LECTURE 15
MATH 261A

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1. Clarifications

We will continue our discussion of representations of $\mathfrak{sl}(3,\mathbb{C})$. But first some clarifications. We saw $\Lambda = \mathbb{Z}\langle L_1, L_2, L_3\rangle$ and $\Lambda^+ = \mathbb{Z}_{\geq 0}\langle L_1, -L_3\rangle$ concretely, but now we offer a sort of invariant definition.

1.1. **Weight lattice.** Any time we have $\mathfrak{g}/\mathbb{C}$ a Lie algebra, we can associate to this a simply-connected complex Lie group $G/\mathbb{C}$. For example, for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ we get $G = \mathrm{SL}(n,\mathbb{C})$. Now if we choose a borus in $\mathfrak{g}$, we will get subgroups of G that play a similar role. So choosing $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ we will get subgroups $H \subseteq B \subseteq G$.

Now we want to compare $\mathfrak{h}^* = \mathrm{Hom}_{\mathbf{Vect}}(\mathfrak{h},\mathbb{C})$ to $\Lambda = \mathrm{Hom}_{\mathbf{Ab}}(H,\mathbb{C}^\times)$. We know $\mathfrak{h}^*$ consists of the eigenvalues of the (irreducible) $\mathfrak{h}$ representations, and Λ consists of the eigenvalues of the irreducible H-representations. This tells us that $\mathfrak{h}^* \simeq \mathbb{C}^{\dim \mathfrak{h}}$ and $\Lambda \simeq \mathbb{Z}^{\dim H}$. Note also that $\mathfrak{h} \simeq \mathbb{C}^{\dim\mathfrak{h}}$ and $H \simeq (\mathbb{C}^\times)^{\dim H}$.

In fact, we naturally have that the weight lattice Λ is contained in $\mathfrak{h}^*$. This map is differentiation, since Λ consists of maps of Lie groups, and $\mathfrak{h}^*$ consists of maps of Lie algebras. But this is not equality, since there are plenty such maps of Lie algebras that don't come from maps of Lie groups.

Warning 1. This inclusion is proper since H is not simply-connected.

1.2. **Dominant weight lattice.** Recall the roots R are the nonzero eigenvalues of the adjoint representation ad. Half of these will be in our choice of Borel subalgebra. We call these roots the positive roots $\Delta^+ \subseteq R$. Equivalently these are the roots in the adjoint representation of $\mathfrak{b}$. Then there are the simple roots $\Sigma^+ \subseteq \Delta^+$ which form a basis. Now write R^+ for the positive root cone $R^+ = \mathbb{Z}_{\geq 0}\Delta^+$.

Recall the killing form is an inner product on $\mathfrak{g}$.

Exercise 1. Show that $\mathfrak{g}$ is semisimple iff the killing form is nondegenerate. This is called Cartan's criterion.

This means it induces an inner product on $\mathfrak{g}^*$, and in particular on $\mathfrak{h}^*$ by restriction. Then the dominant cone consists of the lattice points which are non-negative when paired with the positive root cone. Explicitly:

$$\Lambda^+ = \left\{\lambda \in \Lambda \subseteq \mathfrak{h}^* \,|\, \forall \alpha \in \Delta^+, \langle\lambda,\alpha\rangle \geq 0\right\}$$

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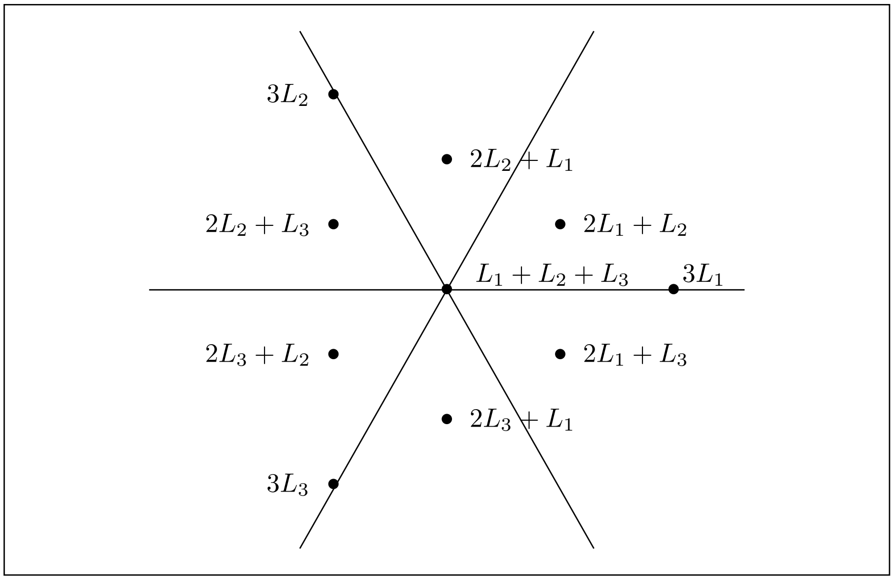

FIGURE 1. Roots of $V_{3L_1} = \operatorname{Sym}^3 V_{L_1}$.

