LECTURE 13
MATH 261A

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The second midterm will be Tuesday October 30th.

1. Root systems

1.1. **Cartan subalgebras.** The "biggest" abelian thing inside $\mathfrak{sl}(3, \mathbb{C})$ is generated by:

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In particular, set $\mathfrak{h} = \langle H_{12}, H_{23} \rangle$.

Fact 1. $\mathfrak{h}$ *is a maximal abelian subalgebra. It also has the property that it is diagonalizable under the adjoint action* ad*.*

The fact that this is abelian means we can simultaneously diagonalize them. Such subalgebras are called *Cartan subalgebras.*

Warning 1. Though this is a convenient Cartan subalgebra it is not unique. However, as we will eventually see, this is actually unique up to conjugation.

1.2. **Roots.** We want to generalize the notion of an eigenvector/eigenvalue for one operator to an algebra. Write $\mathfrak{h}^*$ for the dual of $\mathfrak{h}$. This is the space of possible eigenvalues of $\mathfrak{h}$. Explicitly:

$$\mathfrak{h}^* = \{\lambda : \mathfrak{h} \to \mathbb{C} \text{ linear}\}$$

i.e. in higher dimensions, we should think of eigenvalues as being elements of the dual space.

Define L_1 to be a complex valued function on $\mathfrak{h}$ as follows:

$$L_1(H) = (1,1) \text{ entry of } H$$

for example $L_1(H_{12}) = 1$, and $L_1(H_{23}) = 0$. Define L_2 and L_3 similarly. Note that $L_1 + L_2 + L_3 = 0$.

If we consider $\mathfrak{h} \subseteq \mathbb{C}^3$, then

$$\mathfrak{h}^* = \left(\mathbb{C}^3\right)^* / \mathbb{C} \langle (1,1,1) \rangle$$

where we have quotiented out by the diagonal. We can sort of think of this like looking at the corner of a room as in fig. 1. We will use L_1, L_2, and L_3 as a basis of the dual space.

Date: October 9, 2018.

Now we restrict the adjoint representation to $\mathfrak{h}$. For $H \in \mathfrak{h}$, consider the operator $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$. First let's fix a basis of eigenvectors.

$$X_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad X_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad Y_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 1. The nonzero eigenvalues of $\mathrm{ad}|_{\mathfrak{h}} \circlearrowright \mathfrak{g}$ are called *roots*.

Exercise 1. Check these are eigenvectors.

Solution. We check X_{12} first. Since we are taking H_{12}, H_{23} as our basis for $\mathfrak{h}$, and since $\mathrm{ad}_H = [H, -]$, we need to calculate:

$$[H_{12}, X_{12}] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2X_{12}$$

$$[H_{23}, X_{12}] = 0 - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -X_{12}$$

so we need to find an element of $\mathfrak{h}^*$ which maps $H_{12} \mapsto 2$, and $H_{23} \mapsto -1$. In particular, $L_1 - L_2$ is the root. We write this as α_{12}. The picture here is as in fig. 1.

For X_{13} we have:

$$[H_{12}, X_{13}] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 0 = X_{13}$$

$$[H_{23}, X_{13}] = 0 - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -X_{13}$$

so this has root $L_1 - L_3$. A similar calculation holds for the remaining X_{ij} and Y_{ij}.

So from either brute force or cleverness we get that the roots are all $\alpha_{ij} = L_i - L_j$ for $i \neq j$. These form a hexagon as in fig. 1.

1.3. **Fundamental calculation.** The following lemma plays the role of the "fundamental calculation" that we saw in the $\mathfrak{sl}(2, \mathbb{C})$ case.

Lemma 1. *Suppose V is a representation of $\mathfrak{sl}(3, \mathbb{C})$, and $v \in V$ is an $\mathfrak{h}$ eigenvector with eigenvalue $\lambda \in \mathfrak{h}^*$. Then $X_{ij}v$ is again an $\mathfrak{h}$ eigenvector with eigenvalue $\lambda + \alpha_{ij}$ for $i < j$. Similarly, $Y_{ij}v$ is again an eigenvector with eigenvalue $\lambda + \alpha_{ij} = \lambda - \alpha_{ji}$. for $i > j$*

According to this lemma, the X_{ij}s and Y_{ij}s have sort of "preferred" directions. There is a sort of X-cone which sweeps clockwise between α_{23} and α_{12}, and there is a Y-cone which sweeps clockwise between α_{32} and α_{21}.

