LECTURE 18

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We will continue our discussion of Bernstein-Gelfand-Gelfand (BGG) resolution, and discuss $\mathfrak{sl}(4, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C})$ as generalizations of $\mathfrak{sl}(3, \mathbb{C})$.

1. BGG resolution

Informally speaking this is a resolution of a finite dimensional representation theory by Verma modules. More specifically, if we fix some highest-weight $\lambda \in \Lambda^+$, then we define:

$$\rho = \frac{1}{2} \sum_{\alpha_i \in R^+} \alpha_i$$

and modify the action of the Weyl group to act as:

$$s \tilde{\cdot} \lambda = s(\lambda + \rho) - \rho$$

for $s \in W$. So ρ is now the center of the universe. Now we choose some simple reflections, e.g. for $\mathfrak{sl}(3, \mathbb{C})$, s_{12} and s_{23} generate the group so we set these to be our simple reflections. Then the last ingredient is a length function $l : W \to \mathbb{Z}_{\geq 0}$ where $l(s)$ is the minimum word length of s in terms of simple reflections. Then this gives us the resolution:

$$0 \leftarrow V_\lambda \leftarrow I_\lambda \leftarrow \bigoplus_{l(s)=1} I_{s \tilde{\cdot} \lambda} \leftarrow \cdots \leftarrow I_{w_0 \tilde{\cdot} \lambda} \leftarrow 0$$

At each stage if we ask if it's injective the answer is no[1] since there will be a kernel, and in particular it will be the maximal submodule.

One might be worried about the choice of these reflections depending on the fact that the Weyl group is S_n for $\mathfrak{sl}(n, \mathbb{C})$, but the point is, for every simple root, there will be a Levi $\mathfrak{sl}(2, \mathbb{C})$ living inside the Lie algebra, with that simple root as its positive root, and the reflection for $\mathfrak{sl}(2, \mathbb{C})$ will be a simple reflection for the Lie algebra.

2. Weights and roots of $\mathfrak{sl}(4, \mathbb{C})$

Consider the Lie algebra $\mathfrak{sl}(4, \mathbb{C})$. We proceed the same way as we did for $\mathfrak{sl}(3, \mathbb{C})$ and consider the weights of the standard representation on $\mathbb{C}^4$ as in fig. 1. Now just as α_{12} and α_{23} were simple roots for $\mathfrak{sl}(3, \mathbb{C})$, we now have three simple roots α_{12}, α_{23}, and α_{34} as in fig. 1. Then the other roots are all of the edges of the bigger

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[1]Of course until the last one...

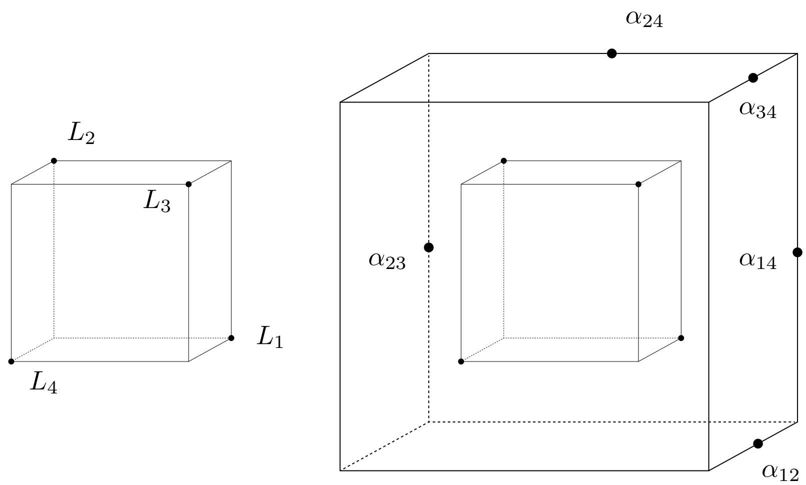


FIGURE 1. (Left) The weights of the standard representation of $\mathfrak{sl}(4,\mathbb{C})$. (Right) The weights of the adjoint representation of $\mathfrak{sl}(4,\mathbb{C})$.

cube on the right of fig. 1. The point here is that if we consider a matrix with a 1 in the spot:

$$\begin{pmatrix} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ & & \alpha_{23} & \alpha_{24} \\ & & & \alpha_{34} \\ & & & \end{pmatrix}$$

it will be an eigenvector with eigenvalue as in the right of fig. 1. Then there are six more on the other edges of the cube.