2. Constructing irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$

We want to construct irreducible representations V_λ for $\lambda \in \Lambda^+$ for $\mathfrak{sl}(3, \mathbb{C})$. Recall we have already seen $V_{L_1} = \mathbb{C}^3$ is the standard representation and $V_{-L_3} = V_{L_1+L_2}$. When this is written in the first way it is supposed to be dual to the standard, and the second way suggests it is $\wedge^2 \mathbb{C}^3$. We also saw that $V_{\alpha_{13}}$ was the adjoint representation. Finally we saw that we have $\operatorname{Sym}^n V_{L_1}$ has highest weight nL_1 and the other weights are as in the following example.

Example 1. Consider $\operatorname{Sym}^3 V_{L_1}$. This has weights as in fig. 1.

A similar story holds for $-L_3$.

Example 2. Consider $\operatorname{Sym}^2 V_{-L_3} = V_{-2L_3}$. This has weights as in fig. 2.

The question that remains, is what if we want a representation which is a linear combination of L_1 and L_2 such as $mL_1 - nL_3$. The idea here is that this highest lives in $\mathfrak{h}^*$ as in fig. 3 and then we claim the following:

Claim 1. The non-zero weights of V_λ lie in the convex hull of $W \cdot \lambda$ as in fig. 3.

Remark 1. Notice that in the case of V_{nL_1} and V_{-mL_3} we have a sort of degenerate hexagon.

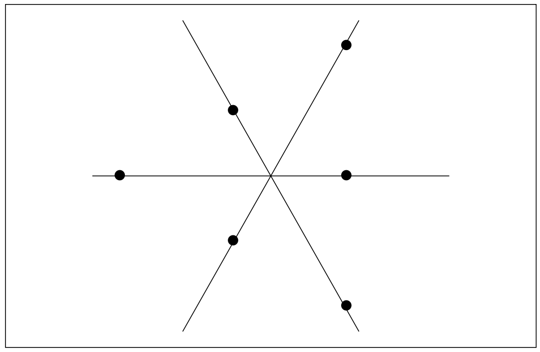

FIGURE 2. Weights of V_{-2L_3}.

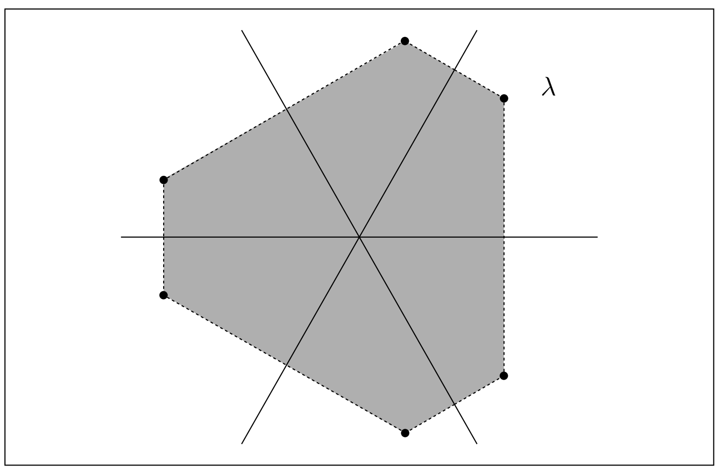

FIGURE 3. The convex hull $W \cdot \lambda$.

The idea of this proof will be to restrict to the copies of $\mathfrak{sl}(2, \mathbb{C})$ inside $\mathfrak{sl}(3, \mathbb{C})$. In particular the block diagonal matrices:

$$\mathfrak{l}_{12} = \begin{pmatrix} * & * & \\ * & * & \\ & & \end{pmatrix} \qquad \mathfrak{l}_{23} = \begin{pmatrix} & & \\ & * & * \\ & * & * \end{pmatrix} \qquad \mathfrak{l}_{13} = \begin{pmatrix} * & & * \\ & & \\ * & & * \end{pmatrix}$$

These are examples of *Levi subalgebras*. The roots in $\mathfrak{l}_{12}$ are as in fig. 4.

We want to think of these subalgebras as moving along the line spanned by their roots in the same sense that $\mathfrak{sl}(2, \mathbb{C})$ moved along the real line[1]. For example if we restrict a representation of $\mathfrak{sl}(3, \mathbb{C})$ to (say) $\mathfrak{l}_{12}$. We get these lines running diagonal all parallel to the line connected α_{12} and α_{21}.

