LECTURE 22
MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

1. $\mathfrak{sl}(3, \mathbb{C})$

Recall for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ we have the Cartan subalgebra $\mathfrak{h} = \mathbb{C}\langle H_{12}, H_{23}, H_{31}\rangle$ where $H_{12} + H_{23} + H_{13} = 0$. Then $\mathfrak{h}^* = \langle L_1, L_2, L_3\rangle$, and we want to describe invariant functions on $\mathfrak{h}^*$. We can regard the H_{ij} as linear functions on $\mathfrak{h}^*$. For example, H_{12} is the function which is $+1$ at L_1, -1 at L_2, and zero on the span of L_3. Now to find functions invariant under the Weyl group action we want a basis of $\mathfrak{h}^*$ for which the Weyl group permutes the elements. Unfortunately the usual basis is not such a basis since, for example, (12) takes H_{12} to $-H_{12}$.

We need to take different coordinates in order for this to be a permutation action. The idea is that we don't want functions which are vanishing on the these particular hyperplanes. Instead we will set:

$$a = H_{12} - H_{31} \qquad b = H_{23} - H_{12} \qquad c = H_{31} - H_{23} .$$

Now these functions take values as in fig. 1.

Claim 1. $W = \Sigma_2$ permutes the functions a, b, and c.

"Proof" by example. Take $\sigma = (12)$, then this takes $a \mapsto b$, $b \mapsto a$, and $c \mapsto c$. □

As a result of this observation, we can write:

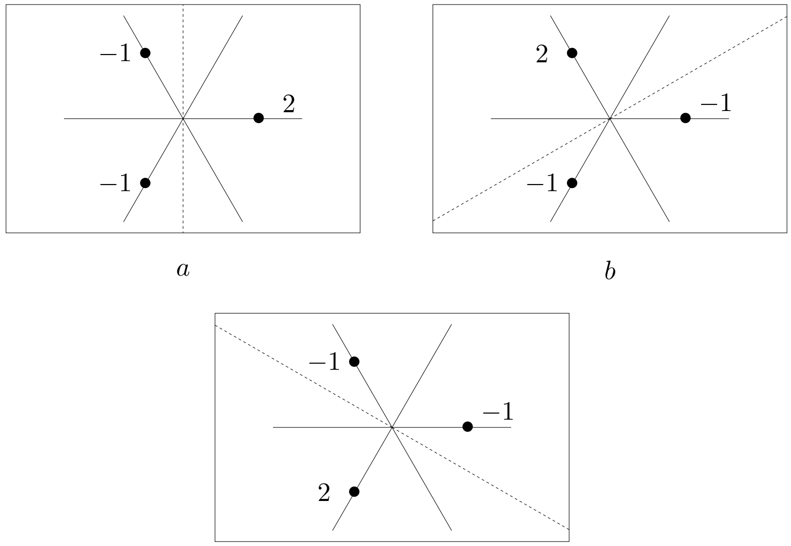
$$\operatorname{Sym}(\mathfrak{h})^W \simeq \mathbb{C}[a, b, c]^W \simeq \mathbb{C}[\sigma_2, \sigma_3]$$

where

$$\sigma_2 = ab + bc + ca \qquad \sigma_3 = abc$$

Now we want to find the image of the hyperplanes under taking W invariants. First notice that L_1, L_2, and L_3 all map to a point. The hyperplanes are the vanishing locus of the H_{ij}, but in the a, b, c basis, the hyperplanes are instead where $a = b$ or $b = c$ or $c = a$. Then the claim is that the image of these hyperplanes is given by some equation of order 3 in σ_2 and order 2 in σ_3, i.e. $\left\{c_2\sigma_2^3 + c_3\sigma_3^2\right\}$ for some c_2 and c_3. In particular for $h = (a - b)(b - c)(c - a)$ we have that h^2 is exactly the equation cutting out this cusp.

