LECTURE 23
MATH 261A

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We will have lecture this week and Tuesday of next week. There will be office hours this week and next week as usual. The final will be posted this week, and it will be due on Monday December 10. The topic for the remaining lectures will be D-modules and Beilinson-Bernstein localization.

1. Harish-Chandra

For any complex simple Lie group G, we have the actions $G \circlearrowright G/N \circlearrowleft B/N \simeq H$ where G/N is the fundamental affine space.

Example 1. For $G = \mathrm{SL}(2, \mathbb{C})$, the fundamental affine space is $\mathbb{C}^2 \setminus \{0\}$, where $\mathrm{SL}(2, \mathbb{C})$ acts by fractional linear transformations, and $\mathbb{C}^\times \simeq H$ acts on the right by dilations.

We know we can map $\mathfrak{g} \to (G/N) \leftarrow \mathfrak{h}$ and extend this to $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}(G/B) \leftarrow \mathcal{U}\mathfrak{h}$.

Theorem 1. *$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \hookrightarrow \mathrm{Diff}(G/N)$ with image*

$$\mathrm{Diff}(G/N) \hookleftarrow (\mathcal{U}\mathfrak{h})^{(W,\tilde{\cdot})} \simeq (\mathrm{Sym}\,\mathfrak{h})^{(W,\tilde{\cdot})} \simeq \mathbb{C}[\mathfrak{h}^*]^{(W,\tilde{\cdot})}$$

so

$$\mathfrak{z}(\mathcal{U}\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}\left[\mathfrak{h}^\times\right]^{(W,\tilde{\cdot})}$$

Example 2. $\mathfrak{z}(\mathcal{U}\mathfrak{sl}(2, \mathbb{C})) \simeq \mathbb{C}[K]$ where $K = H^2 + 2(XY + YX)$ is the Casimir. We should then think of its image under this isomorphism, $(H+1)^2 \in \mathbb{C}[\mathfrak{h}^\times]^{(W,\tilde{\cdot})}$, as the quadratic function that vanishes to order 2 at -1.

2. Beilinson-Bernstein localization

Let $G \circlearrowright G/B$ be a simple complex Lie group acting on this flag variety. As usual we can map $\mathfrak{g} \to \mathrm{Vect}(G/B)$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}(G/B)$.

2.1. Algebraic vector fields and differential operators.

Algebraic vector fields and differential operators. We want to restrict our attention to algebraic vector fields and algebraic differential operators. So let's see what those things are. G/B can be covered by affine spaces $\mathbb{C}^d$ where $d = \dim(G/B)$. We can do this by taking a coordinate flag $E_w^\bullet$ for $w \in W$. Recall the standard flag is:

$$E^\bullet_{\mathrm{std}} = \{\langle 0\rangle \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \mathbb{C}^n\}$$

and then

$$E^\bullet_2 = w\left(E^\bullet_{\mathrm{std}}\right) = \left\{\langle 0\rangle \subset \left\langle e_{w(1)}\right\rangle \subset \left\langle e_{w(1)}, e_{w(2)}\right\rangle \subset \cdots\right\} \; .$$

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So given any coordinate flag we can define the affine space

$$A_w = \{E^\bullet \pitchfork E_w^\bullet\}$$

which means $E^k \pitchfork E_w^{n-k}$ for all k.

Exercise 1. Show that each of these $A_w \simeq \mathbb{C}^d$.

Now "algebraic" means that we only allow polynomial functions on the coordinate patches. So algebraic vector fields are vector fields such that on any coordinate patch, it will look like a polynomial function times $\partial/\partial x_i$ rather than a generic complex analytic function times these $\partial/\partial x_i$. Explicitly they are of the form:

$$x = \sum_{i=1}^{d} p_i\left(x\right) \partial_{x_i}$$

where the p_i are polynomials. We will write $\mathrm{Vect}^{\mathrm{alg}}\left(G/B\right)$ and $\mathrm{Diff}^{\mathrm{alg}}\left(G/B\right)$ for the algebraic vector fields and algebraic differential operators respectively.

Exercise 2. Show that the maps $\mathfrak{g} \to \mathrm{Vect}\left(G/B\right)$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}\left(G/B\right)$ land in algebraic vector fields and algebraic differential operators.

Example 3. For $\mathfrak{g} = \mathfrak{sl}\left(2, \mathbb{C}\right)$, in one of the coordinate patches we saw that $X \mapsto \partial_x$, $H \mapsto -2x\partial_x$, and $Y \mapsto -x^2\partial_x$ which are of course algebraic.

