LECTURE 12
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

We will meet 4 games people play with representations.

1. Tensor products

Recall last time we were playing some games with representations of $\mathfrak{sl}(2,\mathbb{C})$. In particular, we saw that for $m \geq n$,

$$V_m \otimes V_n = \bigoplus_{\substack{l=m-n+2k \\ 0\leq k\leq n}} V_l$$

2. Characters

Consider $\mathbb{C}[\mathbb{Z}]$, the collection of compactly supported $\mathbb{C}$-valued functions[1] on $\mathbb{Z}$.

Definition 1 (Character)**.** A *formal character* is an element of $\mathbb{C}[\mathbb{Z}]$. We write e_n for the characteristic function of $n \in \mathbb{Z}$.

The e_n form a basis for $\mathbb{C}[\mathbb{Z}]$ as a complex vector space. This can be considered a ring with the operation given by convolution. This effectively just depends on the group structure on $\mathbb{Z}$.

$$(f * g)(n) = \sum_{k+l=n} f(k)\, g(l)$$

Exercise 1. Check that $e_n * e_m = e_{n+m}$.

This is somehow a linear extension of the group structure on $\mathbb{Z}$.

2.1. **More invariant origin.** Return to representation theory. We want to think about $\mathbb{Z}$ as integer weights in the H eigenline $\mathbb{C}$.

Definition 2 (Character of a representation)**.** The *formal character* of a finite dimensional representation V is $V \mapsto \chi_V \in \mathbb{C}[\mathbb{Z}]$ where

$$\chi_V(n) = \dim_{\mathbb{C}} V_{\lambda=n}$$

where $V_{\lambda=n}$ is the eigenspace at $\lambda = n$.

Example 1. The irreducibles from before have characters:

$$\chi_{V_n} = \sum_{\substack{l=-n+2k \\ 0\leq k\leq n}} e_l$$

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[1] Or maybe distributions. Or maybe not compactly supported. The mathematics is smart and will tell us what's right.

Exercise 2. Check the following:

$$\chi_{V \oplus W} = \chi_V + \chi_W \qquad \chi_{V \otimes W} = \chi_V * \chi_W$$

So characters somehow take a representation and return an element of $\mathbb{C}[\mathbb{Z}]$. But this statement isn't formal, since this is somehow mixing levels - representations are objects in a category, and $\mathbb{C}[\mathbb{Z}]$ is just a ring. One way to formalize this is to instead consider this as a map from the Grothendieck group to the ring of formal characters:

$$K_0 (\mathbf{Rep}_{\mathrm{fd}} (\mathfrak{sl}(2, \mathbb{C}))) \otimes \mathbb{C} \to \mathbb{C}[\mathbb{Z}]$$

Recall **Rep** is an abelian category. The Grothendieck group is what you get when you ask for a group whose elements are the objects of your category, and direct sum becomes addition. It's somehow the universal version of a group resulting from only insisting that exact sequences make sense.

We can think of elements of this as being some sort of formal difference $V - W$ of two objects of the original category.

Proposition 1. *χ is injective, and in particular,*

$$\chi : K_0 \left(\mathbf{Rep}_{fd} \mathfrak{sl}(2, \mathbb{C})\right) \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}[\mathbb{Z}]^{\Sigma_2}$$

is an isomorphism, where $\Sigma_2 \simeq \mathbb{Z}/2$ acts by $\sigma(n) = -n$.

Proof. Injectivity follows from the fact that up to isomorphism, representations are determined by their characters. To see this is surjective, we just have to check that $e_n + e_{-n}$ is in the image, which is

$$\chi (V_n - V_{n-2})$$

so we are done. □

3. Character formulas

The game is the following. Put $n \in \mathbb{N}$ into the machine, and the machine is supposed to give you χ_{V_n}

$$\mathbb{N} \ni n \rightsquigarrow \chi_{V_n} \in \mathbb{C}[\mathbb{Z}]^{\Sigma_2}$$

The answer for $\mathfrak{sl}(2, \mathbb{C})$ is just the sum of the weights as above in example 1, but in general it won't be this easy. So we will consider in a complicated but beautiful way to do it for $\mathfrak{sl}(2, \mathbb{C})$ which will turn out to generalize.

