LECTURE 19

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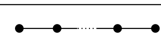
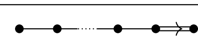
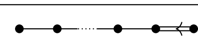
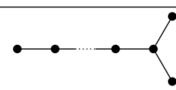
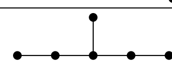
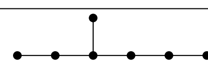
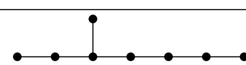
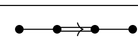
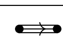
Today we will talk a bit more about the classification of semisimple Lie algebras, root systems, and Dynkin diagrams. Going forward, we will take a more geometric approach via D-modules.

1. Classification of simple Lie algebras

Recall we saw the following classification of Lie algebras according to their rank.[1]

	$\mathfrak{g}$	Diagram	$Z(G)$	$\pi_1(G)$
$A_n\ (n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$		$\mathbb{Z}/(n+1)\mathbb{Z}$	0
$B_n\ (n \geq 2)$	$\mathfrak{so}(2n+1, \mathbb{C})$		0	$\mathbb{Z}/2\mathbb{Z}$
$C_n\ (n \geq 3)$	$\mathfrak{sp}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	0
$D_n\ (n \geq 4)$	$\mathfrak{so}(2n, \mathbb{C})$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
E_6	–		$\mathbb{Z}/3\mathbb{Z}$	–
E_7	–		$\mathbb{Z}/2\mathbb{Z}$	–
E_8	–		0	–
F_4	–		0	–
G_2	–		0	–

We will now see what these cartoons mean mathematically. The strategy will be to go from simple Lie algebras, extract root systems, and get a list of Lie algebras out of that.

1.1. Root systems.

Definition 1. A root system is a real euclidean[2] vector space $(V, \langle \cdot, \cdot \rangle)$ equipped with some subset of roots R which satisfy the following properties:

(1) The roots span V.
(2) If $\alpha \in R$ then $-\alpha \in R$.
(3) α and $-\alpha$ are the only roots on $\mathbb{R} \cdot \alpha$.

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[1] Recall the rank of a Lie algebra is just the dimension of a Cartan subalgebra. These of course all have to be the same since they're related by conjugation by the unique connected, simply-connected Lie group.

[2] Non-degenerate inner product.

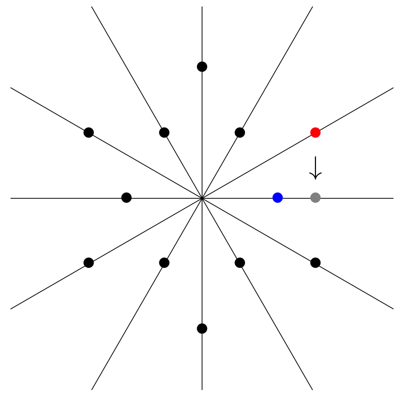

FIGURE 1. The root system G_2. Note that the projection of the red root onto the horizontal axis is 1.5 times the blue root. The projection is pictured in gray.

(4) Reflection across $\alpha^\perp$ for $\alpha \in R$ preserves the set of roots.
(5) Orthogonal projection to $\mathbb{R} \cdot \alpha$ takes R to $\{\pm\alpha, \pm\alpha/2, 3\alpha/2\}$.

Example 1. The roots of $\mathfrak{sl}(3,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C})$ comprise root systems.

Proposition 1. *If $\mathfrak{g}$ is a semisimple complex Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then the roots $R \subset \mathfrak{h}^*$ satisfy all the axioms of a Root system in $V = \mathrm{Span}_{\mathbb{R}}(R) \subseteq \mathfrak{h}^*$ with respect to the Killing inner product.*

Remark 1. If axiom 1 fails for the roots of some Lie algebra, then this means $\mathfrak{g}$ must have a nontrivial center, so it can't be semisimple. Motivation from the second axiom is meant to come from the fact that for every root which somehow comes from an upper triangular location, there should be a corresponding root which came from a lower triangular location. This is again because we are semisimple here. If we were talking about solvable Lie algebras, we would somehow only have roots on one side. For axiom 4, recall we have the Weyl group action. For any root, we have a copy of $\mathfrak{sl}(2,\mathbb{C})$ with that root as its root. Then we can reflect according to the this particular $\mathfrak{sl}(2,\mathbb{C})$. To see the last axiom 5, consider any root γ, and reflect it with respect to α. Then we need the difference between γ and its reflections to be in the $\mathbb{Z}$-linear span of the roots.