[1] Professor Nadler says the secrets to the universe come from understanding the Cartan subalgebra, and understanding $\mathfrak{sl}(2)$.

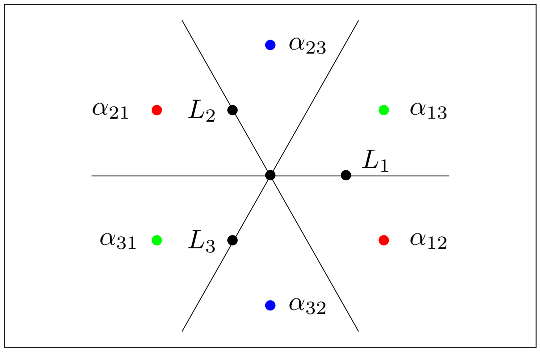

FIGURE 4. In red we have the roots of $\mathfrak{l}_{12}$, in green we have the roots of $\mathfrak{l}_{13}$, and in blue we have the roots of $\mathfrak{l}_{23}$.

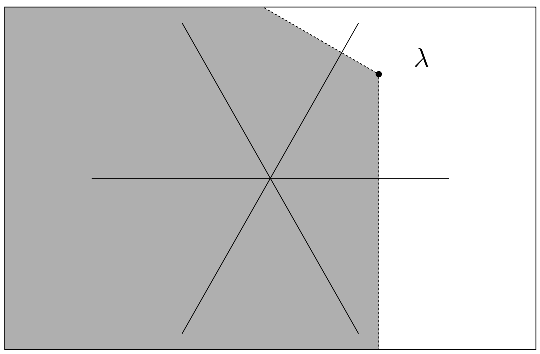

FIGURE 5. Because λ is the highest weight, and because all of the weights are given by acting Y on the highest weight, we know that every weight must be contained in this hull.

Proof. Say we have some highest weight, then the Xs all bring it to zero. We haven't shown that repeatedly acting Y on the highest weight gives us everything, but taking that for granted, nothing is nonzero outside of the hull pictured in fig. 5. But now it must be symmetric about the lines which W reflects over since all of these parallel lines are $\mathfrak{sl}(2,\mathbb{C})$ representations. This is exactly the convex hull in fig. 3. □

3. Constructing irreducible representations of $\mathfrak{sl}(3,\mathbb{C})$

Recall we saw:

$$\mathfrak{sl}(2,\mathbb{C}) \circlearrowright \mathbb{C}^2 \rightsquigarrow \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n\left(\mathbb{C}^2\right) = \mathcal{O}\left(\mathbb{C}^2\right)$$

where $\mathcal{O}\left(\mathbb{C}^2\right)$ consists of polynomial functions. Now we have $\mathfrak{sl}(3,\mathbb{C}) \circlearrowright \mathbb{C}^3$ standard, as well as the dual to this and we want to build a similar picture.

3.1. **Fundamental affine space.** The fundamental affine space is:

$$X_{\mathfrak{sl}(3,\mathbb{C})} = \left\{(v,\lambda) \in \mathbb{C}^3 \times (\mathbb{C}^*)^3 \mid \lambda(v) = 0\right\}$$

Another, more general, way of thinking about this is:

$$X_{\mathfrak{sl}(3,\mathbb{C})} = \left\{(v, w_1 \wedge w_2) \in \mathbb{C}^3 \times \wedge^2\mathbb{C}^3 \,|\, v \wedge (w_1 \wedge w_2) = 0\right\}$$

this is nice since it looks like a version of a flag.

Claim 2.

$$\mathcal{O}\left(X_{\mathfrak{sl}(3,\mathbb{C})}\right) = \bigoplus_{\lambda \in \Lambda^+} V_\lambda$$

where every irreducible appears exactly once.

So every simple Lie algebra has such a fundamental affine space, so this should give some hint as to how we should generalize this.

Remark 2. The proof next time will follow from the Peter-Weyl theorem.

Let's find some of our favorite representations in this.

Example 3. Write v and λ in coordinates:

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \lambda = (y_1, y_2, y_3)$$

The trivial representation is given by constant functions, $V_{L_1} = \mathbb{C}^3 = \mathbb{C}\langle x_1, x_2, x_3\rangle$, and similarly $V_{-L_3} = \left(\mathbb{C}^3\right)^* = \mathbb{C}\langle y_1, y_2, y_3\rangle$. The adjoint representation is given by:

$$V_{L_1 - L_3} = \mathfrak{sl}(3,\mathbb{C}) = \mathbb{C}\langle x_i y_j\rangle$$

where these satisfy:

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$$

so this object is 8-dimensional.