LECTURE 24
MATH 261A

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Recall from last time we stated the following theorem:

Theorem 1. *For $\mathfrak{g}$ a semisimple complex Lie algebra, we have an isomorphism:*

$$\mathcal{U}_0\mathfrak{g} \xrightarrow{\sim} \mathrm{Diff}^{alg}(G/B)$$

Remark 1. The G action on G/B leads to a map $\mathfrak{g} \to \mathrm{Vect}^{\mathrm{alg}}(G/B)$, which leads to a map $\mathcal{U}\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$. Last time we discussed why this factors through $\mathcal{U}\mathfrak{g}$ to $\mathcal{U}_0\mathfrak{g} = \mathcal{U}\mathfrak{g}/\mathfrak{z}^0\mathfrak{g}$, where $\mathfrak{z}^0\mathfrak{g}$ is the ideal I_0 under the identification $\mathfrak{z}\mathfrak{g} \simeq \mathbb{C}[\mathfrak{h}^\times]^{(W,\tilde{\cdot})}$. And from the same argument, is then injective.

1. Why is this map surjective

We will now try to see why we should expect this map to be surjective.

Remark 2. If this is indeed surjective, we might wonder what hits the functions on the right, but the answer is that they're all constant, since G/B is compact. The example to keep in mind is $\mathbb{P}^1$.

Remark 3. If we're somehow only concerned with representation theory of $\mathfrak{g}$, we don't really need this to be surjective, but we will make this comment anyway.

First note that the above maps are actually maps of filtered algebras. I.e.

$$\mathcal{U}\mathfrak{g} \to \mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$$

respect the natural filtrations on these objects. The first has the tensor algebra filtration, the second has the tensor algebra filtration modulo some filtered ideal, and the filtration on $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ is given by the order of a given differential operator. These all somehow look like

$$\sum_\alpha p_\alpha(x)\,\partial_x^\alpha$$

where $p_\alpha(x)$ are polynomials where for big enough α, they're all zero. Then the filtration is given by the order of α.

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Now we can pass to the associated graded algebras:

$$\begin{array}{ccccc}
\mathrm{Sym}(\mathfrak{g}) & \longrightarrow & \mathrm{Sym}(\mathfrak{g}) / \left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 \simeq \mathbb{C}[\mathfrak{g}_{\chi=0}] & \longrightarrow & \mathrm{Gr}\,\mathrm{Diff}^{\mathrm{alg}}(G/B) \\
\downarrow \simeq & & \downarrow \simeq & & \\
\mathbb{C}[\mathfrak{g}^*] & & \left(\mathbb{C}[\mathfrak{g}^*]^G\right)^0 & & \\
 & & \downarrow \simeq & & \\
 & & \left(\mathbb{C}[\mathfrak{h}^*]^W\right)^0 = I^0 & &
\end{array}$$

where $\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0$ is as follows. Recall we wrote down an isomorphism of G-representations between $\mathcal{U}\mathfrak{g}$ and $\mathrm{Sym}\,\mathfrak{g}$. And under this isomorphism, the center goes to the G-invariant piece. Then we take the ideal of zero inside of this, i.e. the ideal of things which vanish when we just look at their constant piece. More precisely,

$$\mathrm{Sym}(\mathfrak{g}) \simeq \mathbb{C} \oplus \mathfrak{g} \oplus \mathrm{Sym}^2\,\mathfrak{g} \oplus \cdots$$
$$\mathrm{Sym}(\mathfrak{g})^G \simeq \mathbb{C} \oplus \langle 0 \rangle \oplus \mathrm{Sym}^2\,\mathfrak{g} \oplus \cdots$$
$$\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 \simeq \langle 0 \rangle \oplus \langle 0 \rangle \oplus \mathrm{Sym}^2\,\mathfrak{g} \oplus \cdots$$

We can think of this quotient by $\left(\mathrm{Sym}(\mathfrak{g})^G\right)^0$ as being functions $\mathbb{C}[\mathfrak{g}_{\chi=0}]$ where $\chi : \mathfrak{g}^* \to \mathfrak{h}^*$ is the characteristic polynomial map, and $\mathcal{N} = \mathfrak{g}^*_{\chi=0}$ consists of the matrices whose eigenvalues are all 0. The sort of geometric picture is:

$$\begin{array}{ccc}
\mathfrak{g}^* & \hookleftarrow & \mathfrak{g}^*_{\chi=0} = \mathcal{N} \\
\downarrow \chi & & \downarrow \\
\mathfrak{h}^*/W & \ni & 0
\end{array}$$

Note that this diagram is the fiber in this category.

