LECTURE 2
MATH 261A

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1. Examples of Lie groups and representations

Recall that a Lie group is a smooth manifold G, which is also a group such that the group multiplication and inverse map is smooth with respect to the manifold structure. These of course have to be associative and unital.

Also recall the nature of a group action on a space. We will always have in mind that the space we are acting on is some smooth manifold X. The action is a smooth map $G \times X \to X$. This action must also satisfy associativity and that the identity acts as the identity diffeomorphism.

We should keep the following examples in mind.

Example 1. The group $G = \mathrm{GL}(n, \mathbb{C})$ is a Lie group consisting of $n \times n$ invertible matrices.

Example 2. A representation is a special case of a group action on a manifold. For any vector space V, $G \times V \to V$ is given by linear diffeomorphisms which are of course associative and unital.

Example 3. In particular, consider $\mathrm{GL}(2, \mathbb{C})$. This consists of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $ad - bc \neq 0$. Take $X = \mathbb{CP}^1$. The action is $\mathrm{GL}(2, \mathbb{C}) \times \mathbb{CP}^1 \to \mathbb{CP}^1$ which maps

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, l = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mapsto \qquad \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

If we think about this in terms of slope, this says that the line with slope y/x goes to the line with slope $(cx + dy) / (ax + by)$. This is what is called a fractional linear transformation.

Example 4. For $V = \mathbb{C}^2$ we get an example of a representation where $\mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2 \to \mathbb{C}^2$ is the natural action.

2. Higher dimensional examples

We now generalize to any n. For $G = \mathrm{GL}(n, \mathbb{C})$, $V = \mathbb{C}^n$, what are the potential spaces X we might consider? We will consider complex projective space, the Grassmannian of n planes, and flag manifolds.

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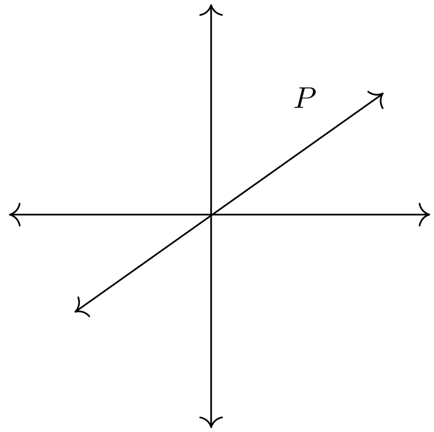

FIGURE 1. The horizontal axis is $\mathbb{C}^k$, and the vertical axis is $\mathbb{C}^{n-k}$. The line P is such that $P \cap \mathbb{C}^{n-k} = \{0\}$.

2.1. **Complex projective space.** Well first we can have $X = \mathbb{CP}^{n-1}$.

Exercise 1. What is the analogue of the "slope" in this higher dimensional case?

How do we see this is a manifold? We cover this with copies of $\mathbb{C}^{n-1}$. For all $i \in \{1, \cdots, n\}$ write

$$\mathcal{U}_i \simeq \mathbb{C}^{n-1} \simeq \{(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n)\}$$

Exercise 2. Check that these are all appropriately compatible in their intersections.

2.2. **Grassmannian.** Another possibility, is to consider the "Grassmannian" which is

$$X = \mathrm{Gr}(k, n, \mathbb{C}) = \{k\text{-planes in } \mathbb{C}^n \text{ through } 0\}$$

Exercise 3. For what k,k',n,n' do we have a diffeomorphism between $\mathrm{Gr}(k, n, \mathbb{C}) \simeq \mathrm{Gr}(k', n', \mathbb{C})$.

How do we see this is a manifold? Consider the following chart in $\mathrm{Gr}(k, n)$. Consider a k-plane P as in fig. 1. such that $P \cap \mathbb{C}^{n-k} = \{0\}$. Then consider $\mathcal{U}$ to be the set of all such k-planes. Note that $\mathcal{U} \simeq \mathrm{Hom}\left(\mathbb{C}^k, \mathbb{C}^{n-k}\right)$. This is of course just a collection of matrices, so $\mathcal{U} \simeq \mathbb{C}^{k(n-k)}$. Now we need to check that these objects actually cover $\mathrm{Gr}(n, k)$. We will take two approaches.

This open set $\mathcal{U}$ can be defined anytime we break this up into k coordinates, and the complement. That is, for any $I \subset \{1, \cdots, n\}$ such that $|I| = k$, we can split this space into $\mathbb{C}^I$ and $\mathbb{C}^{I^c}$ and now we can define $\mathcal{U}_I$ for this choice of I.

The second approach is to note the following:

Exercise 4. $\mathrm{GL}(n, \mathbb{C})$ acts transitively on all three of the spaces we are considering here.

Then we can use the group action to move $\mathcal{U}$ around to cover.

