LECTURE 21
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. Harish Chandra center

Recall we are exploring the Harish Chandra center, and the isomorphism given by the theorem:

Theorem 1. *For $\mathfrak{g}$ a reductive complex Lie algebra,*

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \mathbb{C}\left[\mathfrak{h}^*\right]^{(W,\tilde{\cdot})}$$

Recall that one way to think of the RHS is as $\mathbb{C}\left[\mathfrak{h}^* / \left(W, \tilde{\cdot}\right)\right]$. And then one of the main results of geometric representation theory is Chevalley's theorem which tells us that $\mathbb{C}\left[\mathfrak{h}^*\right]^{(W,\tilde{\cdot})}$ is a polynomial algebra which implies this quotient $\mathfrak{c}^*$ is an affine space. The point is, the center is functions on unordered eigenvalues.

Remark 1. The LHS is important because any time we want to study modules over something, we can ask what its center is and how it acts. The Algebraic geometers will succinctly say that $\operatorname{Spec}\left(\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right)\right) \simeq \mathfrak{h}^* / \left(W, \tilde{\cdot}\right)$.

Recall the key idea is that we have the following isomorphism as vector spaces:

$$\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\mathcal{U}\mathfrak{g}\right)^G \quad .$$

This was because if we write

$$\operatorname{Ad}_g\left(v_1 \otimes \cdots v_k\right) = \operatorname{Ad}_g\left(v_1\right) \otimes \cdots \otimes \operatorname{Ad}_g\left(v_k\right)$$

then differentiating this gives us that the ad action is trivial on such elements. Then using PBW we saw $\left(\mathcal{U}\mathfrak{g}\right) \simeq \left(\operatorname{Sym}\mathfrak{g}\right)$ as vector spaces which means we have

$$\left(\mathcal{U}\mathfrak{g}\right)^G \simeq \left(\operatorname{Sym}\mathfrak{g}\right)^G$$

as vector spaces. Then Chevalley tells us that this is a polynomial algebra, and in fact:

$$\left(\mathcal{U}\mathfrak{g}\right)^G \simeq \operatorname{Sym}\left(\mathfrak{h}\right)^W$$

so they have sort of the same size.

Example 1. We will see why this is true for $\mathfrak{sl}(2,\mathbb{C})$. As usual $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\langle X, H, Y\rangle$. First we look for invariants in $\operatorname{Sym}(\mathfrak{g})^G$. We know $\mathbb{C}$ is an invariant, and the adjoint representation $\mathfrak{g}$ is irreducible. Next we take the tensor:

$$\mathfrak{g} \otimes \mathfrak{g} = V_4 \oplus V_2 \oplus V_0 \simeq V_4 \oplus \mathfrak{g} \oplus \mathbb{C}$$

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and the interesting thing is that this $\mathbb{C}$ is a new invariant. Inside of $\mathfrak{g}^{\otimes 2} \simeq \operatorname{Sym}^2(\mathfrak{g}) \oplus \wedge^2\mathfrak{g}$, the first thing to notice is that

$$\mathfrak{g}^{\otimes 2} \simeq \operatorname{Sym}^2(\mathfrak{g}) \oplus \wedge^2\mathfrak{g} \simeq (V_0 \oplus V_4) \oplus V_2$$

so this $\mathbb{C}$ actually lives in $\operatorname{Sym}^2\mathfrak{g}$. Now we have a favorite element of $\operatorname{Sym}^2\mathfrak{g} \simeq \operatorname{Sym}^2\mathfrak{g}^*$, which is the Killing form.

Exercise 1. Show that $V_0 = \mathbb{C}\langle\kappa\rangle$ where κ is the Killing form.

Solution. We can explicitly write the Killing form as $\kappa = H^2 + 4XY$. Now we want to see if this is invariant. It is in the zero weight space so we just need to calculate:

$$\begin{aligned}
[X,\kappa] &= \left[X, H^2\right] + [X, 4XY] \\
&= [X,H] \otimes H + H \otimes [X,H] + \frac{4}{2}\left(X \otimes [X,Y] + [X,Y] \otimes X\right) \\
&= -2X \otimes H - 2H \otimes X + 2H \otimes X + 2H \otimes X \\
&= 0
\end{aligned}$$

This is the unique invariant vector in $\operatorname{Sym}^2(\mathfrak{g})$.