The cone of elements which pair positively with the positive roots can be seen in fig. 2. This is somehow a triangle on the back face coned off to the origin. Note that it takes 24 such triangles to cover the face of the cube, which is of course what we would expect since $|S_4| = 24$, which makes sense since these cones should correspond to the Borel subalgebras which are acted on simply freely by the Weyl group.

We can view this cone as coming from three copies of $\mathfrak{sl}(3,\mathbb{C})$ as being the sort of intersections of the three figure in fig. 3. Reflections over these planes are the simple reflections for $\mathfrak{sl}(4,\mathbb{C})$.

3. Representations of $\mathfrak{sp}(4,\mathbb{C})$

3.1. **Definition.** Recall the Lie algebra $\mathfrak{sp}(2n,\mathbb{C}) \subseteq \mathfrak{sl}(2n,\mathbb{C})$, is the Lie algebra of the Lie group $\mathrm{Sp}(2n,\mathbb{C}) \subseteq \mathrm{SL}(2n,\mathbb{C})$ which is the group of $2n \times 2n$ matrices which preserve the standard symplectic form in the sense that

$$\mathrm{Sp}(2n,\mathbb{C}) = \left\{ A \in \mathrm{SL}(2n,\mathbb{C}) \mid A^T J A = J \right\}$$

where we have fixed a symplectic form

$$\omega = \sum_i e_{2i-1} \wedge e_{2i}$$

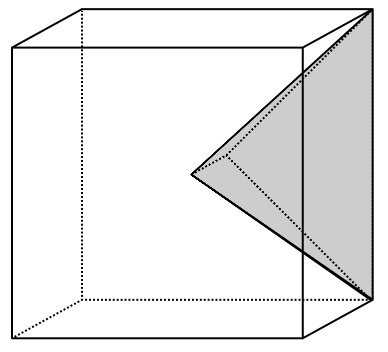
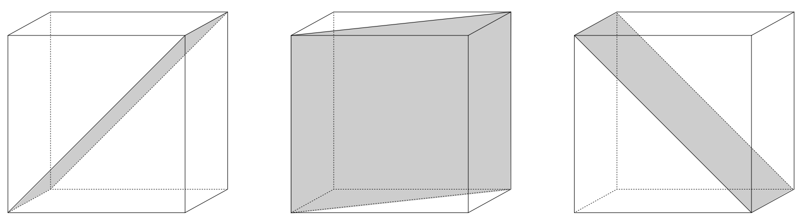


FIGURE 2. The cone of dominant weights in $\mathfrak{sl}(4, \mathbb{C})$.



$$\begin{pmatrix} * & * & & \\ * & * & & \\ & & & \\ & & & \end{pmatrix} \qquad \begin{pmatrix} & & & \\ & * & * & \\ & * & * & \\ & & & \end{pmatrix} \qquad \begin{pmatrix} & & & \\ & & & \\ & & * & * \\ & & * & * \end{pmatrix}$$

FIGURE 3. The three Levi $\mathfrak{sl}(2, \mathbb{C})$s living inside of $\mathfrak{sl}(4, \mathbb{C})$ give us reflections over these three planes.

which, as a matrix, looks like

$$J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

But it doesn't really matter, as long as we take something skew-symmetric and non-degenerate, since we have the following.

Exercise 1. Show that all symplectic forms are equivalent.

Then the Lie algebra is

$$\mathfrak{sp}(2n, \mathbb{C}) = \left\{ X \in \mathfrak{sl}(n, \mathbb{C}) \mid JX = -X^T J \right\}$$

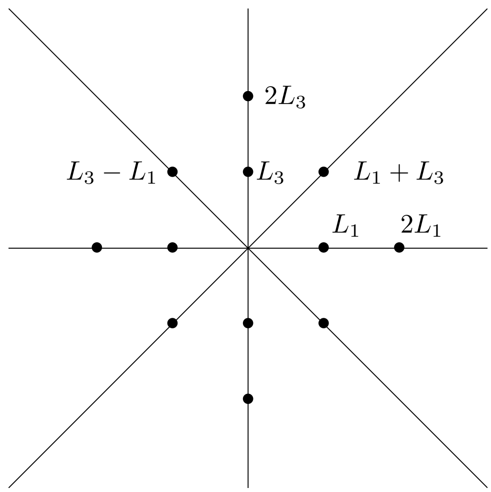


FIGURE 4. The dual space $\mathfrak{h}^*$ of our Cartan subalgebra with weights L_1, L_3, and the eight roots.

Exercise 2. Show that $\mathrm{Sp}(2n, \mathbb{C})$ is simply connected, so this is indeed the unique simply connected Lie group associated to this Lie algebra. Also show $Z(\mathrm{Sp}(2n, \mathbb{C})) = \mathbb{Z}/2$.

