LECTURE 20
MATH 261A

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1. Harish Chandra center

We will continue our discussion of the Harish-Chandra center. The setup will be $\mathfrak{g}$ a semi-simple complex Lie algebra, and G the unique connected, simply-connected Lie group. Inside $\mathfrak{g}$ we have a Cartan subalgebra $\mathfrak{h}$, and on this we have this Weyl group action. Then we have the following theorem:

Theorem 1.
$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathcal{U}\mathfrak{h}\right)^{W,\tilde{\cdot}}$$

2. The RHS

The first thing to notice is that we have a canonical isomorphism $\mathcal{U}\mathfrak{h} \simeq \mathrm{Sym}\left(\mathfrak{h}\right)$. Recall
$$\mathcal{U}\mathfrak{h} \coloneqq \bigoplus_{n\geq 0} \mathfrak{h}^{\otimes n} / \left(H_1 \otimes H_2 - H_2 \otimes H_1 - [H_1, H_2] = 0\right).$$
Since $\mathfrak{h}$ is abelian, we are actually just killing transpositions, so we get a symmetric algebra.

2.1. A useful point of view.

A useful point of view is to think of $\mathrm{Sym}\,\mathfrak{h}$ as polynomial functions on $\mathfrak{h}^*$, written $\mathbb{C}\left[\mathfrak{h}^*\right]$. Now we have a W action[1] only we want to re-center at $-\rho$ to get the this $\tilde{\cdot}$ action.

To get this action we had to choose a Borel subalgebra, which told us the positive roots, and then we write down half the sum of the positive roots, and this is ρ. More specifically,
$$w\tilde{\cdot}\lambda = w \cdot (\lambda + \rho) - \rho \ .$$
The reason this point of view is useful, is that we have
$$\left(\mathcal{U}\mathfrak{h}\right)^{W,\tilde{\cdot}} = \mathrm{Sym}\left(\mathfrak{h}\right)^{W,\tilde{\cdot}} = \mathbb{C}\left[\mathfrak{h}^*\right]^{W,\tilde{\cdot}}$$
so we should think of this as consisting of the polynomials invariant under this twisted W action.

Remark 1. This $\mathbb{C}\left[\mathfrak{h}^*\right]^{W,\tilde{\cdot}}$ is what we would write in algebraic geometry as
$$\mathbb{C}\left[\mathfrak{h}^*//W,\tilde{\cdot}\right]$$
Note that this space, which we now write as
$$\mathfrak{c}^* \coloneqq \mathfrak{h}^*//W \ ,$$

Date: November 8, 2018.

[1] Professor Nadler's polarity changed after his week away and now he gets shocked by the blackboard.

is again an affine space.

Fact 1 (Fantastic fact)**.** $(\mathcal{U}\mathfrak{h})^{W,\tilde{\cdot}}$ *is again a polynomial algebra.*

Example 1. In $\mathfrak{sl}(n,\mathbb{C})$, $W = \Sigma_n$ is the symmetric group. We will ignore the $\tilde{\cdot}$ part for now just to get a feeling for this. Here $\mathfrak{h}^*$ just consists of n-tuples of eigenvalues of trace 0. Now we can ask for all of the polynomial functions of the eigenvalues that are invariant under their permutations. As it turns out, these are the polynomials in the elementary symmetric functions:

$$\mathbb{C}[\mathfrak{h}^*//W] \simeq \mathbb{C}[\sigma_2, \cdots, \sigma_n] \ .$$

So we should think of this guy as another affine space.

2.2. **Motivation/what this is telling us.** Let's stop for a second and enjoy what this theorem is telling us. It is saying that any time you write down a module over this algebra $\mathcal{U}\mathfrak{g}$, i.e. a representation of $\mathfrak{g}$, finite or infinite dimensional, you have an action of the center on the module. So the entire representation theory of this algebra lives over functions on an affine space. So we can talk about when a module is supported at a point of an affine space, based on acting on it by functions.

The upshot of all this is that the category $\mathcal{U}\mathfrak{g}$-**Mod** is linear over the polynomial algebra $\mathbb{C}[\mathfrak{c}^*]$. So if we are given some $\mathcal{U}\mathfrak{g}$ module M, we can take some[2] $\bar{\lambda} \in \mathfrak{c}^*$, and then consider functions which vanish exactly at this point, and multiply the module by them. Now we can ask if the action is, say, always an isomorphism. And if it is, that means the module would somehow live only here.