2.2. **A fundamental theorem.**

Theorem 2. *The map $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{alg}\left(G/B\right)$ is an isomorphism, where we define*

$$\mathcal{U}_0\mathfrak{g} = \mathcal{U}\mathfrak{g}/\mathfrak{z}^0\left(\mathcal{U}\mathfrak{g}\right)$$

where $\mathfrak{z}^0\left(\mathcal{U}\mathfrak{g}\right)$ is the augmentation ideal, i.e. the ideal of $\mathfrak{z}\left(\mathcal{U}\mathfrak{g}\right) \simeq \mathbb{C}\left[\mathfrak{h}^\times\right]^{(W,\cdot)}$ vanishing at $0 \in \mathfrak{h}^\times$.

This is compatible with the HC isomorphism in the following sense. Recall that when discussing HC we mapped $\mathfrak{g} \to \mathrm{Vect}^{\mathrm{alg}}\left(G/N\right) \leftarrow \mathfrak{h}$ and $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}\left(G/N\right) \leftarrow \mathcal{U}\mathfrak{h}$. But now we can obtain $\mathrm{Diff}^{\mathrm{alg}}\left(G/B\right)$ from $\mathrm{Diff}^{\mathrm{alg}}\left(G/N\right)$ by doing "quantum Hamiltonian reduction." In particular we following the steps:

(1) Take H invariant differential operations on G/N, $\mathrm{Diff}^{\mathrm{alg}}\left(G/N\right)^H$.
(2) Quotient by $\mathcal{U}^0\mathfrak{h}$, which is the H-invariant differential operators along the fibers. Note that $\mathcal{U}^0\mathfrak{h}$ is the kernel of the map $\mathcal{U}^0\mathfrak{h} \to \mathcal{U}\mathfrak{h} \simeq \mathbb{C}\left[\mathfrak{h}^\times\right] \xrightarrow{\mathrm{ev}_0} \mathbb{C}$.

So in the end we get:

$$\mathrm{Diff}^{\mathrm{alg}}\left(G/B\right) = \mathrm{Diff}^{\mathrm{alg}}\left(G/N\right)^H/\mathcal{U}^0\mathfrak{h}$$

So the map $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}\left(G/B\right)$ certainly must send the ideal $\mathfrak{z}^0\left(\mathcal{U}\mathfrak{g}\right) \subseteq \mathcal{U}^0\mathfrak{h}$ to 0.

Remark 1. Similarly we can prove that $\mathcal{U}_\lambda\mathfrak{g} \xrightarrow{\sim} \mathrm{Diff}_\lambda^{\mathrm{alg}}\left(G/B\right)$ where $\mathcal{U}_\lambda\mathfrak{g} = \mathcal{U}\mathfrak{g}/\mathfrak{z}^\lambda\left(\mathcal{U}\mathfrak{g}\right)$, and

$$\mathrm{Diff}_\lambda^{\mathrm{alg}}\left(G/B\right) = \mathrm{Diff}^{\mathrm{alg}}\left(G/N\right)^H/\mathcal{U}^\lambda\mathfrak{h}\ .$$

This new object consists of what are called "twisted differential operators." We can think of this as taking the fiber of the moment map at λ rather than at 0. The most general version of this theorem that one might consider is:

Theorem 3. $\mathcal{U}\mathfrak{g} \otimes_{\mathfrak{z}\mathfrak{g}} \mathcal{U}\mathfrak{h} \xrightarrow{\sim} \mathrm{Diff}^{alg}\left(G/N\right)^H$

So in this case we somehow skip the second step of the Hamiltonian reduction above.

The point is that if we want to understand $\mathcal{U}\mathfrak{g}$ modules, or in particular irreducible $\mathcal{U}\mathfrak{g}$ modules, each of them will have a fixed central character, so each of them will come with some $\mathcal{U}_\lambda\mathfrak{g}$. And this theorem is telling us that the theory of $\mathcal{U}_\lambda\mathfrak{g}$ modules is the same as the theory of modules over the differential operators, so something very geometric. We will spend the next week or so talking about what it means to be a module over differential operators.

3. Modules over differential operators

We could tell this whole story for twisted differential operators, which is necessary to understand all $\mathcal{U}\mathfrak{g}$ modules, since studying differential operators only tells us about $\mathcal{U}_0\mathfrak{g}$. But we will keep it simple and just study $\mathrm{Diff}^{\mathrm{alg}}(G/B)$. Now we have the following key idea:

Key idea: To obtain modules over global differential operators from local modules over differential operators by taking global sections.

