the middle of proving the following theorem:

Theorem 1. $\mathbf{Rep}_{fd}(\mathfrak{sl}(2,\mathbb{C}))$ *is semisimple, and the irreducibles are*

$$V_n = \mathrm{Sym}^n(V_1)$$

where V_1 *is the standard representation.*

Continued proof. Recall we're trying to use this weight picture to show this. Let's do some examples to get a feeling for this.

Example 1. The standard representation $\mathbb{C}\langle u, v\rangle$. This has eigenvalue -1 with eigenvector v, and 1 with eigenvector u.

Example 2. For $V_3 = \mathrm{Sym}^3\left(\mathbb{C}^2\right)$, we have the following eigen-vectors/values:

$$\lambda = -3, v_\lambda = v^3 \qquad \lambda = -1, v_\lambda = uv^2 \qquad \lambda = 1, v_\lambda = u^2 v \qquad \lambda = 3, v_\lambda = u^3$$

Recall the Verma module is:

$$I_\lambda = \mathcal{U}\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$$

for $\mathfrak{b} = \mathbb{C}\langle H, X\rangle$. Also recall that we saw:

$$I_\lambda \simeq \mathbb{C}\left\langle v_\lambda, Y v_\lambda, Y^2 v_\lambda, \cdots\right\rangle$$

The Verma module also has the following universal property:

$$\mathrm{Hom}_{\mathfrak{g}}(I_\lambda, V) = \langle v \in V \mid Xv = 0, Hv = \lambda v\rangle$$

since v_λ has to go to something that is killed by X, and is an eigenvector of H with eigenvalue λ. The set on the RHS consists of highest weight vectors, and λ eigenvectors. In particular, if V is irreducible then there is a nonzero map $p : I_{\lambda_{\mathrm{hw}}} \to V$. This must be surjective because V is irreducible, and now we just need to figure out what the kernel is.

Proposition 1. *(1) If* $\lambda \notin \{0, 1, 2, \cdots\} \subset \mathbb{C}$*, then* I_λ *is irreducible.*
(2) For $n = 0, 1, \cdots$*, there exists a short exact sequence:*

$$0 \longrightarrow I_{-n-2} \hookrightarrow I_n \overset{p}{\twoheadrightarrow} V_n \longrightarrow 0$$

The first part implies that V irreducible must have $\lambda_{\mathrm{hw}} \in \{0, 1, 2, \cdots\}$. The second implies that once you get to $-n-2$, we see I_{-n-2} is sitting inside, so now we can quotient I_n / I_{-n-2} to get the finite dimensional representation V_n.

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Warning 1. The sequence in the above proposition does not split.

Example 3. For $\lambda = 0$, we have: $I_{-2} \subset I_0$, now let's imagine if there's a complement of I_{-2} in I_0, but this can't be, since if we have v_0 and act by Y we immediately are moved out of this subspace.

Exercise 1. Prove the above proposition. The idea is to remember that $I_\lambda = \langle v_\lambda, Y v_\lambda, \cdots \rangle$, and then just apply X to see if there is any invariant subspace. So see if there's any way to come back.

Example 4. Let $\lambda = 0$. Then the basis of I_0 is $v_0, Y v_0, \cdots$ and $X v_0 = 0$, so we can calculate the following:

$$XYv_0 = \cancel{YXv_0} + [X, Y]\, v_0 = Hv_0 = 0v_0 = 0$$

However, as we saw last time:

$$XY^2 v_0 = (2\lambda - 2)\, Y v_0 = -2Y v_0$$

So once we have applied Y enough times, we reach the I_{-n-2} subspace, which in this case is I_{-2}. Then the quotient I_0/I_{-2} is the trivial representation.

Example 5. If $\lambda = 1$, we can calculate that:

$$XYv_1 = \cancel{YXv_1} + [X, Y]\, v_1 = Hv_1 = v_1$$

and similarly,

$$\begin{aligned} XY^2 v_1 &= (2\lambda - 2)\, Y v_1 = 0 \\ XY^3 v_1 &= 3\,(\lambda - 2)\, Y^2 v_1 = -3Y^2 v_1 \end{aligned}$$

So again, if we apply Y enough times we reach $I_{-n-2} = I_{-3}$, and then we can't get out. Then quotienting I_1/I_{-3} gives us the standard representation.

Exercise 2. Generalize these formulas.

Now we just need to prove the representations are semisimple. There are two approaches, one is kind of algebraic, and one is kind of geometric.[1] We will prove it the second way.