So on associated graded algebras, $\mathcal{U}\mathfrak{g}$ becomes functions on $\mathfrak{g}^*$, and $\mathcal{U}^0\mathfrak{g}$ becomes functions on $\mathcal{N}$. In other words, if we pass to algebraic functions, we get

$$\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}^*] & \longrightarrow & \mathbb{C}[\mathcal{N}] \\
\| & & \| \\
\mathrm{Sym}(\mathfrak{g}) & \longrightarrow & \mathrm{Sym}(\mathfrak{g}) / \left(\mathrm{Sym}(\mathfrak{g})^G\right)^0 \\
\chi^* \uparrow & & \uparrow \\
\mathbb{C}[\mathfrak{h}^*]^W & \xrightarrow{\mathrm{res}_0} & \mathbb{C}
\end{array}$$

So on the level of associated graded algebras, when we restrict to $\mathcal{U}_0\mathfrak{g}$, instead of looking at functions on all of $\mathfrak{g}^*$, we're just looking at functions on $\mathcal{N}$. Now if we quantize, we get

$$\begin{array}{ccc} \mathcal{U}\mathfrak{g} & \longrightarrow & \mathcal{U}_0\mathfrak{g} \\ \uparrow & & \uparrow \\ \mathfrak{z}\mathfrak{g} & \longrightarrow & \mathbb{C} \end{array}$$

Remark 4. We have seen that when we pass to associated graded, $\mathcal{U}_0\mathfrak{g}$ goes to functions on $\mathcal{N}$. For $\mathcal{U}_\lambda\mathfrak{g}$, this goes to functions on the fiber above λ as in the above diagram.

Now finally, the associated graded algebra of $\mathrm{Diff}^{\mathrm{alg}}(G/B)$ is

$$\mathrm{Gr}\,\mathrm{Diff}^{\mathrm{alg}}(G/B) \simeq \mathcal{O}^{\mathrm{alg}}(T^*G/B)\ .$$

This somehow has nothing to do with G/B, this is just a general fact. This is a canonical construction.

Remark 5. In a kind of naive way, we can say that any time we have ∂_x, we replace it by the functions linear along the fiber, which to a covector, pairs with ∂_x.

Remark 6. Certinaly every vector field gives you a function on the cotangent bundle where you just pair with the vector field. Now if all of those commute, then you get all functions on the cotangent bundle, and if they don't, then you're back to thinking about differential operators.

To convince ourselves of this, we can consider the following local picture. Recall that locally, differential operators on some $\mathbb{C}^d$, are $\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) = \mathbb{C}\left[x_1, .., x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right]$. Then the filtration is given by the order of α. In particular,

$$\mathrm{Diff}^{\leq 0}\left(\mathbb{C}^d\right) = \mathcal{O}\left(\mathbb{C}^d\right) \subset \mathrm{Diff}^1\left(\mathbb{C}^d\right) = \mathcal{O}\left(\mathbb{C}^d\right)\langle \partial_{x_1}, \cdots, \partial_{x_d}\rangle \subset \cdots$$

and in general

$$\mathrm{Diff}^{\leq n}\left(\mathbb{C}^d\right) = \mathrm{Diff}^{\leq (n-1)}\left(\mathbb{C}^d\right)\langle \partial_{x_1}, \cdots, \partial_{x_d}\rangle$$

Now when we pass to associated graded, we get

$$\begin{aligned} \mathrm{Gr}^0 &\simeq \mathcal{O}\left(\mathbb{C}^d\right) \\ \mathrm{Gr}^1 &\simeq \mathcal{O}\left(\mathbb{C}^d\right)\langle \xi_1, \cdots, \xi_d\rangle \\ \mathrm{Gr}^2 &\cong \mathcal{O}\left(\mathbb{C}^d\right)\langle \xi_i\xi_j \forall i, j\rangle \end{aligned}$$

where the ξ_i are fiber-wise linear functions on the cotangent bundle which act as $\xi_i(x, \eta) = \eta_i$ and the $\xi_i\xi_j$ are fiber-wise quadratic functions which act as $\xi_i\xi_j(x, \eta) = \eta_i\eta_j$. Now we can just glue this local picture together.