The point here is that we will construct modules by gluing together "local" modules. We couldn't do this in $\mathcal{U}_0\mathfrak{g}$ itself, but in G/B we can since

$$G/B = \bigcup_{w\in W} A_w$$

is just the union of affine pieces which are each $\mathbb{C}^d$ for $d = \dim G/B$. In particular, we will study differential operators on these pieces A_w and then glue them all together.

3.1. Local story: algebraic differential operators and their modules on $\mathbb{C}^d$.

Exercise 3. Show that the algebraic differential operators are exactly the Weyl algebra:

$$\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \simeq \mathbb{C}\left\langle x_1, \cdots, x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right\rangle$$

where the x_i all commute, the ∂_{x_i} commute, and then

$$[\partial_{x_i}, x_j] = \partial_{x_i} x_j - x_j \partial_{x_i} = \begin{cases} 0 & i \neq j \\ 1 + x_i\partial_{x_i} - x_i\partial_{x_i} = 1 & i = j \end{cases} .$$

Now we will think about what modules are like over this algebra. First we will focus on the case $d = 1$ and do some examples. So in this case, $\mathrm{Diff}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}\langle x, \partial_x\rangle$.

Example 4. The free module $\mathbb{C}\langle x, \partial_x\rangle^{\oplus n}$ is of course a module.

Example 5. Polynomial functions $\mathcal{O}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}[x]$ is a module over $\mathrm{Diff}^{\mathrm{alg}}(\mathbb{C})$ where x acts as x and ∂_x differentiates x. Note that $\mathbb{C}\langle x, \partial_x\rangle / (\partial_x) \xrightarrow{\sim} \mathbb{C}[x]$ where $1 \mapsto 1$. This isn't free, but it's still nice. This is a somehow small module since it's the quotient of a rank-one free module.

Example 6. We can also consider the rational functions

$$\mathcal{K}^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}(x) = \left\{ \frac{p}{q} \mid q \not\equiv 0 \right\} .$$

This is also a module, but it is not finitely generated, because whenever you think you've finitely generated it, some function walks into the room with a deeper pole.

Example 7. We can consider

$$\mathcal{O}^{\mathrm{alg}}(\mathbb{C})\langle e^x\rangle = \{p \cdot e^x \mid p \text{ polynomial}\}$$

but we have to check that

$$\partial_x (pe^x) = p'e^x + pe^x = (p'+p)\, e^x \ .$$

This is somehow a small module, so we expect it to be somehow surjected upon by a rank 1 free module. Indeed, $\mathbb{C}\langle x, \partial_x\rangle / (\partial_x - 1) \xrightarrow{\sim} \mathcal{O}^{\mathrm{alg}}(\mathbb{C})\langle e^x\rangle$ where we map $1 \mapsto e^x$.

Example 8. We can also consider any sufficiently differentiable function space on $\mathbb{C}$, but these are somehow huge and not so algebraic.

Now we might wonder if there is a module M with $\dim_{\mathbb{C}} M < \infty$. Certainly none of them so far have satisfied this. An easier question to ask might be to forget about ∂_x, and think of finite dimensional $\mathbb{C}[x]$ modules.

Exercise 4. Show that any finite dimensional $\mathbb{C}[x]$ module is a finite dimensional vector spaces equipped with an endomorphism. Show that

$$V = \bigoplus_{i=1}^{k} \mathbb{C}[x] / (x-\lambda_i)^{d_i} \ .$$

Note that each of these is a Jordan block.

So now we need to ask ourselves how to add ∂_x into the picture and act on such a V.

Claim 1. Finite dimensional $\mathbb{C}\langle x, \partial_x\rangle$-modules M are all trivial.

First we need to check how ∂_x must act on an eigenvector, then we just need to notice what happens when we keep applying ∂_x.

Remark 2. We can also show it more algebraically. If we have some finite dimensional module M, then we can represent x and ∂_x as two matrices A and B which act as linear operators on M. Then since $[x, \partial_x] = 1$, $[A, B] = I$, but if M is finite dimensional we have a well defined Tr, so

$$n = \mathrm{Tr}(I) = \mathrm{Tr}(AB - BA) = 0$$

so M must be zero dimensional, or Tr is not well defined.