Date: November 15, 2018.



c

FIGURE 1. The values taken by our new basis a, b, and c. The functions vanish along the dotted hyperplanes now rather than the hyperplanes that the Weyl group reflects over.

2. Isomorphism of $\mathcal{U}\mathfrak{g}$ and $\operatorname{Sym}\mathfrak{g}$ as vector spaces but NOT representations

If we choose a PBW basis, we can get an identification $\mathcal{U}\mathfrak{g} \simeq \operatorname{Sym}(\mathfrak{g})$ as vector spaces, but certainly not as G-representations. The following example shows this.

Example 1. Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the usual PBW basis. If we look at $XY \in \mathcal{U}\mathfrak{g}$, then under this isomorphism with $\operatorname{Sym}\mathfrak{g}$ this element $XY \mapsto XY \in \operatorname{Sym}\mathfrak{g}$. But if we instead take $YX \in \mathcal{U}\mathfrak{g}$, the prescription is to rewrite this as $YX = XY + [Y, X] = XY - H$ which is in "PBW form" so this gets mapped to $XY - H \in \operatorname{Sym}\mathfrak{g}$. This is exactly the point of PBW, it is somehow telling you how to break symmetry.

Consider

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \ .$$

Then we're hoping that if we conjugate YX and map it to $\operatorname{Sym}\mathfrak{g}$, this is the same as mapping it and then conjugating it. We can calculate the action of Ad_g to be

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & a \end{pmatrix}$$

which means

$$H \mapsto -H \qquad X \mapsto -Y \qquad Y \mapsto -X$$

and therefore we have

$$\operatorname{Ad}_g(YX) = \operatorname{Ad}_g(Y)\operatorname{Ad}_g(X) = (-X)(-Y) = XY \ .$$

Therefore we have seen that if we first conjugate and then map to $\operatorname{Sym}\mathfrak{g}$ versus mapping to $\operatorname{Sym}\mathfrak{g}$ and then conjugating, we don't get the same result:

$$\begin{array}{ccccc} YX & \longmapsto & XY - H & & \\ \downarrow \mathrm{Ad}_g & & & \searrow & \\ XY & \xrightarrow{\text{PBW}} & XY & \neq & YX + H \end{array}$$

Therefore these things are not isomorphic as G representations in this way.

3. Isomorphism of $\mathcal{U}\mathfrak{g}$ and $\operatorname{Sym}\mathfrak{g}$ as adjoint G representations

The goal is now to construct an isomorphism of these as representations. We will not be using PBW at all. We know $\mathcal{U}\mathfrak{g}$ is filtered, and $\operatorname{Sym}\mathfrak{g}$ is even graded, so it's certainly filtered.

Claim 2 (Good news)**.** There exists an isomorphism of adjoint G representations so that in particular, for any piece of our filtration of $\mathcal{U}\mathfrak{g}$ we have the following isomorphism:

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \xrightarrow{\sim} & \operatorname{Sym}\mathfrak{g} \\ \uparrow & & \uparrow \\ F^k\mathcal{U}\mathfrak{g} & \xrightarrow{\sim} & \bigoplus_{i=0}^{k} \operatorname{Sym}^i \mathfrak{g} \end{array}$$

Proof. We will prove this by induction on the filtration. The base case is just:

$$F^0\mathcal{U}\mathfrak{g} \simeq \mathbb{C} \simeq \operatorname{Sym}^0(\mathfrak{g})$$