We know we can take V_n which has a surjective map $I_n \to V_n$ from the Verma module, and in particular, we have the exact sequence:

$$0 \to I_{-n-2} \to I_n \to V_n \to 0$$

Remark 1. This is a special case of the Bernstein-Gelfand-Gelfand (BGG) resolution.

This sequence implies that the character[2] of V_n is the character of I_n minus the character of I_{-n-2}:

$$\chi_{V_n} = \chi_{I_n} - \chi_{I_{-n-2}}$$

[2] This isn't really a character because the dimension of each eigenspace to the infinite negative side has dimension 1 for these Verma modules. Therefore the associated character is not compactly supported.

But taking inspiration from:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

we can write this is a more clever way:

$$\chi_{V_n} = \frac{e_n}{1-e_{-2}} - \frac{e_{-n-2}}{1-e_{-2}} = \frac{e_n - e_{-n-2}}{1-e_{-2}}$$

Now rewriting this, we get:

$$\chi_{V_n} = \frac{e_{n+1} - e_{-n-1}}{e_1 - e_{-1}}$$

4. Tannakian formalism

Suppose $\mathbf{C}$ is a $\mathbb{C}$-linear abelian $\otimes$-category. Suppose

$$F : \mathbf{C} \to \mathbf{Vect}$$

is a "forgetful functor." This means this is a $\otimes$-functor which is exact, faithful, and maybe a few more things that the actual forgetful functor is.

To this, we can associate a group

$$G = G_{\mathbf{C},F} = \operatorname{Aut}^{\otimes}(F)$$

which is the group of tensor automorphisms of F. For $g \in G$, we get an automorphism

$$g_V : F(V) \xrightarrow{\sim} F(V)$$

for every $V \in \mathbf{C}$, which respects the tensor structure in the sense that:

$$g_{V \otimes W} = g_V \otimes g_W$$

This is called the Tannakian group of $\mathbf{C}$ with respect to the fiber functor F.

Exercise 3. For $\mathbf{C} = \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(2,\mathbb{C}))$, and $F = \text{Forget}$, then this says for every representation of $\mathfrak{sl}(2,\mathbb{C})$, forget it down to a vector space, then everything in $\operatorname{Aut}^{\otimes}(F)$ is a choice of automorphisms of these vector spaces. Show that $G_{\mathbf{C},F} \simeq \mathrm{SL}(2,\mathbb{C})$.

Solution. First start with an element $A \in \mathfrak{sl}(2,\mathbb{C})$, then we want to get an element $g \in G$, i.e. a collection of automorphisms

$$g_V \circlearrowright V \in \mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(2,\mathbb{C}))$$

It is enough to specify this on the irreducibles $V_n = \operatorname{Sym}^n V_1 = \operatorname{Sym}^n \mathbb{C}^2$.

Remark 2. If you have an abelian category you're trying to learn something about, try calculating the Tannakian group. By the above discussion, the category will then be the representations of this group, though the new group might be something terrible you've never seen before.

5. Classification of simple Lie algebras over $\mathbb{C}$

This is somehow the general answer over algebraically closed fields, but we will just do it over $\mathbb{C}$. This is called the Cartan classification.

5.1. **Classical Lie algebras.** The first type is A_n for $n \geq 1$, and these are

$$\mathfrak{sl}(n+1,\mathbb{C}) = \langle \operatorname{tr} A = 0 \rangle$$

The next series is B_n for $n \geq 2$, and these are the odd orthogonal Lie algebras

$$\mathfrak{so}(2n+1,\mathbb{C}) = \left\langle -A = A^T \right\rangle$$

This one starts at $\mathfrak{so}(5)$ because $\mathfrak{so}(3)$ is already on the list since:

Proposition 2. $\mathfrak{sl}(2) \simeq \mathfrak{so}(3)$

The next is C_n for $n \geq 3$, which are $\mathfrak{sp}(2n,\mathbb{C})$. These preserve the standard symplectic inner product:

$$\mathfrak{sp}(2n,\mathbb{C}) = \left\langle \omega A = -A^T \omega \right\rangle$$

for ω some nondegenerate skew-symmetric matrix/inner product. So these are linear automorphisms of a symplectic vector space.

