# LECTURE 14
# MATH 261A

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## 1. Recall

Recall we have the space of all weights $\mathfrak{h}^*$ which contains the dominant integral weights $\mathbb{Z}_{\geq 0} \langle L_1, -L_3 \rangle$. This consists of non-negative integral multiples of the fundamental weights. Recall the fundamental weights are the highest weights for the standard representation and the standard dual representation.

Also recall we had the theorem:

**Theorem 1.** *Irreducible representations are in bijection with the dominant weights. In particular, we send an irreducible representation to its $\mathfrak{b}$ highest weight.*

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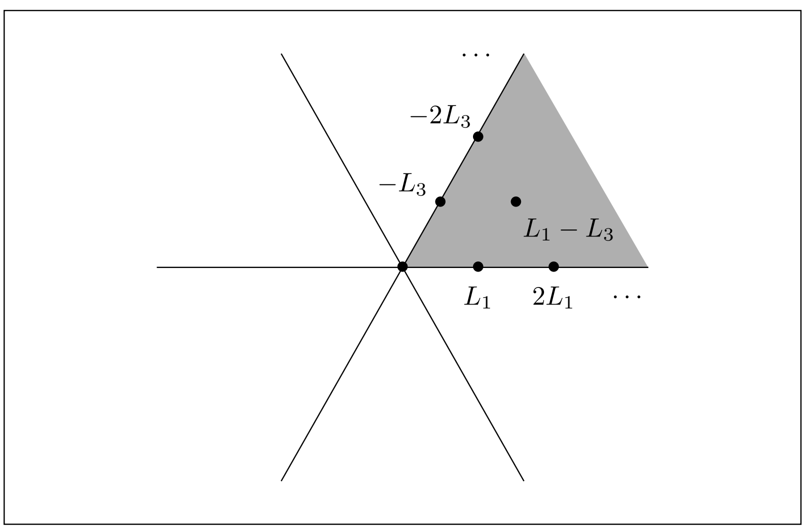


Figure 1. Dominant weights for $\mathfrak{sl}(3, \mathbb{C})$ if we take $\mathfrak{b}$ to be generated by the $H_{ij}$ and $X_{ij}$.

## 2. What choices have we made so far

Professor Nadler says it's relatively fair to say that representation theory is the study of choices. So we now review some of the choices we have made so far in our discussion of $\mathfrak{sl}(3, \mathbb{C})$.

### 2.1. Cartan subalgebra.

Recall we chose a Cartan subalgebra $\mathfrak{h}$ to be some maximal abelian subalgebra. This is not a unique choice, but we do have the following:

**Proposition 1.** *Let $G$ have Lie algebra $\mathfrak{g}$.*

1. *All Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ are conjugate by $G$ under* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.
2. *The* Weyl group
$$W_{\mathfrak{g}} = N_G(\mathfrak{h}) / Z_G(\mathfrak{h}) = N_G(\mathfrak{h}) / H$$
*is a finite, where $H \subseteq G$ is the subgroup with Lie algebra $\mathfrak{h}$.*

*Remark* 1. We don't need to take the unique simply-connected $G$ since whether or not we quotient out by the center $Z(G)$ won't affect the action Ad, so it won't change whether or not these things are related by conjugation.

*Remark* 2. It is very beautiful when the action of a group is transitive, since it is somehow enough to only understand the action on one element. But then we have to ask another very important question, which is what the stabilizer of this one is, and that's what led us to the second half of this proposition.

*Remark* 3. The ambiguity of making a certain choice of Cartan subalgebra is somehow recorded by the $W_{\mathfrak{g}}$ action on $\mathfrak{g}$ by conjugation.

**Example 1.** We now calculate the Weyl group for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. In this case
$$W_{\mathfrak{g}} = \Sigma_n$$
The action of this on $\mathfrak{h} \subseteq \mathfrak{g}$, i.e. the traceless diagonal matrices, is called the standard representation of $W_{\mathfrak{g}}$.

The action is explicitly given by permutation matrices. For example under $\sigma = (12)$,
$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{pmatrix}$$
So this acts on the space of eigenvalues by permuting them as expected.

In the language of the diagrams we have been drawing, the three lines that we were just sort of using to orient ourselves are really representing the hyperplanes over which the elements of $W_{\mathfrak{g}}$ are reflecting. For example, $\sigma = (12)$ is reflecting across the $-L_3$ line.