3.2. **Roots.** We can take our Cartan subgroup to be

$$H = \left\langle \begin{pmatrix} a & & & \\ & a^{-1} & & \\ & & b & \\ & & & b^{-1} \end{pmatrix} \right\rangle$$

The idea is that if we are going to preserve the area, we need to spin neighbouring coordinates by opposite amounts. Technically we should check that this is not only abelian but actually maximal abelian, but we know this is rank 2, and we've already seen the classification so we already know this is maximal.

This means our Cartan subalgebra $\mathfrak{h}$ is

$$\mathfrak{h} = \left\langle \begin{pmatrix} r & & & \\ & -r & & \\ & & s & \\ & & & -s \end{pmatrix} \right\rangle$$

Again we have $\mathfrak{h}^* \simeq \mathbb{C}^2$, so we have an analogous picture in fig. 4.

Life is a little better here than it was in $\mathfrak{sl}(3, \mathbb{C})$, because we have two favorite functionals. We can take the functional which returns out the first diagonal entry, L_1, and the functional which returns the third, L_3. Now we can generate the roots by calculating commutators, and draw them as in fig. 4. The Weyl group here is $S_2 \times \mathbb{Z}/2 = \mathbb{Z}/2 \times \mathbb{Z}/2$ which is of course the dihedral group D_2. Now we want to find positive and simple roots. To do this we pick the Borel:

$$B = \mathrm{Sp}(2n, \mathbb{C}) \cap \{ \text{ upper triangular matrices } \}$$

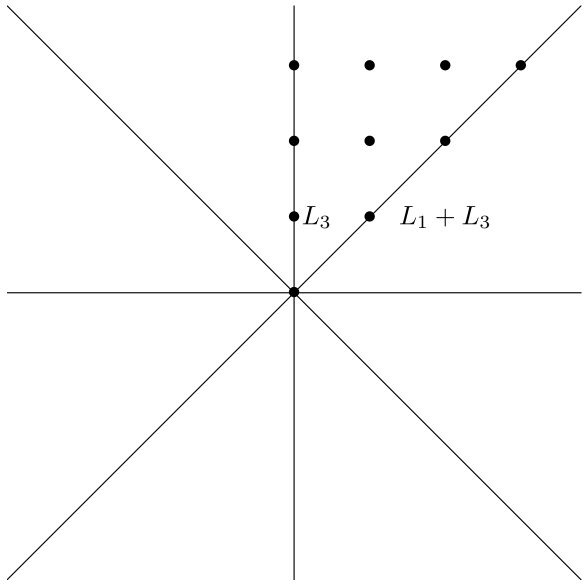


FIGURE 5. The dominant weight lattice of $\mathfrak{sp}(4, \mathbb{C})$.

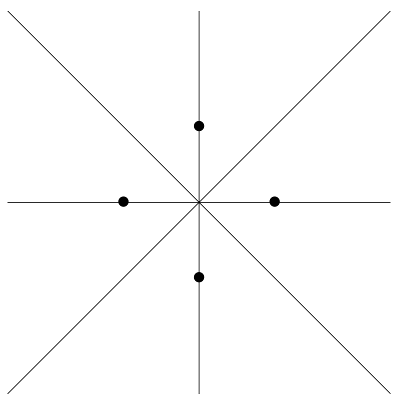


FIGURE 6. Weights of the standard representation of $\mathfrak{sp}(4, \mathbb{C})$.

Then the positive roots are $L_3 - L_1$, $2L_3$, $L_1 + L_3$, and $2L_1$, and out of these the two simple roots are $2L_1$ and $L_3 - L_1$. The dominant weights are then $\mathbb{Z}$ multiples of $L_1 + L_3$ and L_3 as in fig. 5.

3.3. **Constructing representations.** Now we want to construct some representations of this by hand. Recall in $\mathfrak{sl}(3, \mathbb{C})$, we constructed the standard representation and the dual standard representation, and then we could just get everything from tensoring these. Analogously, the two most important constructions here have highest-weights L_3 and $L_1 + L_3$. The representation corresponding to L_3 is the standard representation $\mathbb{C}^4$. The weights of this are as in fig. 6.