Remark 3. We got this condition on elements of $\mathfrak{sp}$ by differentiating

$$(gv_1)^T \omega g v_2 = v_1^T \omega v_2$$

with respect to g which gives:

$$0 = (Av_1)^T \omega v_2 + v_1^T \omega A v_2 = A^T \omega + \omega A$$

For $n = 1$ we get $\mathfrak{sp}(2)$, which just consists of area preserving matrices, but this is $\mathfrak{sl}(2)$ so this is already on the list. And for $n = 2$ we have:

Exercise 4. Show that $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$.

Solution. *Proof.* Take (V,ω) to be a four-dimensional symplectic vector space. Then we have an action of $\mathrm{Sp}(4)$ on $\wedge^2 V$, which is 6-dimensional and preserves the symmetric pairing

$$\wedge^2 V \times \wedge^2 V \to \wedge^4 V = \mathbb{C}$$

So we have a map

$$\mathrm{Sp}(V) \to \mathrm{SO}\left(\wedge^2 V\right)$$

The element ω is fixed and its norm $\omega \wedge \omega \neq 0$, so $\mathrm{Sp}(V)$ fixes the 5-dimensional orthogonal complement $I^\perp$ and we have an induced map $\mathrm{Sp}(4,\mathbb{C}) \to \mathrm{SO}(5,\mathbb{C})$. Check it is surjective and at the level of Lie algebras induces the required isomorphism. □

Next we have D_n for $n \geq 4$ which corresponds to $\mathfrak{so}(2n,\mathbb{C})$. This indexing starts here because:

Proposition 3.

$$\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \qquad \mathfrak{so}(6,\mathbb{C}) \simeq \mathfrak{sl}(4,\mathbb{C})$$

and

Proposition 4. $\mathfrak{so}(2,\mathbb{C})$ *is one-dimensional and commutative, and therefore it is not semisimple.*

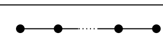
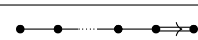
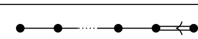
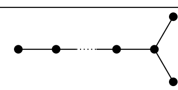
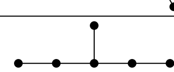
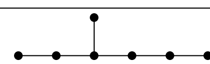
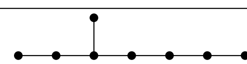
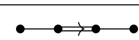
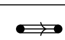
Note that the B_n and D_n are both $\mathfrak{so}(n)$ for odd/even n. So we could hypothetically lump these in the same list, but as we have seen, these algebras have different behaviours. In particular, the simply connected Lie groups corresponding to these Lie algebras have different centers. One has $\mathbb{Z}/4$, and one has $\mathbb{Z}/2 \times \mathbb{Z}/2$.

This is the full list of classical Lie algebras. Looking at this we can sort of ask what kinds of geometries we can do. And this tells us we can do classical euclidean geometry, which has to do with the orthogonal matrices, or you can do symplectic geometry, which of course has to do with the symplectic matrices.

5.2. **Exceptional Lie algebras.** Now we have the exceptional E_6, E_7, E_8, F_4, and finally G_2, and now this is everything.

Remark 4. In a certain sense, if you're a usual algebraist that likes to understand simple things and view them as atoms, these are somehow the atoms that things will be built out of.

5.3. **Dynkin diagrams.** We will come back to these later, but for now we will just see them as "hieroglyphics" which will help us remember this classification.

	$\mathfrak{g}$	Diagram	$Z(G)$	$\pi_1(G)$
$A_n\ (n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$		$\mathbb{Z}/(n+1)\mathbb{Z}$	0
$B_n\ (n \geq 2)$	$\mathfrak{so}(2n+1, \mathbb{C})$		0	$\mathbb{Z}/2\mathbb{Z}$
$C_n\ (n \geq 3)$	$\mathfrak{sp}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	0
$D_n\ (n \geq 4)$	$\mathfrak{so}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	–		$\mathbb{Z}/3\mathbb{Z}$	–
E_7	–		$\mathbb{Z}/2\mathbb{Z}$	–
E_8	–		0	–
F_4	–		0	–
G_2	–		0	–

These appear all over mathematics.