Remark 2. What professor Nadler is trying to convey here in basic terms is the following. If you've ever taken Spec of a module in algebraic geometry you know you get a sheaf on this thing. So it tells us that all modules have some expansion over this affine space.

In particular, if you're irreducible, we already know what the irreducible modules are for a polynomial algebra. They're just given by the maximal ideals, i.e. just the points. So the upshot is that all of the irreducible modules live over points of $\mathfrak{c}^*$. If a module "spreads out" then we can just multiply it by a function $x - \bar{\lambda}$ and get a submodule.

So again, the representation theory of this lives over an affine space, whose points are somehow sets of eigenvalues. So what remains for us, as representation theorists trying to understand all $\mathcal{U}\mathfrak{g}$ modules, is to fix any single point in $\mathfrak{c}^*$ and just study all of the modules that live above this.

For fixed $\bar{\lambda} \in \mathfrak{c}^*$ we set the following notation:

$$\mathcal{U}_{\bar{\lambda}}\mathfrak{g} = \mathcal{U}\mathfrak{g}/I_{\bar{\lambda}}$$

where $I_{\bar{\lambda}}$ is the ideal of $\mathbb{C}[\mathfrak{c}^*]$ which consists of functions vanishing at $\bar{\lambda}$. So the only nonzero things are what's happening here.

Example 2. This is supposed to be like taking $k_\lambda \simeq k[t]/(t-\lambda)$ to get the sky scraper (copy of scalars) at λ.

So now we have the following observation: $\mathcal{U}_{\bar{\lambda}}\mathfrak{g}$-**Mod** just consists of $\mathcal{U}\mathfrak{g}$-**Mod** on which the center acts by the character. I.e. we map

$$\mathfrak{z} \to \mathfrak{z}/I_{\bar{\lambda}} \simeq k_{\bar{\lambda}}$$

[2] We call this $\bar{\lambda}$ to convey that it somehow comes as the image of some set of eigenvalues. Points of $\mathfrak{h}^*$ are like ordered eigenvalues, and points of $\mathfrak{c}^*$ are like unordered eigenvalues.

Remark 3. What do we mean by acts by the character? If we take any $\mathcal{U}\mathfrak{g}$ module we can ask how the center acts. Remember we want to think of this as functions on $\mathfrak{c}^*$. So now we're just doing algebraic geometry. We have a polynomial algebra, and we're asking how it acts on a module. One of our favorite ways is to take a polynomial, restrict it to $\bar{\lambda}$ to get its value, and then scale the module by that value. But this is mathematically the same as saying the polynomial algebra maps to its quotient by the ideal of functions vanishing at $\bar{\lambda}$ which is just the values at $\bar{\lambda}$.

The whole representation theory of the enveloping algebra, which is to say the representation theory of $\mathfrak{g}$, has this giant center in it, and we can somehow talk about those representations that live over any point in Spec of the center. We define this algebra to be the quotient of the entire algebra which is given by just looking at the point $\bar{\lambda}$. The whole representation theory is then somehow an "integral" over the representation theory at these $\bar{\lambda}$.

Remark 4. This is somehow the reason one likes centers. If $\mathfrak{g}$ was commutative, it would just be its own center, and we would be back in algebraic geometry where we know how to classify irreducibles (given by maximal ideals) so there's a whole structure there. And the next best thing is the (very big) center of $\mathcal{U}\mathfrak{g}$. So in this part things are just algebraic geometry, and then for each $\bar{\lambda}$ you have to do the algebraic geometry of this new particular algebra which has center just consisting of scalars.

Remark 5. This is somehow a general paradigm. Any time someone gives you a mathematical object, you should ask what its endomorphisms are, and then find the center of the endomorphisms. Then spread it out over Spec of the center. Professor Nadler says you can understand almost everything in mathematics by asking that question.