We might keep searching for invariants, but Chevalley tells us that this is a polynomial algebra, so

$$\operatorname{Sym}(\mathfrak{g})^G = \mathbb{C} \oplus \mathbb{C}\cdot\kappa \oplus \mathbb{C}\cdot\kappa^2 \oplus \cdots \simeq \mathbb{C}[\kappa]$$

Now we need to lift this element into the enveloping algebra. So we seek a central element $\tilde{\kappa} \in \mathcal{U}\mathfrak{g}$ such that under the PBW isomorphism:

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\mathcal{U}\mathfrak{g})^G \simeq \mathbb{C}[\kappa]$$

we have that $\tilde{\kappa} \mapsto \kappa$. But now the ambiguity here is that we don't know where XY lifts, i.e. this depends on our choices in the PBW setup. But we can just choose $\tilde{\kappa} = H^2 + 2(XY + YX)$, and we already checked this is a good lift.

2. The ρ shift

Now let's find the ρ shift in this picture. Our goal is a more explicit isomorphism in the example of $\mathfrak{sl}(2,\mathbb{C})$. As usual, consider $G \circlearrowright G/N$ where N is the fundamental affine space. From this we get a map $\mathfrak{g} \to \mathbf{Vect}(G/N)$, and now we naturally get a map $\mathcal{U}\mathfrak{g} \to \operatorname{Diff}(G/N)$ where $\operatorname{Diff}(G/N)$ denotes the differential operators on G/N. We also have a commuting right H-action on G/N. All together, we have G/N with a left G action, and a right H-action, and these actions commute. So we get a map $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{h} \to \operatorname{Diff}(G/N)$.

Exercise 2. Prove that H is exactly the symmetries of G/N that commute with G.

Now if we take the center $\mathfrak{zg} \subseteq \mathcal{U}\mathfrak{g}$, then we have the following diagram:

$$\begin{tikzcd}
\mathfrak{zg} \arrow[d, hook] \arrow[dr] \arrow[drr, hook, "\varphi"] & & \\
\mathcal{U}\mathfrak{g} \arrow[r] & \operatorname{Diff}(G/N) & \mathcal{U}\mathfrak{h} \arrow[l, hook']
\end{tikzcd}$$

This factorization φ results from the facts:

(1) $\mathfrak{z}\mathfrak{g}$ commutes with $\mathcal{U}\mathfrak{g}$

(2) $\mathcal{U}\mathfrak{h}$ is exactly what commutes with $\mathcal{U}\mathfrak{g}$.

Then the Harish Chandra isomorphism is the map

$$\varphi : \mathfrak{z}\mathfrak{g} \to (\mathcal{U}\mathfrak{h})^{W,\tilde{\cdot}}$$

Example 2. We will do this explicitly for $\mathfrak{sl}(2, \mathbb{C})$. In this case $G/N \simeq \mathbb{C}^2 \setminus \{0\}$ with coordinates u and v, and with the left action, we get

$$H \mapsto -u\partial_u + v\partial_v \qquad X \mapsto -v\partial_u \qquad Y \mapsto -u\partial_v$$

Now let's calculate this Casimir $K = H^2 + 2(XY + YX)$:

$$\begin{aligned} K &= (u\partial_u - v\partial_v)^2 + 2(v\partial_u u\partial_v + u\partial_v v\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 - u\partial_u v\partial_v - v\partial_v u\partial_u + 2(v\partial_u u\partial_v + u\partial_v v\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 - uv\partial_u\partial_v - vu\partial_v\partial_u + 2(vu\partial_u\partial_v + v\partial_v + uv\partial_v\partial_u + u\partial_u) \\ &= (u\partial_u)^2 + (v\partial_v)^2 + 2uv\partial_u\partial_v + 2(u\partial_u + v\partial_v) \end{aligned}$$

Now for the right action,

$$H \mapsto u\partial_u + v\partial_v$$

and we want to write K as a polynomial in H, and indeed:

$$K = H^2 + 2H = (H+1)^2 - 1$$

which is of course ρ-shifted.