Now we want to understand the last map $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$ on the level of associated graded algebras. Recall we've identified these with $\mathcal{O}^{\mathrm{alg}}(\mathcal{N})$ and $\mathcal{O}^{\mathrm{alg}}(T^*G/B)$ respectively. Then the claim is that the map $\mathcal{U}_0\mathfrak{g} \to \mathrm{Diff}^{\mathrm{alg}}(G/B)$ is an isomorphism. In particular, we claim that this is an isomorphism iff it is an isomorphism on the level of the associated graded algebras, and that this is an isomorphism.

Exercise 1. The G action on G/B gives you a moment map $\mu : T^*G/B \to \mathfrak{g}^*$. Show:

(1) $\mathrm{im}(\mu)$ is $\mathcal{N} \subseteq \mathfrak{g}^*$
(2) μ^* is $\mathcal{O}^{\mathrm{alg}}(\mathcal{N}) \to \mathcal{O}^{\mathrm{alg}}(T^*G/B)$.

(3) μ is a resolution, in particular proper and surjective, and conclude μ^* is an isomorphism.

In summary, we have the following:[1]

$$\begin{array}{ccc} & \mathfrak{g}^* \hookleftarrow \mathcal{N} \twoheadleftarrow T^*G/B \\ \Big\downarrow \text{quantize} & \\ & \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}_0\mathfrak{g} \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{alg}}\left(G/B\right) \end{array}$$

This is a very nice picture, because we have these geometric spaces, and then consider functions on them, and then we deform these functions to be non-commutative and we get this picture.

2. Local modules

Now we return to the question of modules over differential algebraic operators on $\mathbb{C}^d$. Recall

$$\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \simeq \mathbb{C}\left[x_1, \cdots, x_d, \partial_{x_1}, \cdots, \partial_{x_d}\right]$$

In particular, we were wondering if there are any finite dimensional representations over this. Suppose M is a finite dimensional representation, then x_i and ∂_{x_i} are just matrices and then there is a well defined trace, so $\mathrm{tr}\left(\left[x_i, \partial_{x_i}\right]\right) = 0 = \mathrm{tr}\left(I\right) = n$. So the "smallest" modules are the size of the examples from last time which were all somehow like $\mathcal{O}\left(\mathbb{C}^d\right) \simeq \mathbb{C}\left[x_1, \cdots, x_d\right]$ and $\mathcal{O}\left(\mathbb{C}^d\right) e^{ax} \simeq e^{ax}\mathbb{C}\left[x_1, \cdots, x_d\right]$.

There are more of this sort when we consider the x_i and the ∂_{x_i} on the same footing. In particular, we have some Fourier transform symmetries $\mathrm{FT}_i : \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) \to \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right)$ which map $x_j \mapsto x_j$ and $\partial_{x_j} \to \partial_{x_j}$ for $j \neq i$ and $x_i \mapsto \partial_{x_i}$ and $\partial_{x_i} \mapsto x_i$. So another example is:

$$\Delta\left(0\right) \simeq \mathbb{C}\left[\partial_{x_1}, \cdots, \partial_{x_d}\right] \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / \left(x_1, \cdots, x_d\right) \ .$$

This is like $\mathcal{O}\left(\mathbb{C}^d\right) \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / \left(\partial_{x_1}, \cdots, \partial_{x_d}\right)$ except we instead quotient out by the x_i. We also have

$$\Delta\left(p\right) \simeq \mathbb{C}\left[\partial_{x_1}, \cdots, \partial_{x_d}\right] \simeq \mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right) / \left(x_1 - p_1, \cdots, x_d - p_d\right) \ .$$

These are called the "delta functions" and can be thought of as distributions.

