LECTURE 16

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Fundamental affine space

Definition 1. The fundamental affine space X_n of $\mathfrak{sl}(n,\mathbb{C})$ is contained in

$$X_n \subset \mathbb{C}^n \times \wedge^2\mathbb{C}^n \times \cdots \times \wedge^{n-1}\mathbb{C}^n$$

In particular, it comprises collections of the following form:

$$(a_1, b_1 \wedge b_2, c_1 \wedge c_2 \wedge c_3, \cdots)$$

These are elementary forms in the sense that they aren't sums of such elements. We also insist on the "inclusions" $a_1 \wedge (b_1 \wedge b_2) = 0$ and the higher-dimensional analogues[1] i.e. $a_1 \wedge (c_1 \wedge c_2 \wedge c_3)$ etc. I.e. the spans are included in the larger if nonzero.

This is supposed to look like the flag variety.

To see that this isn't so mysterious, consider the open subset $X_n^0 \subset X_n$ where all terms are nonzero. This space has a natural projection

$$\begin{array}{c} X_n^0 \\ \downarrow \\ \mathcal{B} \end{array}$$

to the flag variety of flags in n-space, $\mathcal{B}$, where we map these primitive forms to their span. This makes sense since we required them to be nonzero.

This is surjective, and in fact a fibration with fiber as follows. Each time we sort of introduce a new vector, all we care about is preserving the "volume" of the parallelepiped, there's sort of $\mathbb{C}^\times$ many choices. So the fibers are $(\mathbb{C}^\times)^{n-1}$.

Note the following:

$$\dim \mathcal{B}_n = \frac{n(n-1)}{2} \qquad \dim X_n^0 = \frac{n(n-1)}{2} + n - 1 = \frac{(n+2)(n-1)}{2}$$

Now we have the following lemma:

Lemma 1. *$X_0^n \simeq G/N$ where $N = [B,B] = \left\langle b_1 b_2 b_1^{-1} b_2^{-1} \in B \right\rangle$ consists of upper diagonal matrices with 1 on the diagonal.*

Proof. We need to show G acts transitively, and the stabilizer is N. I.e. we take any list of such forms to any other list of forms using G. So we write down our favorite element of X_0^n:

$$(e_1, e_1 \wedge e_2, \cdots, e_1 \wedge \cdots \wedge e_{n-1})$$

Date: October 18, 2018.

[1] These are called Plücker equations.

and then any other one is:

$$(a_1, b_1 \wedge b_2, c_1 \wedge c_2 \wedge c_3, \cdots)$$

and we need a matrix in $\mathrm{SL}(n)$ which takes us there. The first column should just be a_1. Now because of the inclusion equations, we can write $b_1 \wedge b_2$ as $a_1 \wedge b_2'$ for some b_2'. Explicitly, we can write a matrix with columns:

$$\begin{pmatrix} a_1 & b_2' & \cdots \end{pmatrix}$$

this is basically just a change of basis matrix.

Now what group elements fix this nested sequence? We better have $1, 0, \cdots, 0$ in the first column, and we have to maintain the span of b_1 and b_2, so we have to have 0 after the second coordinate, so we get $(*, 1, 0, \cdots, 0)$ in the second column, since we have 1 on the diagonal. All together we get:

$$\begin{pmatrix} 1 & * & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is of course N. □

Corollary 1. *Recall we already saw $\mathcal{B}_n = G/B$, so this is a fibration for $B/N \simeq T$.*

Example 1. The fundamental affine space of $\mathfrak{sl}(2, \mathbb{C})$ is $X_2 = \mathbb{C}^n$ and $X_2^0 = \mathbb{C}^2 \setminus \{0\}$. The projection maps $v \mapsto l \in B_n \simeq \mathbb{P}^1$ where $l \simeq \mathbb{C}\langle v \rangle$.

Remark 1 (For algebraic geometers). We have these two creatures X_n^0 and X_n and we might sort of wonder why we're considering both of them. Dealing with just G/N is very nice, but it is not an affine variety, as we saw in the previous example: $\mathbb{C}^2 \setminus 0$ is not affine (though it is quasi-affine). The affine closure of X_n^0 is X_n.

Example 2. We know the dimension of $\mathfrak{sl}(3, \mathbb{C})$ is 8, and then the dimension of X_3^0 is $8 - 3 = 5$. T is two dimensions so when we divide by this we get down to the three-dimensional flag variety. We start with $a_1 = ae_1$ then $b_2' = be_2$, and all together

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a^{-1}b^{-1} \end{pmatrix}$$

since we need determinant one. So this is a map from points above to determinant one diagonal matrices.

The fiber living above the standard flag:

$$E_1 = \mathbb{C}\langle e_1 \rangle \qquad E_2 = \mathbb{C}\langle e_1, e_2 \rangle$$

is $(\mathbb{C}^\times)^2$, so it is somehow missing the axes. Then the closure is $\overline{T} \simeq \mathbb{C}^2$, but we got this only from paying attention to a and b, but we really want something which pays equal attention to all coordinates. More democratically, T is naturally a subset of $(\mathbb{C}^\times)^3$ cut out by $\det = 1$. The quotient picture was like a photograph of the corner of the room, and this is like the slice of the corner of the room.