### 2.2. Borel subalgebra.

We also saw that we have to choose a Borel subalgebra inside of $\mathfrak{g}$ which contains $\mathfrak{h}$. We have a similar proposition for this choice:

**Proposition 2.** *Let $G$ have Lie algebra $\mathfrak{g}$.*

1. *All Borel subalgebras are related by conjugation under* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.
2. *The stabilizer of any $\mathfrak{b}$ is the subgroup $B \subseteq G$ with Lie algebra $\mathfrak{b}$.*

**Definition 1.** The *flag variety* $\mathcal{B}$ of $\mathfrak{g}$ is the space of Borel subalgebras.

The proposition tells us that the flag variety is just $G/B$ since $G$ acts transitively on this space, and the stabilizer is $B$.

*Remark* 4. The flag variety is the space of choices for Borel subalgebras.

**Example 2.** For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we calculate the flag variety. So consider all of the Borel subalgebras inside $\mathfrak{g}$. This is just an ambient vector space, and each Borel subalgebra is a subspace, so we should think of this flag variety as being a submanifold of the Grassmannian of subspaces of $\mathfrak{g}$ i.e.

$$\mathcal{B} \subseteq \mathrm{Gr}(\dim \mathfrak{b}, \dim \mathfrak{g})$$

This all seems a bit abstract, but we're just looking for like $k$ planes in $l$ space, and then some of these are Borel subalgebras and that's what we want.

So let $G = \mathrm{SL}(n, \mathbb{C})$, and $B$ be the upper triangular matrices in $\mathrm{SL}(n, \mathbb{C})$.

**Claim 1.** $G/B$ is naturally isomorphic to flags

$$\langle 0 \rangle = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n$$

where $\dim E_i = i$.

**Exercise 1.** Prove this claim. That is, show that $\mathfrak{sl}(n)$ acts transitively on flags, and that $\mathfrak{b}$ is the stabilizer of the standard flag where $E_i = \mathrm{Span}\langle e_1, \cdots, e_i \rangle$.

**Solution.** This is somehow a standard exercise in linear algebra. Show every flag can be split into a basis, and then $\mathfrak{sl}(n)$ acts transitively on the basis.

**Example 3.** For $n = 1$ the flag variety is a point. For $n = 2$, we are studying flags in $\mathbb{C}^2$. Since the ends are fixed, every flag is just a choice of lines, which is just $\mathbb{CP}^1$.

For $n = 3$, this consists of lines $E_1$ inside planes $E_2$, inside $\mathbb{C}^3$. The collection of these isn't anything special we have seen before, but it is inside the collection of choices of lines crossed with choices of planes:

$$\mathbb{CP}^2 \times \left(\mathbb{CP}^2\right)^*$$

The line is represented by a vector $v$, and the plane is represented by a covector $w$. The condition is just that the line must be inside the plane. In particular, $E_1$ is the span of $v$ and $E_2$ is orthogonal to $w$, the kernel of $w$. So the flag variety is cut out by the equation $w(v) = 0$. So we start with four dimensions, and insisting on this equation gives us three dimensions, which is good since $\mathfrak{sl}(3)$ is eight-dimensional, and $B$ is 5-dimensional.

There are many beautiful things to be said about flag varieties, but we just state one more thing. Recall we really liked $\mathbb{CP}^1$ since it was just projective space. But as it turns out, we can think of this flag variety as a sort of iterated projective space. So let's say we forget the line and remember the hyperplane, so we're projecting:

$$\begin{array}{c} \mathrm{SL}(n, \mathbb{C}) / B(n) \\ \big\downarrow \\ \left(\mathbb{CP}^{n-1}\right)^* \end{array}$$

and then the rest of the data for this fixed hyperplane is just a flag in this hyperplane, so the fiber is $\mathrm{SL}(n-1, \mathbb{C}) / B(n-1)$.

The $n = 2$ example was literally $\mathbb{CP}^1$, the $n = 2$ example was just sort of roughly a $\mathbb{CP}^1$ and a $\mathbb{CP}^2$, and the next one is put together as a $\mathbb{CP}^1$, $\mathbb{CP}^2$, and a $\mathbb{CP}^3$. This is however not to say that these aren't somehow put together in an interesting way, because they are.

2.3. **Borus.** The two choices of a Cartan and Borel subalgebra together make the choice of a Borus, and now we have the following proposition bringing them together:

**Proposition 3.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, then*

*(1) All "boruses" are conjugate by $G$.*
*(2) The stabilizer is isomorphic to $Z_G(\mathfrak{h}) \simeq H$.*

2.4. **Back to representation theory.** Now we have the following as a result of these propositions:

**Corollary 1.** *The Weyl group acts simply transitively on the Borel subalgebras containing $\mathfrak{h}$.*

**Exercise 2.** Prove that the above three propositions imply this corollary.

So note that the choice of a borus that we made last time determined which chamber was the dominant one. There are actually four other choices of chambers that are just as good. We illustrate this with some examples. For all of them fix $\mathfrak{h} \subseteq \mathfrak{sl}(n, \mathbb{C})$ the usual diagonal Cartan subalgebra.