3. D-modules

Definition 1. A D-module M on G/B is a compatible collection of $\mathrm{Diff}^{\mathrm{alg}}\left(A_w\right)$-modules.

More precisely we want

$$M_w|_{A_w \cap A_{w'}} \simeq M_{w'}|_{A_w \cap A_{w'}}$$

given by $\varphi_w^{w'}$ and then there's a cocycle condition.

By restriction we mean the following. Given a $\mathrm{Diff}^{\mathrm{alg}}\left(\mathbb{C}^d\right)$-module M and an open $\mathcal{U} = \{p_1 \neq 0, \cdots, p_l \neq 0\} \subseteq \mathbb{C}^d$ then set

$$M|_U = M\left[\frac{1}{p_1}, \cdots, \frac{1}{p_l}\right]$$

[1] Professor Nadler says we should be screaming this in the middle of the night. He also says this is potentially tattoo worthy mathematics.

Theorem 2 (Beilinson-Bernstein localization)**.** $\mathcal{U}_0$-**Mod** *is equivalent to D-modules on G/B where we send a D-module M to its global sections* $\Gamma(G/B, M)$*.*

Note that we define the global sections to be the equalizer of the following diagram:

$$\Gamma(G/B, M) \longrightarrow \prod_{w\in W} M_w \rightrightarrows \prod_{w,w'} M_w|_{A_{w'}}$$

3.1. *D***-modules on** $\mathbb{P}^1$**.** We will focus on $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, so $G/B \simeq \mathbb{P}^1 \simeq A_1 \cup A_\sigma \simeq \mathbb{C} \cup \mathbb{C}$ where $W \simeq \mathbb{Z}/2 \simeq \{1, \sigma\}$. Recall that since $E^\bullet_{\text{std}} = \langle e_1 \rangle$, A_1 is the subset of lines l in $\mathbb{P}^1$ where l transverse to e_1, and then $E^\bullet_\sigma = \langle e_2\rangle$, so A_σ is the subset of lines l in $\mathbb{P}^1$ which are transverse to e_2. A_1 has coordinate $t = s^{-1}$, and A_σ has coordinate s, where s is the slope of the line, i.e. the intersection with the line at $x = 1$.

Therefore a D-module M has two parts, it is a pair M_1 and M_σ which are modules over $\mathbb{C}[t, \partial_t]$ and $\mathbb{C}[s, \partial_s]$ respectively. They also come with an isomorphism

$$M_1\left[t^{-1}\right] \simeq M_\sigma\left[s^{-1}\right] =: M_{1,\sigma}$$

as modules over $\mathbb{C}\left[t, t^{-1}, \partial_t\right] \simeq \mathbb{C}\left[s, s^{-1}, \partial_s\right]$.

The global sections are:

$$\Gamma\left(\mathbb{P}^1, M\right) = \ker\left[M_1 \times M_\sigma \to M_{1,\sigma}\right]$$

so these are pairs which map to $m_1, m_\sigma \mapsto m_1 - m_\sigma$. Recall $\mathfrak{sl}(2,\mathbb{C}) \to \mathrm{Vect}^{\mathrm{alg}}\left(\mathbb{P}^1\right)$.

Example 1. $\mathcal{O}^{\mathrm{alg}}\left(\mathbb{P}^1\right) = \Gamma\left(\mathbb{P}^1, \mathcal{O}\right)$ for some D-module $\mathcal{O}$, defined by $\mathcal{O}(A_1) = \mathbb{C}[t]$ and $\mathcal{O}(A_\sigma) = \mathbb{C}[s]$. So this is saying all polynomials on a compact space are constant. This is a complicated way of telling you the trivial representation of $\mathfrak{sl}(2,\mathbb{C})$.

Example 2. The D-module $\Delta_{s=0}$ at $s = 0$ is given by the pieces $\Delta_{s=0}(A_1) = \langle 0\rangle$, and $\Delta_{s=0}(A_\sigma) = \mathbb{C}[s, \partial_s]/(s)$.

Exercise 2. What representation of $\mathfrak{sl}(2,\mathbb{C})$ does this correspond to?

The BGG resolution will come from the Schubert cells which we will see on Tuesday.