2.3. **Calculation for $\mathfrak{sl}(2,\mathbb{C})$.**

Example 3. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Then $\mathfrak{h} = \mathbb{C} \cdot H$ and $\mathfrak{h}^* = \mathbb{C} \cdot L_1$ where $L_a(aH) = a$. We want to think about L_1 as a point, and then functions on this line $\mathfrak{h}^*$ are polynomials in H, $\mathbb{C}[\mathfrak{h}^*] \simeq \mathbb{C}[H]$. Recall the usual $W = \mathbb{Z}/2 = \{1, \sigma\}$ action is just reflection wrt 0, so $\mathbb{C}[H]^W$ is all polynomials invariant under $H \mapsto -H$, which is of course $\mathbb{C}\left[H^2\right]$. This is again a polynomial algebra, and we should think of this as being like a double cover by the square map. Here $\rho = L_1$, so $-\rho = -L_1$, so the action of $W = \mathbb{Z}/2$ by $\tilde{\cdot}$ is really reflection over -1:

$$\sigma \tilde{\cdot} (aL_1) = -(aL_1 + L_1) - L_1 = (-a-2)\, L_1\ .$$

Now the invariant functions under this action are:

$$\mathbb{C}[\mathfrak{h}^*]^{W,\tilde{\cdot}} = \mathbb{C}\left[(H+1)^2\right]$$

Remark 6. One might be annoyed by this $\tilde{\cdot}$ action because it's an extra thing to keep track of. Professor Nadler says that often times in mathematics it is best to respect structures like this and be their friends so they can guide us.

3. The LHS

Now we want to think about the LHS of the theorem. Recall again that:

$$\mathcal{U}\mathfrak{g} = \left(\bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}\right) / \left(X \otimes Y - Y \otimes X - [X,Y]\right)$$

But this quotient doesn't respect the grading of the tensor algebra since it effectively sets degree 2 things equal to degree 1 things. We do however maintain the filtration:

$$F^0\mathcal{U}\mathfrak{g} \subset F^1\mathcal{U}\mathfrak{g} \subset \cdots$$

where $F^i\mathcal{U}\mathfrak{g}$ somehow consists of tensors of degree i and lower. We can canonically write that

$$F^0\mathcal{U}\mathfrak{g} = \mathbb{C} \qquad\qquad F^1\mathcal{U}\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g}$$

are tensors of degree 0 and degree 1, but after this there is no canonical splitting of the filtration. PBW does however give us a non-canonical splitting.

Recall that this PBW story was that $\mathcal{U}\mathfrak{g}$ has a basis of ordered monomials of the form $X^{\alpha_1}H^{\alpha_2}Y^{\alpha_3}$ which gives us a splitting of this filtration. This tells us that we can set

$$F^2\mathcal{U}\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g} \oplus \{\text{degree two s.t. linear combination of PBW basis}\} \ .$$

Now if we pass to the associated graded algebra:

$$\mathrm{Gr}_F(\mathcal{U}\mathfrak{g}) = \bigoplus_{n=0}^{\infty} F^n\mathcal{U}\mathfrak{g}/F^{n-1}\mathcal{U}\mathfrak{g} \simeq \mathrm{Sym}(\mathfrak{g})$$

we get a symmetric algebra on $\mathfrak{g}$.

3.1. Organizing $\mathcal{U}\mathfrak{g}$ and $\mathrm{Sym}(\mathfrak{g})$.

What we did above doesn't really have anything to do with PBW. Any time we have a filtered algebra like this we can do what's called the Rees construction, which builds a new algebra which depends on both the initial algebra and the filtration. Explicitly we define the algebra:

$$R\mathfrak{g} = \bigoplus_{n=0}^{\infty} F^n\mathcal{U}\mathfrak{g} \cdot \hbar^n$$

where the multiplication is given by:

$$(\tau_n \cdot \hbar^n) \cdot (\tau'_m \cdot \hbar^m) := \tau_n \tau'_m \cdot \hbar^{n+m}$$

Note that $\hbar$ is central, which means this algebra lives over the $\hbar$ line. Now we can ask about the fibers of this algebra at different points. When $\hbar = 0$, we recover the associated graded algebra $\mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$, and at $\hbar = 1$ we recover $\mathcal{U}\mathfrak{g}$. In general we get:

$$R_{\mathfrak{g}}|_a = R\mathfrak{g}/(\hbar - a)$$

Exercise 1. Show that with this definition $R_{\mathfrak{g}}|_0 = \mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$ and $R_{\mathfrak{g}}|_1 = \mathcal{U}\mathfrak{g}$.