Exercise 5. Check that these agree on the intersections.

2.3. **Flag manifolds.** Another example is a flag manifold. Let's write some subset

$$\underline{k} \subseteq \{1, \cdots, n-1\} \ .$$

Then we can consider the flag manifold $\mathrm{Fl}\,(k)$ which consists of nested sequences of subspaces $E_{k_1} \subset E_{k_2} \subset \cdots$ with dimension of $E_{k_i} = k_i$ where k_i is the ith element of $\underline{k}$.

To see this is a manifold, we can consider it as a subspace of the following:

$$\mathrm{Fl}\,(\underline{k}, n, \mathbb{C}) \subseteq \prod_{k_i} \mathrm{Gr}\,(k_i, n, \mathbb{C})$$

Exercise 6. Show that $\mathrm{Fl}\,(\underline{k}, n, \mathbb{C})$ is cut out of the above as a regular value of a smooth map so it is a submanifold.

3. Types of group actions

We now introduce some terminology for different types of group action. We will write an action $G \times X \to X$ as $G \circlearrowright X$. We say an action is transitive if for every $x, y \in X$, there exists some $g \in G$ such that $g \cdot x = y$. This is somehow saying G is bigger than X. We can also ask that the action is free, which means for every $x \in X$, if $g \cdot x = x$, then $g = e$. This is somehow saying X is bigger than G.

Define the orbit of $x \in X$ to be

$$X \supseteq G \cdot x \coloneqq \{y \in X \mid \exists g \in G \text{ s.t. } y = g \cdot x\}$$

Then the stabilizer is

$$G \supseteq G_x \coloneqq \{g \in G \mid g \cdot x = x\}$$

Note that an action is transitive iff there is only one orbit, and an action is free iff every stabilizer is trivial.

Lemma 1. *Stabilizers are closed subgroups. In addition, for $y = gx$, we have $G_y = gG_xg^{-1}$.*

Proof. The second statement is effectively obvious so we focus on the first statement. The fact that the stabilizer is a subgroup is immediate. We prove it is closed. The stabilizer G_x is the fiber at x of the map $g \mapsto g \cdot x$. Since X is Hausdorff, points are closed, so the fiber is closed, so the stabilizer is closed. $\square$

Example 5. Consider $G = \mathrm{GL}\,(n, \mathbb{C}) \circlearrowright \mathrm{Fl}\,(\underline{k}, n, \mathbb{C})$. This is a transitive action so there is only one orbit. The stabilizer G_x of a point

$$x = \{E_{k_1} = \mathrm{Span}\,\{e_1, \cdots, e_{k_1}\}, \cdots, E_{k_i} = \mathrm{Span}\,\{e_1, \cdots e_{k_i}\}\}$$

is the collection of matrices such that the top left k_i block has zeros beneath it for every i. Note that for full flags we get the collection of all upper-triangular matrices in $\mathrm{GL}\,(n, \mathbb{C})$.

Based on lemma 1 we introduce the following definition:

Definition 1. A *Lie subgroup* $H \subset G$ is a subgroup, which is also closed.

Example 6 (Non-example)**.** We offer a subgroup $H \subset G$ which is not closed. Take $G = T^2$, and then $H = \mathbb{R} \times \{\text{irrational slope}\}$ then we get a subgroup which is not closed.

From now on we assume all subgroups are Lie subgroups.

Lemma 2. *Lie subgroups are Lie groups. In particular we have a bijection between Lie subgroups and transitive G actions.*

Exercise 7. Prove lemma 2. I.e. show that Lie subgroups are in fact submanifolds.

Example 7. Consider $\mathrm{GL}(2,\mathbb{C}) \circlearrowright \left(\mathbb{CP}^1\right)^k$. For what k is this transitive? For what k is this free? For what k does this have finitely many orbits? What are the stabilizers?

The case $k = 0$ is trivial. For $k = 1$ this is transitive but not free. It is not free because the diagonal matrices scale the vectors without changing the line, so the stabilizer of any point contains the diagonal matrices which comprise $\mathbb{C}^\times$. For $x = (1, 0)$,

$$G_x = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

For $k = 2$, this action is not transitive, and it has two orbits which consist of pairs of matching lines $l_1 = l_2$, and different lines $l_1 \neq l_2$. What about the stabilizers? First take x to be $l_1 = l_2 = [e_1]$, and we get

$$G_x = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

and then for x consisting of $l_1 = [e_1]$,$l_2 = [e_2]$ we get

$$G_x = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

Exercise 8. Complete the same analysis for $k = 3$.

Exercise 9. Consider $\mathrm{GL}(2,\mathbb{R}) \circlearrowright \mathbb{CP}^1$. Calculate the orbits. Calculate the stabilizers.

Next time we start with G acting on itself by left/right translations. This will lead us to Lie algebras.