Remark 5. One of the last things Grothendieck did before "leaving" mathematics to become a farmer, is that he found these Dynkin diagrams in resolutions of surface singularities. One can look at algebraic surfaces, and there are these nice classical du Val singularities, and they have natural resolutions, and then these diagrams show up in the geometry of their resolutions.[3]

These pictures bring to light a clear duality called Langlands duality, that isn't made apparent from the list itself. If we reverse the direction of the bar, then A_n, D_n, and E_n are self dual. The diagrams A_n, D_n, and E_n are called *simply-laced*. These are somehow the most basic ones. Then B_n and C_n are dual to one another. Then F_4 and G_2 are said to be *twisted self-dual*.

[3] We won't talk about this, but one can ask Professor Nadler some other time if one is interested.

One might want to play a game where we start with A_n, D_n, and E_n and recover B_n, C_n, F_4, and G_2 form some operations. There's a whole "game" called folding Lie algebras which allows you to take D_n, and sort of collapse the end together to get these double bars in B_n and C_n. Similarly, we can take D_5 and sort of collapse it down to F_4, and collapse D_4 into G_2.

5.4. **Associated groups.** In the table above we have written the centers of the associated Lie groups. Note however that these centers are of the "usual" group associated with the algebra. This is however not the unique simply connected one in the case of $\mathfrak{so}(2n+1)$ and $\mathfrak{so}(2n)$. In this case the unique simply connected one is $\mathrm{Spin}(2n+1)$ and $\mathrm{Spin}(2n)$ respectively. We list the centers so we can determine all of the groups which can be associated to these algebras since we just have to quotient out by subgroups of the center to get these.

In the case of A_n we can quotient out by any subgroup of $\mathbb{Z}/(n+1)\mathbb{Z}$, which is of course just any divisor of $n+1$. In the case of B_n we can take the universal cover, and then these are the only two: $\mathrm{SO}(2n+1)$ and $\mathrm{Spin}(2n+1)$. For C_n, we just get $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})/\mathbb{Z}/2$. Finally, for D_n we get $\mathrm{SO}(2n,\mathbb{C})$ and $\mathrm{SO}(2n,\mathbb{C})/\mathbb{Z}/2$, and $\mathrm{Spin}(2n)$. Only now $\mathrm{Spin}(2n)$ has center $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ if n is even and center $\mathbb{Z}/4\mathbb{Z}$ if n is odd.

Remark 6. One might wonder what Lie groups give rise to the exceptional Lie algebras. We can play the usual game, and take the adjoint representation, then since the algebras are simple, they have no center, so the adjoint representation puts it inside endomorphisms of some vector space, then we can exponentiate these matrices and get a group.

G_2 is the smallest, so it's sort of easiest to get our hands on. If we look at the unit octonions, we can then consider the automorphisms of the non-associative algebra of unit octonions, and this is G_2. In fact all of them arise as automorphisms of something. E_8 is probably the most important one in all of Math, it's somehow the biggest.

6. Finite dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$

It's somehow the case that once one understands $\mathfrak{sl}(2,\mathbb{C})$, and then how to generalize this to $\mathfrak{sl}(3,\mathbb{C})$, there isn't much left to do to understand simple Lie algebras.

We want a similarly natural set of operators to act as a basis like we had for $\mathfrak{sl}(2,\mathbb{C})$. First we define:

$$H_{12}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\qquad H_{23}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

These will again generate a subalgebra:

$$\mathfrak{h}=\mathbb{C}\langle H_{12},H_{23}\rangle\subseteq\mathfrak{sl}(3,\mathbb{C})$$

which is a 2-dimensional abelian subalgebra. Now we can consider all of the following matrices:

$$X_{12}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\qquad X_{13}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\qquad X_{23}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$Y_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad Y_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We choose these since it's a basis of eigenvectors for $\mathfrak{h}$ acting on $\mathfrak{g}$ with the adjoint action.