**Example 4.** Let $n = 2$. Then $\Sigma_2 \circlearrowright \langle \mathfrak{b} \text{ containing } \mathfrak{h} \rangle$ acts simply transitively.

$$\mathfrak{b} = \left\langle \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\rangle \qquad \mapsto \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathfrak{b} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\rangle = \mathfrak{b}^{\mathrm{op}}$$

**Example 5.** Let $n = 3$. Then $\Sigma_3 \cdot \langle \mathfrak{b} \text{ containing } \mathfrak{h} \rangle$. We know $\mathfrak{b}$ upper triangular matrices works. Now we can conjugate to find the others. We know $\Sigma_2 = \langle (12), (23) \rangle$. first we lift these to matrices:

$$(12) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

FIGURE 2. These are two distinct tangles which represent products of transpositions which are the same in $\Sigma_3$.

Now we conjugate to get:

$$
\begin{tikzcd}
\langle e_1\rangle \subset \langle e_1, e_2\rangle & \langle e_2\rangle \subset \langle e_1, e_2\rangle & \langle e_3\rangle \subset \langle e_1, e_3\rangle & \langle e_2\rangle \subset \langle e_1, e_2\rangle \\
& \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \arrow[r, "(23)"] & \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \arrow[dr, "(12)"] & \\
\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \arrow[ur, "(12)"] \arrow[dr, "(23)"] & & & \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \\
& \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \arrow[r, "(12)"] & \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & 0 & * \end{pmatrix} \arrow[ur, "(23)"] & \\
\langle e_1\rangle \subset \langle e_1, e_2\rangle & \langle e_1\rangle \subset \langle e_1, e_3\rangle & \langle e_2\rangle \subset \langle e_2, e_3\rangle & \langle e_2\rangle \subset \langle e_1, e_2\rangle
\end{tikzcd}
$$

**Exercise 3.** Show that this is the case by explicitly conjugating.

We write the flags stabilized by these choices of Borels above and below the diagram. The fact that these paths give the same final result is a result of the fact in fig. 2.

Now fix a Cartan subalgebra $\mathfrak{h}$. Inside of $\mathfrak{h}^*$ we want to talk about the $\mathfrak{b}$-dominant integral weights $\Lambda^+ \subseteq \mathfrak{h}^*$. All integral weights form a lattice inside $\mathfrak{h}^*$, and now we can ask how we chose this cone. We chose those which were "positive" with respect to $\mathfrak{b}$. The cone $\Lambda^+$ consisted of the possible highest weights for $\mathfrak{b}$. So if we conjugate $\mathfrak{b}$ by a permutation matrix, this is just acting this permutation on this cone by reflecting over the $L_i$, so it gives us the alternative cones.

## 3. Construction of irreducible representations

Fix a borus $\mathfrak{h} \subset \mathfrak{b}$. We want to construct the irreducible representation with a given highest weight. Recall in the $\mathfrak{sl}(2, \mathbb{C})$ case we saw

$$V_n = \operatorname{Sym}^2 V_1$$

and

$$\bigoplus_{n=0}^{\infty} V_n = \mathbb{C}[u, v]$$

so since we're trying to construct polynomials we might have guessed that.

So now in the $\mathfrak{sl}(3, \mathbb{C})$ case, we have $V_0 = \mathbb{C}$ is trivial, $V_{L_1} \simeq \mathbb{C}^3$ is the standard representation, and $V_{-L_3} \simeq \mathbb{C}^3$ is the dual standard representation. Then we claim the following:

**Claim 2.** $\mathrm{Sym}^n(V_{L_1})$ is irreducible with highest weight $nL_1$, and $\mathrm{Sym}^n(V_{-L_3})$ is irreducible with highest weight $-nL_3$.

So we have the same $\mathfrak{sl}(2, \mathbb{C})$ picture along the $-L_3$ and $L_1$ lines. Just as before we have the following decomposition:

$$\bigoplus_n \mathrm{Sym}^n V_{L_1} \simeq \mathbb{C}[u, v, w] \qquad \bigoplus_n \mathrm{Sym}^n V_{-L_3} \simeq \mathbb{C}[u^*, v^*, w^*]$$