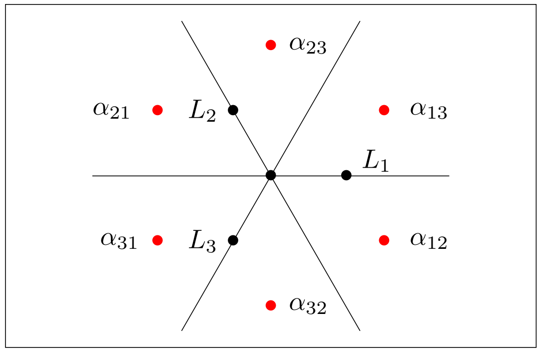


FIGURE 1. The real projection of $\mathfrak{h}^*$. The roots (in red) form a hexagon.

1.4. **Borel subalgebra and positive roots.** Now consider the subalgebra:

$$\mathfrak{b} := \mathfrak{h} + \mathbb{C}\langle X_{ij}\rangle$$

Note $\mathfrak{b}$ is a maximal solvable subalgebra. This makes sense since we somehow know solvable algebras to be upper triangular, and this is upper triangular. This is an example of a *Borel subalgebra.*[1] We will call the roots inside $\mathfrak{b}$ the *positive roots.* These are α_{23}, α_{13}, and α_{12}. We will write the collection of these as R^+.

1.5. **Simple roots.** Notice that $\alpha_{13} = \alpha_{23} + \alpha_{12}$. This somehow indicates that the roots α_{23} and α_{12} are more special. We will call these roots the simple roots. We will write the collection of simple roots as Δ^+. We will eventually see the following fact:

Fact 2. *All of the roots can be recovered from the simple roots.*

2. REPRESENTATIONS OF $\mathfrak{sl}(3, \mathbb{C})$

Now we're finally ready to meet some representations. Recall in the $\mathfrak{sl}(2)$ case the irreducibles were indexed by the natural numbers. We now meet the analogous object.

Definition 2. The dominant (integral) weights are:

$$\Lambda^+ = \mathbb{Z}_{\geq 0}\langle L_1, -L_3\rangle$$

This is an integer lattice of L_1 and $-L_3$ as in fig. 2.
Then the theorem is as follows:

[1] Professor Nadler says that often times in mathematics, when one does something important, ones name becomes a noun forever, however when alive, people typically don't call these objects their own name. For example Hitchin himself never referred to a Hitchin system as such. A more relevant example is that Borel always just called this a "maximal solvable subalgebra" rather than use his own name. Since a choice of such a subalgebra is often accompanied by the choice of a Cartan subgroup, which has something to do with a torus, it is sometimes said that choosing such an $\mathfrak{h}$ and $\mathfrak{b}$ is a choice of a "borus". Professor Nadler wonders if Borel would have preferred this...

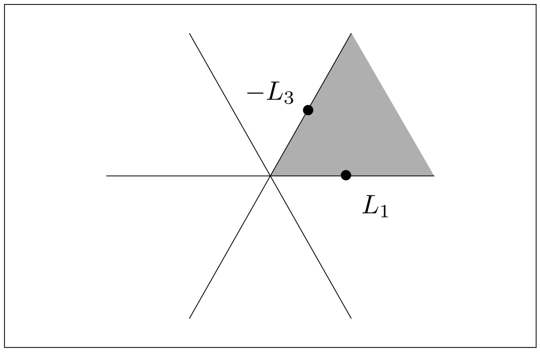


FIGURE 2. Dominant weights for $\mathfrak{sl}(3,\mathbb{C})$.

Theorem 1. *The finite dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$ form a semisimple category $\mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$, and the irreducibles are indexed by Λ^+:*

$$\Lambda^+ \ni \lambda \rightsquigarrow V_\lambda \in \mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$$

where V_λ is some irreducible representation. We can reverse this construction by taking the highest weights with respect to $\mathfrak{b}$.