Recall we have an equivalence between simply connected, connected Lie groups over $\mathbb{C}$ and finite dimensional Lie algebras over $\mathbb{C}$. In particular, this means for any complex vector space V,

$$\mathrm{Aut}\,(V) = \mathrm{GL}\,(V) \mapsto \mathfrak{gl}\,(V) = \mathrm{End}\,(V)$$

This means, for our arbitrary $\mathfrak{g}$, we have

$$\begin{array}{ccc} \mathrm{Hom}_{\textbf{Lie-Alg}}\,(\mathfrak{g}, \mathfrak{gl}\,(V)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\textbf{Lie-Gp}}\,(G, \mathrm{GL}\,(V)) \\ & \searrow \quad \swarrow & \\ & \textbf{Vect} & \end{array}$$

i.e.

$$\textbf{Rep}_{\mathrm{fd}}\,(G) \cong \textbf{Rep}_{\mathrm{fd}}\,(\mathfrak{g})$$

and this preserves the natural forgetful map to **Vect**. Actually to see this, we technically need the following:

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[1] According to Professor Nadler, some sort of higher intelligence might prefer the algebraic approach, but he is not a higher intelligence, so we will take the sort of geometric approach. Also because we will see some interesting important math along the way.

Exercise 3. Show that since G is simply-connected, the map $\operatorname{Hom}(G, \mathrm{GL}(V)) \to \operatorname{Hom}\left(G, \widetilde{\mathrm{GL}}(V)\right)$ is the inverse of the projection in the following diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{\textbf{Lie-Alg}}(\mathfrak{g}, \mathfrak{gl}(V)) & = & \operatorname{Hom}_{\textbf{Lie-Gp}}\left(G, \widetilde{\mathrm{GL}}(V)\right) \\ & \searrow & \downarrow \uparrow \\ & & \operatorname{Hom}_{\textbf{Lie-Gp}}(G, \mathrm{GL}(V)) \end{array}$$

so these things are all equal. I.e. show that if we have a homomorphism of a simply connected group, it naturally lifts to the universal cover. So we have the following diagram:

$$\begin{array}{ccc} & & \widetilde{\mathrm{GL}}(V) \\ & \nearrow & \downarrow \\ G & \longrightarrow & \mathrm{GL}(V) \end{array}$$

Remark 1. The previous exercise holds for any group, not just $\mathrm{GL}(V)$.

Now to finish the proof of the theorem, it is sufficient to prove the following:

Proposition 2. $\mathbf{Rep}_{fd}(\mathrm{SL}(2,\mathbb{C}))$ *is semisimple.*

We will first reduce this to an even easier statement.

Consider $\mathrm{SU}(2) \subseteq \mathrm{SL}(2,\mathbb{C})$. Recall $\mathrm{SU}(2)$ are the matrices

$$\left\langle \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\rangle$$

which also preserve the standard hermitian inner product:

$$\langle a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2 \rangle = \overline{a}_1 b_1 + \overline{a}_2 b_2$$

Exercise 4. Show this is true.

Solution. Let $A \in \mathrm{SU}(2)$. Then

$$\langle Av, Aw \rangle = \overline{Av}^T Aw \overline{v}^T \overline{A}^T Aw = \overline{v} w = \langle v, w \rangle$$

where the last equality uses the fact that $\overline{A}^T A = I$.

Notice the following good properties of $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$:

(1) $\mathrm{SU}(2)$ is compact and isomorphic to S^3
(2) $\mathfrak{su}(2) \otimes \mathbb{C} \simeq \mathfrak{sl}(2,\mathbb{C})$

so the first says it is small, and the seccond says it's big in the sense that it doesn't miss any of the structure of $\mathfrak{sl}(2,\mathbb{C})$.

Remark 2. Subgroups of $G_{\mathbb{C}}$ with these properties are called "maximal compact". I.e. this doesn't really have anything to do with $\mathfrak{sl}(2,\mathbb{C})$.

Lemma 1. *The restriction*

$$\mathbf{Rep}_{fd}(\mathrm{SL}(2,\mathbb{C})) \xrightarrow{\sim} \mathbf{Rep}_{fd}(\mathrm{SU}(2))$$

is an isomorphism.

Proof. Since $\mathrm{SU}(2)$ is simply connected, $\mathbf{Rep}_{\mathrm{fd}}(\mathrm{SU}(2)) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{su}(2))$.

Exercise 5. Show the restriction:

$$\mathbf{Rep}_{\mathrm{fd}}\left(\mathfrak{su}\left(2\right)\otimes_{\mathbb{R}}\mathbb{C}\right)\xrightarrow{\sim}\mathbf{Rep}_{\mathrm{fd}}\left(\mathfrak{su}\left(2\right)\right)$$

is an isomorphism.