So now suppose we have

$$F^{k-1}\mathcal{U}\mathfrak{g} \simeq \bigoplus_{i=0}^{k-1} \operatorname{Sym}^i(\mathfrak{g})$$

as G representations. Consider the following SESs:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{k-1}\mathcal{U}\mathfrak{g} & \longrightarrow & F^k\mathcal{U}\mathfrak{g} & \longrightarrow & \operatorname{Gr}_k \mathcal{U}\mathfrak{g} & \longrightarrow & 0 \\ & & \sim\uparrow & & ?\uparrow & \nwarrow & \sim\uparrow & & \\ 0 & \longrightarrow & \bigoplus_{i=0}^{k-1} \operatorname{Sym}^i(\mathfrak{g}) & \longrightarrow & \bigoplus_{i=0}^{k} \operatorname{Sym}^i(\mathfrak{g}) & \longrightarrow & \operatorname{Sym}^k(\mathfrak{g}) & \longrightarrow & 0 \end{array}$$

where the bottom sequence naturally splits. Then since the category is semi-simple, the top SES splits. □

4. More discussion of Harish Chandra

We won't be explicitly proving the HC isomorphism, but we will discuss it more so we at least feel comfortable with it. Recall the content of the theorem is that

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \simeq (\mathcal{U}\mathfrak{h})^{W,\tilde{\cdot}} .$$

Geometrically, we can think of G/N as having a left action of G by left multiplication, and a right action of H which commutes with this action since

$$(gN)\,h = gh\left(h^{-1}Nh\right) = ghN$$

since H normalizes N.

Example 2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Then $G/N = \mathbb{C}^2 \setminus \{0\}$ and

$$gN \mapsto g \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Recall that $N = \left\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\rangle$ is exactly the stabilizer of this vector $(1, 0)$. Now we can think about what this action does to vectors in $\mathbb{C}^2 \setminus \{0\}$. Take $h = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ inside the Cartan, and now we want to see what $gNh = ghN$ goes to:

$$ghN \mapsto gh \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g \begin{pmatrix} z \\ 0 \end{pmatrix}$$

so $H \simeq \mathbb{C}^\times$ acts by dilation, and $\mathrm{SL}(2, \mathbb{C})$ acts as usual by linear transformations. So we have these two commuting actions on $\mathbb{C}^2 \setminus \{0\}$.

Exercise 1. Show than when we have these two commuting actions $G \circlearrowright G/N \circlearrowleft H$, then G-equivariant automorphisms of G/N are exactly just H acting on the right.

Solution. Since $B = N_G(N)$, we have $H \simeq B/N$. This is just a general fact that in any subgroup if you ask what are the G-equivariant automorphisms of the homogeneous space, the answer will be the normalizer modulo the stabilizer which in this case is H.

The point is that one only needs to know where one point goes since it is G equivariant.

Remark 1. This is very similar to when we were talking about highest weights.

Corollary 1. *G invariant vector fields on G/N are given by vector fields coming from $\mathfrak{h}$.*

Corollary 2 (More generally)**.** *The collection of G invariant differential operators on G/N is isomorphic to differential operators coming from $\mathcal{U}\mathfrak{h}$.*

Example 3. Let's return to $G = \mathrm{SL}(2, \mathbb{C})$ to see what's going on here. In this context this is saying that:

$$\mathrm{Aut}^G \left(\mathbb{C}^2 \setminus \{0\} \right) \simeq \mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts by dilation. So if you need to map a vector to another vector in a way that is G-compatible, i.e. it commutes with linear tranformations, then the only way to do it is by dilation.

4.1. **Application.** The reason this abstract discussion is useful, is the following application. We can map

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \to \mathrm{Diff}^G(G/N)$$

but these must come from $\mathcal{U}\mathfrak{h}$. In other words we have a factorization

$$\begin{array}{ccc} \mathfrak{z}\mathcal{U}\mathfrak{g} & \longrightarrow & \mathrm{Diff}^G(G/N) \\ & \searrow & \nearrow \\ & \mathcal{U}\mathfrak{h} & \end{array}$$

and this is the Harish Chandra homomorphism. In this language the theorem is saying that the image is actually $(\mathcal{U}\mathfrak{h})^{(W, \tilde{\cdot})}$.

To be continued. . . [1]

[1] A fire alarm went off at this point. Probably because the large amount of smoke in the air from the forest fires leaked into the building and set the alarms off.