3. Constructing irreducible representations

Example 1. First we have $V_0 = \mathbb{C}$ is the trivial representation. The picture is just a single weight at 0.

Example 2. $V_{L_1} = \mathbb{C}^3$ will be the standard representation.

$$H = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \circlearrowright \mathbb{C}^3$$

for $a+b+c=0$. The eigenvectors are e_1, e_2, and e_3 which go to ae_1, be_2, and ce_3. The eigenvalues are $L_1(H)=a$, $L_2(H)=b$, and $L_3(H)=c$. The weights are just the L_1, L_2, L_3 that we have seen. Since $X_{ij}e_1 = 0$ for $i<j$ we see that L_1 is the highest weight.

Example 3. The representation $V_{-L_3} = \mathbb{C}^3$ is dual to the standard representation. The weights are as in fig. 3.

Example 4. Now consider the representation $V_{L_1-L_3} = V_{\alpha_{13}}$. We might guess that this is the tensor product $V_{L_1} \otimes V_{-L_3}$. Just like for $\mathfrak{sl}(2,\mathbb{R})$, the eigenvectors of the tensor product are tensors of the eigenvectors, so the weights just add as in fig. 4. Note that this is a nine-dimensional representation. Since V_{-L_3} is the dual of V_{L_1}, we have:

$$V_{L_1} \otimes V_{-L_3} = V_{L_1} \otimes V_{L_1}^* = \mathrm{Hom}(V_{L_1}, V_{L_1})$$

which in particular contains an invariant subspace $\mathbb{C}\left\langle \mathrm{id}_{V_{L_1}} \right\rangle$. From this point of view the identity is:

$$\mathrm{id}_{V_{L_1}} = e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_3^*$$

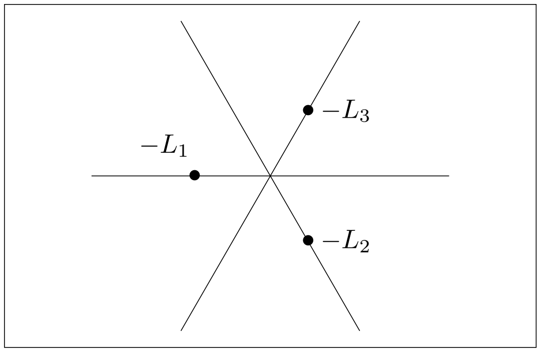


FIGURE 3. The roots of V_{-L_3}.

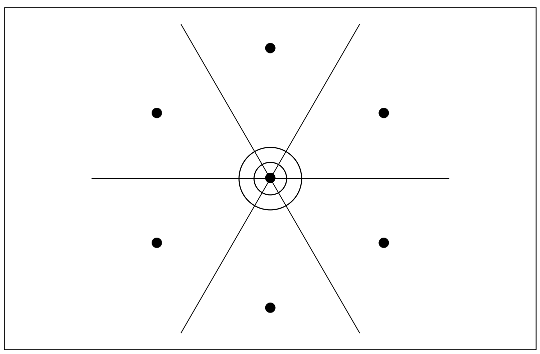


FIGURE 4. Roots of $V_{L_1} \otimes V_{-L_3}$.

This is like the diagonal matrices. Now we can decompose this the tensor into the invariant portion and whatever is left, which we are guaranteed to be a subspace since this is semisimple. The invariant portion just has one weight at 0, and then we are left with an eight-dimensional representation with the same weights, only one less multiplicity at 0. But we can recognize this eight-dimensional representation as the adjoint one, which means $V_{L_1-L_3} = \mathfrak{sl}(3, \mathbb{C})$ is the adjoint representation.

Now the story continues as it did in the case of $\mathfrak{sl}(2, \mathbb{C})$. We can keep tensoring the standard and adjoint representations until we get one that has a desired highest weight, and then we just have to decompose it to find the desired representation.