Example 2. So far we have only ever seen root systems where we only need $\pm\alpha$ and $\pm\alpha/2$ in the list in axiom 5. An example that illustrates the fact that we need the $3\alpha/2$ which can be seen in fig. 1.

As it turns out, the arrow between semi-simple Lie algebras and root systems is really an equivalence.

Lemma 1. *There is a map from based root systems*[3] *to root systems. I.e. we can construct everything from the simple roots.*

[3] These just consist of a basis of roots satisfying some properties.

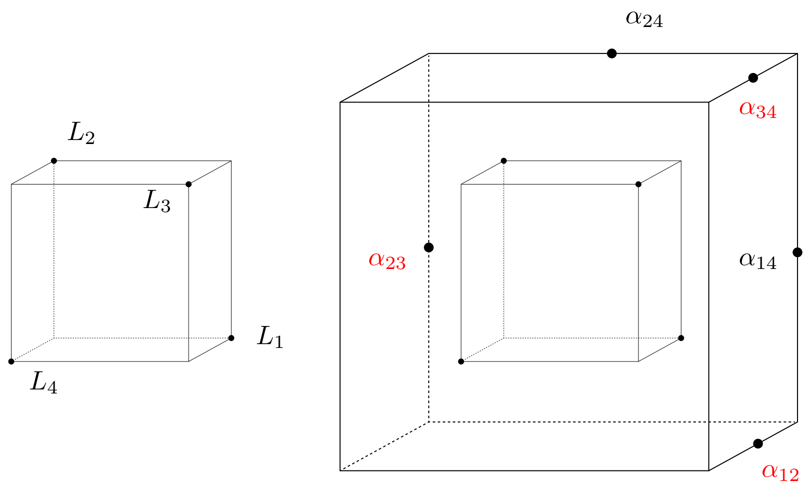

FIGURE 2. (Left) The weights of the standard representation of $\mathfrak{sl}(4, \mathbb{C})$. (Right) The roots of $\mathfrak{sl}(4, \mathbb{C})$, i.e. the weights of the adjoint representation of $\mathfrak{sl}(4, \mathbb{C})$. One choice of simple roots is in red.

1.2. **Dynkin diagrams.** A Dynkin diagram is a graph with a vertex for every $\alpha \in \Delta$. Then there is a single edge edge for an angle between the roots of $2\pi/3$, a double edge for $3\pi/4$, and a triple edge for $5\pi/6$. The direction of the double and triple edges point from longer to shorter roots.

Example 3. Recall $\mathfrak{sl}(4, \mathbb{C})$ has a picture as in fig. 2. One choice of simple roots consists of α_{12}, α_{23}, and α_{34}. The angle between α_{12} and α_{34} is $\pi/2$ so they don't get connected, but the angle between α_{12} and α_{23} is $2\pi/3$, and similarly for α_{23} and α_{34}, so we indeed get

which is the A_3 diagram.

Example 4. First consider $\mathfrak{sp}(6, \mathbb{C})$ which has root system B_3 as in fig. 3. $\mathfrak{so}(7, \mathbb{C})$ has the root system C_3 as in fig. 3. Both of these systems have roots on all edges of the square. The B_3 root system has roots on all surfaces of the cube, whereas the C_3 system has roots above all surfaces of the cube. Therefore they have the same angular relationship, which is that two of the simple roots have an angle of $\pi/2$, two have an angle $2\pi/3$, and two have an angle of $3\pi/4$. But the relationship between the lengths of the roots is different which is why the direction of the arrow is different for B_3 versus C_3:

Example 5. For $\mathfrak{so}(4, \mathbb{C})$ the roots are as in fig. 4. which means the angle between all roots is $\pi/2$, so the Dynkin diagram is just two unconnected points. This is a reflection of the fact that

$$\mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$$

Exercise 1. Come up with a creative way to think about/draw F_4.