1.1. **Relationship with fundamental representations.** We now return to the following proposition:

Proposition 1.

$$\mathcal{O}(X_n) = \bigoplus_{\lambda \in \Lambda^+} V_\lambda$$

where V_λ is a representation of highest weight λ, and each V_λ appears exactly once.

Proof. We need to calculate the highest weights in $\mathcal{O}(X_n)$. Recall these are the invariants under N, so they are in $\mathcal{O}(X_n)^N$ where $N = [B, B]$ consists of the strictly upper triangular matrices as usual. Recall the Lie algebra of N is $\mathbb{C}\langle X_{ij}\rangle$.

Claim 1. There exists an open B orbit in X_n isomorphic to B.

Proof. Take the opposite standard flag,

$$e^{\mathrm{op}} = \{e_n, e_n \wedge e_{n-1}, \cdots\}$$

and then we claim B acts on this with an open orbit. Start with e_3, then take $e_3 \wedge e_2$.

Exercise 1. Show that $B \cdot e^{\mathrm{op}}$ consists of all configurations with nonzero terms with spans transverse to the standard configuration e. Also check that if $b \cdot e^{\mathrm{op}} = e^{\mathrm{op}}$ then $b = 1$.

and we are done. □

This means $B \simeq B \cdot e^{\mathrm{op}} \subseteq X_n$ is an open dense subset. Now take the functions $\mathcal{O}(X_n)$ and restrict them to $\mathcal{O}(B \cdot e^{\mathrm{op}}) = \mathcal{O}(B)$, and since B is dense this must be an inclusion. Now we can also restrict:

$$\mathcal{O}(X_n)^N \hookrightarrow \mathcal{O}(B \cdot e^{\mathrm{op}})^N \simeq \mathcal{O}(N \backslash B) \simeq \mathcal{O}(T)$$

So N invariant functions give us functions on T. In conclusion we have an injection:

$$\mathcal{O}(X_n)^N \hookrightarrow \mathcal{O}(T)$$

But we know the weight lattice Λ is just monomial functions on T, $\Lambda = \mathrm{Hom}_{\mathbf{Ab}}(T, \mathbb{C}^\times)$, so $\mathcal{O}(T)$ is just the $\mathbb{C}$-span of the weight lattice $\mathbb{C}\langle\Lambda\rangle$.

Example 3. The idea here is

$$\mathcal{O}(\mathbb{C}^\times) = \left\{ \sum_{i=-N}^{N} c_i z^i \right\}$$

and in this case $\Lambda \simeq \mathbb{Z} = \{z^i \mid i \in \mathbb{Z}\}$.

So to every N-invariant function we have assigned a linear combination of weights. But we know N-invariants are highest weights, so the actual function we get can't be arbitrary, it has to be highest weight. I.e. the image of any particular highest weight must be a monomial. I.e. the injection above is T-equivariant.

There is a G action $G \circlearrowright \mathcal{O}(X_n)$ where $(g \cdot f)(x) = f(g^{-1}x)$. Now look at $\mathcal{O}(X_n)^N$. Then claim that this still has a T action given by the same formula. So $(t \cdot f)(x) = f(t^{-1}x)$. So now we want to check the following. Look at $n(tf)$ and we want to show that this is just tf:

$$n(tf) = tt^{-1}(ntf) = t(n'f) = tf$$

so there's still a T action.

Since the construction is T-equivariant we have the map

$$\mathcal{O}\left(X_n\right)^N \hookrightarrow \mathcal{O}\left(T\right)$$

and a left T-action on both. This is just since restriction "commuted" with the T-action. A highest weight vector of highest-weight λ must get mapped to some scale of the highest weight monomial z^λ. Therefore we can conclude that the highest weight vectors inject into the possible weights. I.e. there exists at most one dimension of highest weight vector for any given weight.

Now conversely we claim that

$$V_{\text{std}}, \wedge^2 V_{\text{std}}, \cdots, \wedge^{n-1} V_{\text{std}} = V_{\text{std}}^*$$

are all inside $\mathcal{O}\left(X_n\right)$. Recall the highest weights of these representations are a basis for the dominant weights. Now let $f_1, \cdots, f_{n-1}$ be highest weight vectors in each of the $\wedge^i V_{\text{std}}$ inside $\mathcal{O}\left(X_n\right)$. Products of these will be nonzero, and this product will still be N invariant. Therefore it has to contain at least one irreducible of every highest weight. I.e. $f_1^{i_1} f_2^{i_2} \cdots f_{n-1}^{i_{n-1}}$ is a nonzero N-invariant vector of weight $i_1\lambda_1 + \cdots + i_{n-1}\lambda_{n-1}$ i.e. a highest weight of this eigenvalue. This is in Λ^+, therefore for every $\lambda \in \Lambda^+$ there exists highest weight representation V_λ inside $\mathcal{O}\left(X_n\right)$. ■