Example 5. Consider V_{2L_1}. Our first guess might be $V_{L_1} \otimes V_{L_1}$. The weights for this representation will be as in fig. 5. This is not irreducible, since as usual we can write:

$$V_{L_1} \otimes V_{L_1} = \mathrm{Sym}^2(V_{L_1}) \oplus \wedge^2(V_{L_1})$$

Then Sym^2 has the weights

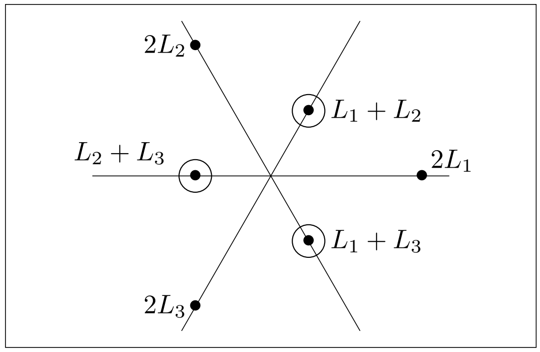
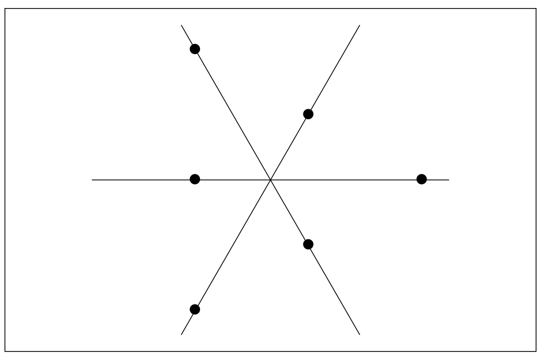
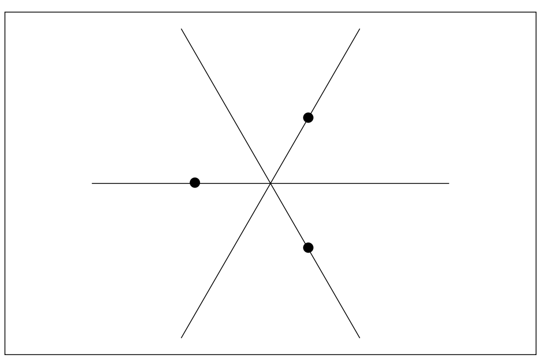


FIGURE 5. Roots of V_{2L_1}.



and $\wedge$ has these



One way to see these pictures is that since order "doesn't matter" in Sym^2, the double multiplicity won't show up, and therefore the remaining three are left to $\wedge^2$. Another way to see this is that the third exterior power is trivial, so

$$V_{L_1}^* \simeq \wedge^2 V_{L_1}$$

In the end we get:

$$V_{2L_1} = \mathrm{Sym}^2\left(V_{L_1}\right)$$

4. A step back

Now we want to generalize this story. To move towards this, we compile a list of "theoretical ingredients" which will be a part of the general story:

(1) $\mathfrak{g}/\mathbb{C}$ simple Lie algebra (could generalize to semi-simple, reductive)
(2) $\mathfrak{h} \subseteq \mathfrak{g}$ Cartan subalgebra (not unique)
(3) $\mathfrak{h}^*$ is a quotient of $\mathfrak{g}^*$, which is the dual space of eigenvalues/weights.
(4) The non-zero eigenvalues of the adjoint representation of $\mathfrak{g}$ form the roots $R \subset \mathfrak{h}^*$.
(5) $\mathfrak{b} \subseteq \mathfrak{g}$ a Borel subalgebra (not unique) (contains $\mathfrak{h}$)
(6) $R^+ \subset R$ positive roots (roots inside Borel)
(7) $\Delta^+ \subset R^+$ simple roots ($R^+ \subset \mathbb{Z}_{\geq 0}\Delta^+$, $R \subset \mathbb{Z}\Delta^+$, Δ^+ linearly independent)
(8) $\Lambda^+ \subset \mathfrak{h}^*$ is the cone of integral dominant weights. (Also the highest weights with respect to $\mathfrak{b}$ for irreducible representations)

Schur functors next time and PBW for $\mathfrak{sl}(3, \mathbb{C})$. We will also discuss how to generalize this whole story, but will likely not prove it in detail.