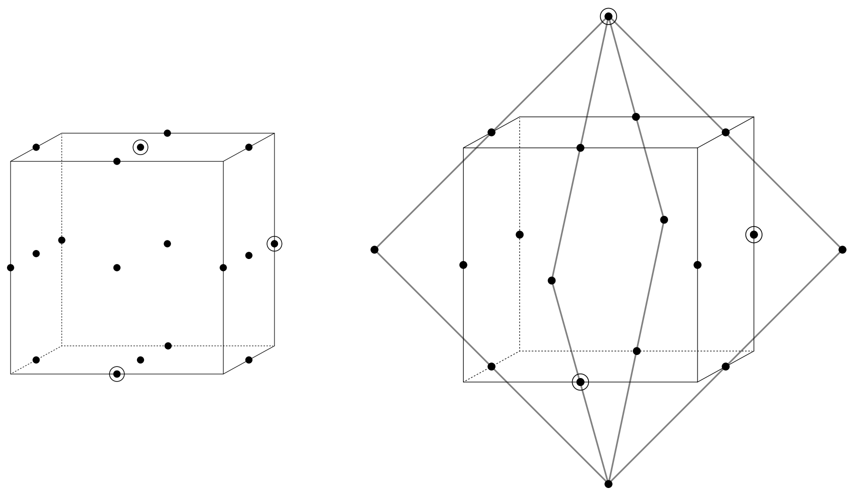

FIGURE 3. (Left) The root system B_3 for $\mathfrak{sp}(6,\mathbb{C})$. The simple roots are the roots of multiplicity 2. (Right) The root system C_3 for $\mathfrak{so}(7,\mathbb{C})$. The simple roots are the roots of multiplicity 2.

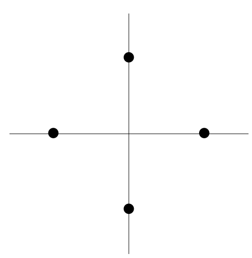

FIGURE 4. The roots of $\mathfrak{so}(4,\mathbb{C})$.

1.3. **Simply-laced Dynkin diagrams.** The A_n, D_n, and E_n Dynkin diagrams are called simply-laced because they only have single bars in their Dynkin diagram. One sample connection to another part of mathematics is as follows.[4] The ADE diagrams are in bijection with rational/du Val surface singularities.

Example 6. The diagrams A_n correspond to

$$V\left(x^2+y^2+z^{1+n}\right)\subseteq\mathbb{C}^3$$

For $n=1$, so $\mathfrak{sl}(2,\mathbb{C})$, we get a nice cone. As n increases this gets worse and worse. Now we might ask how far this is from being a manifold. One way to measure this is to find a minimal resolution $\tilde{X}_n\to X_n$. The idea is that $\tilde{X}_n$ will be a smooth surface, and this map will be an isomorphism away from the singular point which is also proper everywhere.[5] Then the preimage of the singular point is n copies of

[4] This is typically attributed to to Grothendieck but Professor Nadler says the way it goes is that the rich get richer.

[5] This just means the inverse image of compact sets is compact.

$\mathbb{CP}^1$ which intersect in a chain which is the A_n diagram. The same story holds for types D_n and E_n, only in those cases the singularities look a bit different.

2. Harish Chandra center

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let G be the connected simply-connected group with Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, the center $\mathfrak{z}(\mathfrak{g}) = \langle 0 \rangle$. This means $Z(G)$ must be finite.[6] This seems like we can't attack it with much since it's sort of atomic, but as it turns out the enveloping algebra $\mathcal{U}\mathfrak{g}$ has a somehow large center. This is an amazing fact, because if we have a G action, we have a $\mathfrak{g}$ action, so we have a $\mathcal{U}\mathfrak{g}$ action, and this object actually has this large useful center.

Theorem 1 (Harish Chandra)**.** *The center of the enveloping algebra $\mathfrak{z}(\mathcal{U}\mathfrak{g})$ is isomorphic to $(\mathcal{U}\mathfrak{g})^{(W,\tilde{\cdot})}$ where W acts around ρ rather than the origin.*

[6] It is immediate that the center must be discrete, but after a bit of work we could see that it does indeed have to be finite.