LECTURE 25
MATH 261A

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Today we will pick up where we left off with $\mathfrak{sl}(2,\mathbb{C})$ and return to the BGG resolution.

1. D-modules

Recall we were considering $G = \mathrm{SL}(2,\mathbb{C}) \circlearrowright \mathbb{P}^1 \simeq \mathcal{B} \simeq G/B$. This is the usual action by fractional linear transformations. So we get a map $\mathfrak{g} \to \mathrm{Vect}\left(\mathbb{P}^1\right)$. Then we have our favorite elements X, H, and Y, which go to

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto s^2\partial_s \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto 2s\partial_s \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto -\partial_s$$

in the chart $A_\sigma = \{l \pitchfork \langle e_2 \rangle\}$. Recall a D-module M on $\mathbb{P}^1$ is a compatible pair M_1 and M_σ where M_1 is a module over the Weyl algebra $\mathbb{C}\langle s, \partial_s\rangle$ and M_2 is a module over $\mathbb{C}\langle t, \partial_t \rangle$ where $t := s^{-1}$. Then these have to agree on the overlap. Equivalently, the following diagram commutes:

$$\begin{array}{ccc} & \mathbb{C}\langle s,\partial_s\rangle & \\ \nearrow & & \searrow \\ \mathcal{U}\mathfrak{g} & & \mathbb{C}\left\langle s,s^{-1},\partial_s\right\rangle = \mathbb{C}\left\langle t,t^{-1},\partial_t\right\rangle \\ \searrow & & \nearrow \\ & \mathbb{C}\langle t,\partial_t\rangle & \end{array}$$

The global sections of this pair, $\Gamma\left(\mathbb{P}^1, M\right)$, comprise the kernel of the map:

$$M_1 \times M_\sigma \xrightarrow{\mathrm{res}_1 - \mathrm{res}_\sigma} M_1|_{A_\sigma} = M_\sigma|_{A_1}$$

Now because of the above compatibility, this is naturally a $\mathcal{U}\mathfrak{g}$ module. Now we want to match D-modules and $\mathcal{U}\mathfrak{g}$-modules.

One thing we could do is write down some D-modules and see what we get when we take global sections. We could also pick a representation and see some D-module that hits it. There are plenty of ways to play this game.

Example 1. Let's start with our favorite D-module $\mathcal{O}$, which consists of algebraic functions on $\mathbb{P}^1$. In the charts this is $\mathcal{O}_1 = \mathbb{C}[t]$ and $\mathcal{O}_\sigma = \mathbb{C}[s]$. Then the global sections $\Gamma\left(\mathbb{P}^1, \mathcal{O}\right)$ are given by the kernel of $\mathbb{C}[t] \times \mathbb{C}[s] \to \mathbb{C}\left[t, t^{-1}\right]$ which takes $(f(t), g(s)) \mapsto f(t) - g\left(t^{-1}\right)$. But if this is 0, then $f = g$ is a constant, and this is indeed the trivial representation.

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Example 2. Fix $l = \langle e_1 \rangle$, and define $\Delta(l)$ to be the algebraic delta function at l. On charts, $\Delta(l)_1 = \langle 0 \rangle$, and $\Delta(l)_\sigma = \mathbb{C}[s, \partial_s] / (s) = \mathbb{C}[\partial_s]$. Note that we can write this as $\mathbb{C}[\partial_s] = \mathbb{C} \oplus \mathbb{C}\partial_s \oplus \mathbb{C}\partial_s^2 \oplus \cdots$ so 1 is playing the role of a distribution which takes a function and tells you the value at a point. Then ∂_s is playing the roll of the distribution which eats a functions, takes the derivative, and tells you the value of the derivative at a point, and so on. The global sections are

$$\Gamma\left(\mathbb{P}^1, \Delta(l)\right) = \mathbb{C}\langle s, \partial_s \rangle / \langle s \rangle \ .$$

Now to find which representation this is, we look at the image of the basis in $\operatorname{Vect}\left(\mathbb{P}^1\right)$, and act it on this. First notice that

$$X \cdot 1 = s^2 \partial_s \cdot 1 = s(\partial_s s - 1) \cdot 1$$

since we quotiented out by (s). Therefore 1 is an X highest-weight vector.

Now we want to see what the weight of the highest weight vector is by calculating:

$$H \cdot 1 = 2s\partial_s \cdot 1 = 2(\partial_s s - 1) \cdot 1 = -2 \cdot 1$$

so 1 is a highest weight vector of highest weight -2. We already saw Y generates this, so we have that ∂_s has weight -4, ∂_s^2 has weight -6, etc. This shows us that this is the Verma module $V_{-2} = \mathcal{U}\mathfrak{sl}(2, \mathbb{C}) \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_{-2}$.

Remark 1. If we change the pole of the delta function, i.e. we consider $\Delta(l)$ for some $l \in \mathbb{P}^1$, then the representation is a Verma module, but for a different choice of Borel.