This is effectively a tautology. So we are done. □

Proposition 3. $\mathbf{Rep}_{fd}\left(\mathrm{SU}\left(2\right)\right)$ *is semisimple.*

Proof. Let V be a finite dimensional representation of $\mathrm{SU}(2)$. Then we will construct a hermitian inner product on V invariant under $\mathrm{SU}(2)$.

First choose any hermitian inner product $\langle v,w\rangle_0$. Now to make this invariant under the group action, we define:

$$\langle v,w\rangle=\int_{\mathrm{SU}(2)}\langle gv,gw\rangle_0\ dg$$

Here, dg is a nonzero invariant measure[2] on $\mathrm{SU}(2)$. Suppose $W\subseteq V$ is a sub-representation, then we can consider

$$W^{\perp}:=\{x\in V\mid\forall y\in W,\langle x,y\rangle=0\}$$

Exercise 6. Show that $W^{\perp}\subseteq V$ is also a sub-representation, and in particular:

$$V\simeq W\oplus W^{\perp}$$

□

So the strategy was to go from a simple Lie algebra over $\mathbb{C}$, to a simply connected Lie group over $\mathbb{C}$, to maximal compact Lie group:

$$\mathfrak{g}\rightsquigarrow G\rightsquigarrow G_c$$

which all have the same representations. ■

1.1. **Invariant measure.** At the end of the proof of the above theorem we just asserted there was an invariant measure on $\mathrm{SU}(2)$. We now construct this. At every point of $\mathrm{SU}(2)$, we will define a volume form, i.e. a nondegenerate 3-form, and then this will give us a measure.

First pick an inner product on the tangent space at the identity. In particular, choose an Ad-invariant volume m on $\mathfrak{su}(2)$. One such example is the killing form. Now translate this by left multiplication to any $T_g\,\mathrm{SU}(2)$. Finally, observe that this is also right invariant. This is since the initial form was Ad invariant.

2. Playing with the representations

2.1. **Tensor products.** Take $V_n\otimes V_m$. This may or may not be irreducible, but it certainly will be a sum of irreducibles:

$$V_n\otimes V_m=\bigoplus_{k=0}^{\infty}V_k^{d_k}\ .$$

Then the challenge is to determined the d_k.

Example 6. $V_0\otimes V_n=V_n$, so $d_n=1$, and all other $d_i=0$.

[2] See section 1.1.

Example 7. $V_1 \otimes V_1 = V_2 \oplus V_0$, so $d_0 = d_2 = 1$ and all other $d_i = 0$.

The past two examples were somehow easy to do without thinking too hard. The next example effectively generalizes to any case:

Example 8. Let's try to calculate $V_2 \otimes V_3$. The weights of V_2 are 2, 0, and -2. The weights of V_3 are 3, 1, -1, and -3. Then we observe that the restriction of a representation of Lie algebra to a Lie subalgebra, the tensor product is preserved. In particular, $\mathfrak{h} = \mathbb{C}\langle H \rangle \subseteq \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ preserves $\otimes$.

Therefore the weights of $V_2 \otimes V_3$ are the pairwise sums of weights of V_2 and V_3 independently. Therefore the weights are:

$$-5 \qquad \bigcirc\!\!\!\!\!\!-3 \qquad ◎-1 \qquad ◎1 \qquad \bigcirc\!\!\!\!\!3 \qquad 5$$

where we have circled the weights as many times as their multiplicity. Then the multiplicity is how many ways these weights summed to give the new weights. Therefore the multiplicity of 5 is 1, the multiplicity of 3 is 2, the multiplicity of 1 is 3, and the same multiplicities for the negative weights.

Now we can understand the irreducibles just from this. Find the highest weight 5, then this means we must have a copy of V_5 inside, so we can cancel the weights associated with 5, so we have 1 left on ± 1, and 2 left on ± 1, so we have a V_3, and we cancel again, to get only 1 left on ± 1 so we get a V_1 and our answer is:

$$V_2 \otimes V_3 = V_5 \oplus V_3 \oplus V_1$$

Exercise 7. Write this down in general.

Solution. The basic idea is starting at the sum $m+n$ and then just counting down by 2 until you hit their difference. Let $m \geq n$, then:

$$V_m \otimes V_n = \bigoplus_{\substack{i \in 2\mathbb{Z} \\ m-n \leq i \leq m+n}} V_i$$

Next time we will generalize to all simple Lie algebras. In particular, we will write down a list of all such Lie algebras, and then see that the general story references $\mathfrak{sl}(2, \mathbb{C})$, so this is really an important thing to understand.