LECTURE 17
MATH 261A

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Today we will finish discussion representation of $\mathfrak{sl}(3,\mathbb{C})$ and then talk about representations of $\mathfrak{sp}(4)$.

1. Representations of $\mathfrak{sl}(3,\mathbb{C})$

1.1. Recall.

Recall we have the following theorem:

Theorem 1. *$\mathbf{Rep}_{fd}(\mathfrak{sl}(3,\mathbb{C}))$ is semisimple and the irreducibles are in bijection with $\lambda \in \Lambda^+$.*

So far we have done the following:

1. Constructed $V_\lambda \subset \mathcal{O}(X_3)$ for $\lambda \in \Lambda^+$.
2. Some analysis of possible weights. For $\lambda \in \Lambda^+$ we discussed that the possible weights for any irreducible representation with this highest weight will live in some sort of generalize hexagon.
3. Semi-simplicity of this category. This is done in the exact same way as it was done for $\mathfrak{sl}(2,\mathbb{C})$. Recall we said that representations of $\mathfrak{sl}(2,\mathbb{C})$ are the same as representations of $\mathrm{SL}(2,\mathbb{C})$, which are the same as representations of $\mathrm{SU}(2)$, and then we put metrics on everything and decomposed. Similarly, we have:
$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{sl}(3,\mathbb{C})) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathrm{SL}(3,\mathbb{C})) \simeq \mathbf{Rep}_{\mathrm{fd}}(\mathrm{SU}(3))$$
and these all have invariant inner products, so subrepresentations have orthogonal complements. This technique generalizes even further to all simple Lie algebras. All we're really doing here is bring it to its unique simply connected Lie group, then go to the maximal compact subgroup, and then construct an invariant metric.

What we haven't shown is that any two irreducibles with the same highest-weight must be isomorphic. Once we do this, we will be done with the proof of the theorem. Recall for $\mathfrak{sl}(2,\mathbb{C})$ this was accomplished using Verma modules, which we will use in this case as well.

1.2. Verma modules.

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ be a choice of Borus. Recall $\mathfrak{b}$ is maximal solvable, and $\mathfrak{h}$ is maximal abelian, and ad-diagonalizable.

Note that $\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$, and in fact this composition is an isomorphism.[1] I.e. $\mathfrak{h}$ lives in $\mathfrak{b}$ as a subalgebra and a quotient. This tells us that $\mathfrak{b} \simeq \mathfrak{h} \ltimes [\mathfrak{b},\mathfrak{b}]$.

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[1] It makes sense that the quotient $\mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ is abelian since $\mathfrak{b}$ is solvable.

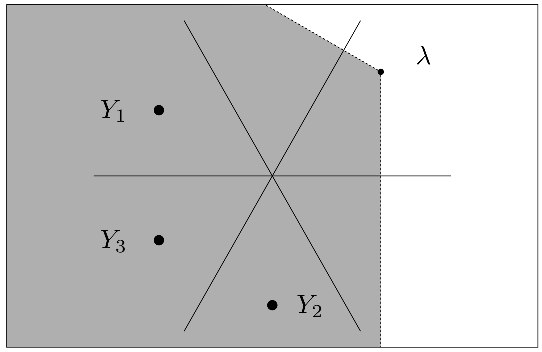

FIGURE 1. The ordering of the negative roots used in the calculations.

Fix a character (i.e. a linear map) $\lambda : \mathfrak{h} \to \mathbb{C}$ of $\mathfrak{h}$. In other words think of $\lambda \in \mathfrak{h}^*$ as a weight. We want to view λ as a 1-dimensional representation $\mathbb{C}_\lambda$ of $\mathfrak{h}$, i.e. C_λ is a complex line where $H \in \mathfrak{h}$ acts as: $H \cdot v = \lambda(H) v$.

Definition 1. The *Verma module* I_λ is

$$I_\lambda := \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$$

Note that $\mathbb{C}_\lambda$ is a representation of $\mathfrak{b}$ via $\mathfrak{b} \to \mathfrak{h}$, so it can of course be a representation of $\mathcal{U}_\mathfrak{b}$. In other words, $X \cdot v = 0$ and $H \cdot v = \lambda(H) v$ for $X \in [\mathfrak{b}, \mathfrak{b}]$, and $H \in \mathfrak{h}$.