Now let's pick a representation and try to construct a D-module M such that when we take the global sections $\Gamma\left(\mathbb{P}^1, M\right)$ we get this representation. Our favorite $\mathcal{U}\mathfrak{g}$ module with trivial central character, is $\mathcal{U}_0 = \mathcal{U}\mathfrak{sl}(2, \mathbb{C}) / \mathfrak{z}^0\mathfrak{sl}(2, \mathbb{C})$ where $\mathfrak{z}^0\mathfrak{sl}(2, \mathbb{C})$ is generated by the Casimir. As it turns out, if we take the D-module of differential operators $M = D$, where $D_1 = \mathbb{C}\langle t, \partial_t \rangle$ and $D_\sigma = \mathbb{C}\langle s, \partial_s \rangle$, then the global sections are indeed $\mathcal{U}_0\mathfrak{g}$.

Remark 2. The collection of global sections of a D-module is the same as

$$\Gamma\left(\mathbb{P}^1, M\right) = \operatorname{Hom}_{D\textbf{-Mod}}(D, M)$$

where D is the free module. This is supposed to be like the story in algebra where we have some ring A and a module over A, then if we want to uncover the underlying abelian group structure of the module, we can just take Hom from the free rank 1 A-module to the module in question.

There is always a left adjoint to such a construction. If we have a representation V, we can tensor $D \otimes_{\mathcal{U}_0\mathfrak{sl}(2,\mathbb{C})} V$ where D is a D-module. We can form this tensor since $\mathcal{U}_0(\mathfrak{sl}(2, \mathbb{C}))$ maps to D. This is what's called localization. This is the usual adjunction between Hom and tensor, which in this case is an equality.

2. BGG Resolution

2.1. $\mathfrak{sl}(2, \mathbb{C})$.

We first do this for $\mathfrak{sl}(2, \mathbb{C})$ and trivial central character. There is only one finite-dimensional representation with trivial central character, which is the trivial representation. Then we saw that $V_0 \twoheadrightarrow \mathbb{C}$, and in particular

$$0 \to V_{-2} \hookrightarrow V_0 \twoheadrightarrow \mathbb{C} \to 0$$

is exact. Now we want to try to understand what is going on with the D-modules. So we will localize to obtain a SES of D-modules:

$$0 \to \Delta(l) \hookrightarrow ?? \twoheadrightarrow \mathcal{O} \to 0$$

where $l = \langle e_1 \rangle$. As it turns out, "??" is:

$$\Delta(U_l) = D \otimes_{\mathcal{U}_0 \mathfrak{sl}(2,\mathbb{C})} V_0$$

which we will specify in each coordinate chart. On the chart A_1, this is just $\mathcal{O} = \mathbb{C}[t]$. On A_σ, this is $\mathbb{C}\langle s, \partial_s \rangle / (s\partial_s)$. We say this consists of the algebraic distributions on $\mathbb{P}^1 \setminus \{l\}$. Now we would need to check that when we invert s and t, these become the same.

On the A_σ chart the SES looks like

$$0 \to \mathbb{C}\langle s, \partial_s \rangle / (s) \hookrightarrow \mathbb{C}\langle s, \partial_s \rangle / (s\partial_s) \twoheadrightarrow \mathbb{C}\langle s, \partial_s \rangle / (\partial_s) \to 0$$

where $1 \mapsto \partial_s$ under the injection, and $1 \mapsto 1$ under the surjection. On A_1 the SES looks like

$$0 \to \langle 0 \rangle \hookrightarrow \mathbb{C}[t] \twoheadrightarrow \mathbb{C}[t] \to 0$$

So to l we have assigned $\Delta(l)$, and to the rest, U_l, we assigned this $\Delta(U_l)$. So we cut $\mathbb{P}^1$ up into a point and its complement, and to each of these we have assigned a canonical module, which is the Verma module. And then the fact that the flag variety is a union of these pieces is manifested by functions on the flag variety being built up in a short exact sequence like this.

2.2. **General story.** In general, we have the Schubert stratification of the flag variety:

$$\mathcal{B} = G/B = \bigcup_{w \in W} \mathcal{B}_w$$

where $\mathcal{B}_w$ consists of the Borels $\mathfrak{b} \subseteq \mathfrak{g}$ such that the relative position[1] of $\mathfrak{b}$ w.r.t. fixed $\mathfrak{b}_0$ is w. Then to each $\mathcal{B}_w$ we can assign the D-module of algebraic distributions on $\mathcal{B}_w$, $\Delta(\mathcal{B}_w)$, and then the collection of global sections is a Verma for this Borel and for some character. And so the BGG resolution is the obvious fact that the D-module of functions can be cut up into distributions/functions on each of the Schubert pieces.

[1] Up to linear algebra, if one flag is standard, then the other can be made to be some coordinate flag, and w is which one.