Now for $L_1 + L_3$, we should first notice that the weights should likely lie in some sort of convex hull that looks like a square, where we have just reflected this

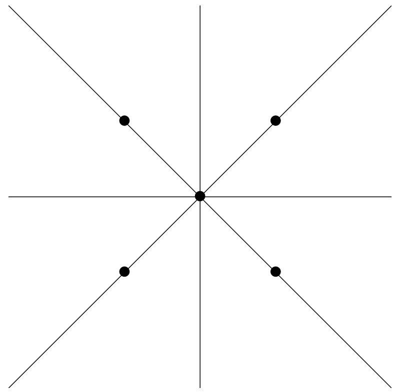
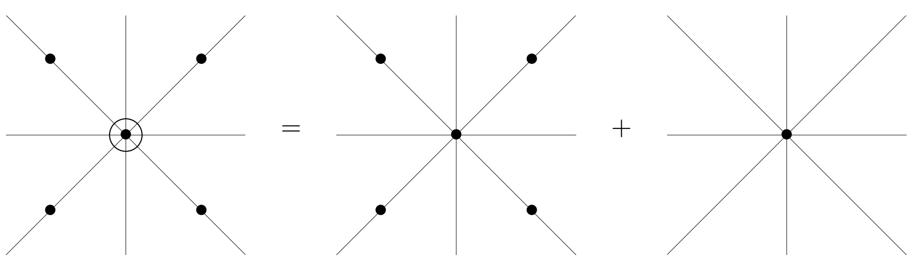


FIGURE 7. The weights of the representation with highest-weight $L_1 + L_3$.

highest weight across these hyperplanes. Then we might wonder if 0 is a weight of this representation. To find out, we can just act with the "lowering" operators and see if we land in it. Applying the root $-L_1 - L_3$, we do land at 0, so it is possible. The answer turns out to be as in fig. 7. So now we know the weights, and we want to find the actual representation. We learned from $\mathfrak{sl}(n, \mathbb{C})$, that once we know the standard representation, the smaller ones are just exterior powers. This inspires us to look at:

$$\wedge^2 \left(\mathbb{C}^4\right) = \mathbb{C} \cdot \omega \oplus W$$

which is a 6-dimensional representation, where W is some five-dimensional irreducible representation. The weights of this exterior power are pairwise sums of weights from the standard where we don't add any weight to itself, so we get a weight of multiplicity 2 at 0, and one at each of the four corners. Then the weights of the decompositions are as follows:



Note that $\mathfrak{sp}(4, \mathbb{C})$ is born as a subalgebra of $\mathfrak{sl}(4, \mathbb{C})$. This means we can project the dual space of the Cartan subalgebras

$$\begin{array}{c} \mathfrak{h}^*_{\mathfrak{sl}(4,\mathbb{C})} \\ \downarrow \\ \mathfrak{h}^*_{\mathfrak{sp}(4,\mathbb{C})} \end{array}$$

according to the dual of the inclusion $\mathfrak{sp} \hookrightarrow \mathfrak{sl}$. Then if we would have picked our basis correctly, the eigenvalues would map exactly to eigenvalues $L_i \mapsto L_i$.

Exercise 3. Draw G_2. This is the other distinct rank two simple Lie algebra.

4. Flag variety and fundamental affine space for $\mathfrak{sp}(4,\mathbb{C})$

In general, the flag variety X should be the moduli of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. For $\mathrm{SL}(n,\mathbb{C})$ we saw that $X \simeq G/B$ since $G \circlearrowright X$ transitively by conjugation with stabilizer B. In this setting we have the following:

Proposition 1. *The flag variety for* $\mathfrak{sp}(4,\mathbb{C})$ *is*

$$X \simeq \left\{\{0\} \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \mathbb{C}^{2n}\right\}$$

where E_i *is isotropic*[2]*, i.e.* $\omega|_{E_i} = 0$ *and* E_n *is Lagrangian.*

Exercise 4. Show that the symplectic group acts transitively on these isotropic flags, and that the stabilizer of the standard isotropic flag is exactly a Borel subgroup.

Example 1. For $\mathfrak{sp}(2,\mathbb{C})$, the flag variety is just $\mathbb{CP}^1$, which is good since $\mathfrak{sp}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$, so they should agree.

Example 2. For $\mathfrak{sp}(4)$, the choices of E_1 are just $\mathbb{CP}^3$, i.e. forgetting E_1 is a map $X \to \mathbb{CP}^3$. Then the fiber is $\mathbb{CP}^1$, which is the choice of E_2 for a fixed E_1. The idea is that once we fix E_1, we are looking for lines symplectically orthogonal to it. So they have to somehow live in $\mathfrak{sp}(2)/B \simeq \mathbb{CP}^1$.

[2] Note that all lines are isotropic, so it's really only a relevant condition for $i \geq 1$.