Solution. For $\hbar = 0$ we can look at the inclusion of something in the ideal $(\mathfrak{h})$ into $F^{n+1}\mathcal{U}\mathfrak{g} \cdot \hbar$ and this will be exactly $\mathrm{Gr}_F(\mathcal{U}\mathfrak{g})$.

Remark 7. The whole point here is that what we did above is completely general and doesn't have anything to do with PBW.

Remark 8. So we start with a symmetric algebra, the functions on $\mathfrak{g}$, and then one can view this as quantizing those functions. A useful point of view is that $R\mathfrak{g}$ is the deformation quantization[3] of the algebra of symmetric functions with respect to the Poisson structure given by $[\cdot,\cdot]$.

[3] This is just a word for deforming a commutative algebra to be something noncommutative.

Remark 9. The general point of view here is that when we're studying $\mathcal{U}\mathfrak{g}$, we're really looking at $\mathfrak{g}^*$ as a vector space, which has a ring of functions with a Poisson bracket, and we're quantizing the Poisson bracket to get $\mathcal{U}\mathfrak{g}$.

3.2. **Key observation.** Now we want to determine the center of $\mathcal{U}\mathfrak{g}$. What is the center of the special fiber? This is commutative, so it is the whole thing. So passing between them something strange happens since the center gets much smaller. We now have the following key observation:

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathcal{U}\mathfrak{g}\right)^G$$

under the adjoint action. So if we take a tensor τ such that $\mathrm{Ad}_g \tau = \tau$, then differentiating with respect to $g = \exp(tX)$ we get $\mathrm{ad}_X(\tau) = 0$ which means $[X, \tau] = 0$ where this extends by the Jacobi identity. Therefore, being an invariant means you're certainly in the center. Now conversely, if something is in the center we can just exponentiate it, which generates a neighborhood of the identity, and therefore this thing is invariant under the adjoint action of G as well. So we just reinterpreted being in the center as a quality which only concerns invariants of this vector space.

Now the PBW splitting gives the isomorphism of vector spaces $\mathcal{U}\mathfrak{g} \simeq \mathrm{Sym}(\mathfrak{g})$ so the subspaces of invariants are the same as vector spaces:

$$\left(\mathcal{U}\mathfrak{g}\right)^G \simeq \mathrm{Sym}\left(\mathfrak{g}\right)^G \ .$$

Warning 1. The LHS is commutative and the RHS is not, so these are certainly not isomorphic as algebras, and in fact they're not even isomorphic as G-representations. We will see an example of this soon.

Now there is a theorem of Chevalley[4] which says:

$$\left(\mathrm{Sym}\,\mathfrak{g}\right)^G \simeq \left(\mathrm{Sym}\,\mathfrak{h}\right)^W$$

Example 4. In $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, this isn't so surprising. In this case $\mathrm{Sym}(\mathfrak{g})^G$ consists of all polynomial functions on traceless $n \times n$ matrices which are invariant under change of basis, i.e. they are conjugation invariant. On the other hand, $\mathrm{Sym}(\mathfrak{h})^W$ consists of symmetric functions on traceless n-tuples of eigenvalues. There is a natural map $\mathrm{Sym}(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{h})^W$ where we just restrict to the diagonal matrices. In the other direction, one needs to convince oneself that any function of $n \times n$ matrices that are conjugation invariant is just going to be a symmetric function on the eigenvalues.

Exercise 2. Show this.

All together we have shown that:

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathrm{Sym}\,\mathfrak{h}\right)^W$$

as vector spaces, which shows us they are somehow the same size, which is what Harish-Chandra said was true. But there is no $\tilde{\cdot}$ in sight, so we need to go back and somehow correct the fact that this was not an isomorphism of algebras, or even of representations. We will see what this looks like for $\mathfrak{sl}(2, \mathbb{C})$, and the main point will be that we need to go and symmetrize the PBW basis. This changes the W action, which finishes the picture. In the process we will calculate the first casimir, which is to say the first interesting invariant.

[4] This is often referred to as Chevalley's restriction theorem.