One might wonder why anyone would bother defining this in the first place. As it turns out, this is the standard way to construct a module which has the following universal property. If you find a vector in your representation such that X kills it, and H acts by λ, then there exists a unique map from I_λ to your representation. I.e.

$$\mathrm{Hom}_\mathfrak{g}(I_\lambda, V) \simeq \mathrm{Hom}_\mathfrak{g}(\mathbb{C}_\lambda, V)$$

which consists of the $[\mathfrak{b}, \mathfrak{b}]$-invariants and $\mathfrak{h}$ eigenvectors of weight λ.

Now the following gives us a basis for the Verma module:

Theorem 2 (PBW)**.** *Choose an ordering of the positive roots R^+, which gives us an ordering of the negative roots. Then a basis of $\mathcal{U}\mathfrak{g}$ is given by ordered monomials $Y^a H^b X^c$.*

This immediately implies that a basis for I_λ is given by the ordered monomials Y^a. This implies that we understand the weights of I_λ. Order the negative roots as in fig. 1. Then we can calculate the dimension of the weight spaces of these weights as in fig. 2. The weight spaces which are given by successive actions of Y_1 or Y_2 are all of dimension 1. However if we act by $Y_1 Y_2$, this is the same as just acting Y_3, so this space has dimension 2. As it turns out, each shell consists exactly of spaces with the same dimension, and every time you venture one shell deeper the dimension increases by 1.

Example 1. The corner of the third shell can be reached by monomials Y_3^2, $Y_1^2 Y_2^2$, and $Y_1 Y_2 Y_3$.

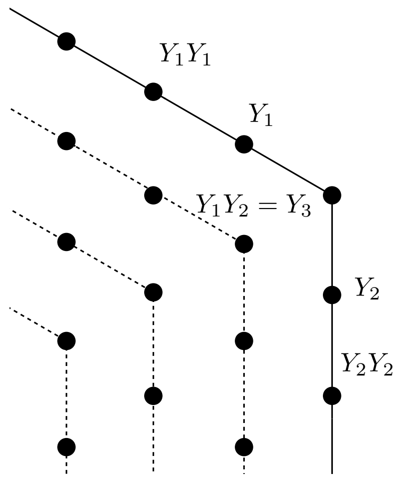

FIGURE 2. The different potential ways of reaching a given weight with ordered monomials gives the dimension of the weight space by PBW.

Exercise 1. Recall the pattern for $\mathfrak{sl}(2,\mathbb{C})$ was somehow linear with slope 0, then this is linear of slope 1. Find the pattern for $\mathfrak{sl}(4,\mathbb{C})$.

Now we want to write a closed formula for the character of I_λ. Well we know that it will somehow be $e_\lambda * (\cdots)$ for something inside. In particular:

$$\mathrm{ch}(I_\lambda) = e_\lambda * \left(\prod_{\alpha_i \in -R^+} \frac{1}{1-e_{\alpha_i}}\right)$$

We would love this to be a function with compact support on Λ, i.e. an element of $\mathbb{C}[\Lambda]$ for $\lambda \in \Lambda$, but we end up taking the completion $\widehat{\mathbb{C}[\Lambda]}$.

1.3. **Back to finite dimensional representations.** Suppose V is an irreducible finite dimensional representation of highest weight $\lambda \in \Lambda^+$. Now we will try to state some facts and arguments which will hopefully show that V is unique up to isomorphism.

Proposition 1. *The natural map $I_\lambda \to V$ given by the highest-weight vector is surjective.*

Proof. This is somehow a tautology, because if it wasn't surjective then the image would be a subspace of V, which is of course impossible since V is irreducible. $\square$

Remark 1. This is saying that we can get to anything in V by applying the Ys.

Now we construct a resolution of V in terms of the Verma modules. Return to $\mathfrak{sl}(3,\mathbb{C})$. We will discuss how it generalizes later. Recall we already know the nonzero weights of some representation with highest-weight λ lie in this sort of generalized hexagon. Then we can consider the first points where the Ys act as 0. To do this we define the following:

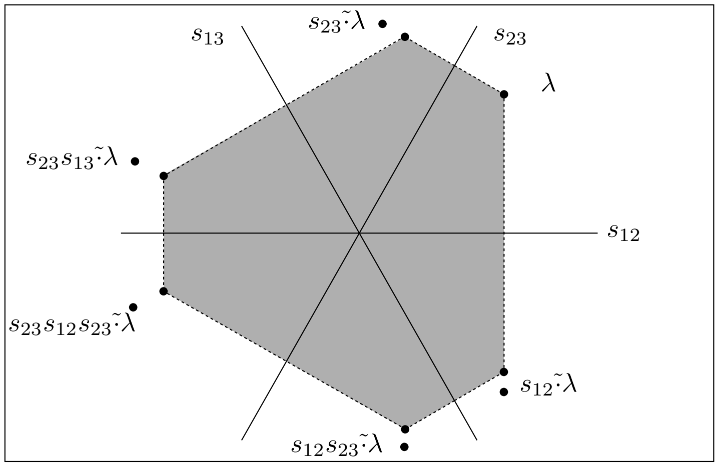

FIGURE 3. The images of λ under the $s_{ij}\tilde{\cdot}$ action.

Definition 2. Let $2\rho = \alpha_{12} + \alpha_{23} + \alpha_{13}$. Then define $s\tilde{\cdot}\lambda$ for $s \in W$ and $\lambda \in \mathfrak{h}^*$ to be the "reflection" with respect to hyperplanes translated by $-\rho$.

Remark 2. The point here is somehow that $-\rho$ was supposed to be the center of the universe all along rather than 0.

Example 2. We write the twisted product explicitly for single group actions:

$$s_{12}\tilde{\cdot}\lambda = s_{12}\lambda - \alpha_{12} \qquad s_{23}\tilde{\cdot}\lambda = s_{23}\lambda - \alpha_{23} \qquad s_{13}\tilde{\cdot}\lambda = s_{13}\lambda - \alpha_{13}$$

This explicitly tells us that $s_{12}\tilde{\cdot}\lambda$ and $s_{23}\tilde{\cdot}\lambda$ are the first time we escape the generalized hexagon from applying Ys as is evident in fig. 3.

This means that we have the following exact sequence:

$$I_{s_{12}\tilde{\cdot}\lambda} \oplus I_{s_{23}\tilde{\cdot}\lambda} \longrightarrow I_\lambda \longrightarrow\!\!\!\!\!\rightarrow V$$

but this isn't short exact since we still haven't somehow killed everything. So we keep considering the kernels to get the full resolution:

$$0 \longrightarrow I_{s_{12}s_{23}s_{12}\tilde{\cdot}\lambda} \longrightarrow I_{s_{12}s_{23}\tilde{\cdot}\lambda} \oplus I_{s_{23}s_{12}\tilde{\cdot}\lambda} \longrightarrow I_{s_{12}\tilde{\cdot}\lambda} \oplus I_{s_{23}\tilde{\cdot}\lambda} \longrightarrow I_\lambda \twoheadrightarrow V$$

Example 3. For $\mathfrak{sl}(2,\mathbb{C})$ the eigenvalue for upper-triangular matrices was 2, so $\rho = 1$, and then reflection about -1 is what gave us the term I_{-n-2} as the kernel in the SES.

Now we are somehow done, because here we just learned that for any irreducible, we are able to resolve it in terms of Verma modules. I.e. the part of the sequence without V has nothing to do with V. This is somehow just taking the maximal proper submodule of an object and repeating the process until we get 0.

1.4. **Weyl character formula.** The resolution from above also gives us the Weyl character formula since the character of V is the alternating sum of the preceding objects in the sequence.

Corollary 1 (Weyl character formula)**.**

$$\begin{aligned}\mathrm{ch}\,(V) &= \sum \mathrm{ch}\,(I_\lambda) - (\mathrm{ch}\,(I_{s_{12}\tilde{\cdot}\lambda}) + \mathrm{ch}\,(I_{s_{23}\tilde{\cdot}\lambda})) + \cdots \\ &= \sum_{w\in W} (-1)^{l(w)} \left(e_{w\tilde{\cdot}\lambda} * \prod_{\alpha_i \in R^+} \frac{1}{1-e_{\alpha_i}} \right)\end{aligned}$$

where l is the length of w.

Example 4. The length $l\,(w)$ in the case of $\mathfrak{sl}\,(3)$ is the number of these simple transpositions to get to w.

We will return to this next time when we will learn why this resolution is true, and that we can view it as some sort of algebraic realization of Schubert decomposition of the flag variety.