LECTURE 3
MATH 261A

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Office hours are now settled to be after class on Thursdays from $12:30-2$ in Evans 815, or still by appointment.

1. The action of $\mathrm{GL}(2,\mathbb{C})$ on $\left(\mathbb{CP}^1\right)^k$

Recall we are studying the action of $\mathrm{GL}(2,\mathbb{C})$ on $\left(\mathbb{CP}^1\right)^k$. We already thought about $k=0,1,2$. When $k=3$, we are studying triples of lines in $\mathbb{C}^2$. There are three orbits of this action. The first is when $l_1=l_2=l_3$. This looks like a copy of $\mathbb{CP}^1$ again. The stabilizer of the configuration $l_1=l_2=l_3=e_1$, consists of upper triangular matrices

$$B=\left\langle\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\rangle$$

Note that $\mathbb{CP}^1 \simeq \mathrm{GL}(2,\mathbb{C})/B$. B is for Borel subgroup. Note B is not Abelian.

Then $l_i=l_j\neq l_k$ is another orbit, which looks like $\left(\mathbb{CP}^1\times\mathbb{CP}^1\right)\setminus\mathbb{CP}^1$ diagonal , so we just remove the diagonal. For $l_i=l_j=e_1$ and $l_j=e_2$, the stabilizer consists of diagonal matrices

$$T=\left\langle\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\rangle$$

Note that, the orbit $\left(\mathbb{CP}^1\times\mathbb{CP}^1\right)\setminus\mathbb{CP}^1\simeq \mathrm{GL}(2,\mathbb{C})/T$. T is for torus.

The final orbit consists of distinct lines. This is an open, dense, orbit. This is all that's left, so it's $\left(\mathbb{CP}^1\right)^3$ minus everything else. The stabilizer of $l_1=e_1$, $l_2=e_2$ and $l_3=(1,1)$ is Z for center consisting of

$$Z=\left\langle\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\rangle$$

For $a\neq 0$.

Note that we have the following exact sequence:

$$1 \longrightarrow \mathbb{C} \longrightarrow B \underset{\curvearrowleft}{\longrightarrow} T \longrightarrow 1$$

So we can write this as a semidirect product $B\simeq \mathbb{C}\rtimes T$.

Exercise 1. Describe $T \circlearrowright \mathbb{C}$.

The third orbit is just $\mathcal{O}=\mathrm{GL}(2,\mathbb{C})/Z$. Since Z is the center, it is normal, which means this is a group. This has a name: $\mathrm{PGL}(2,\mathbb{C})\simeq\mathrm{GL}(2,\mathbb{C})/Z$.

Exercise 2. We know $\mathrm{GL}(n,\mathbb{C})=\mathrm{Aut}_{\mathrm{Vect}}(\mathbb{C}^n)$. Convince yourself that $\mathrm{PGL}(n,\mathbb{C})=\mathrm{Aut}_{\mathbf{AlgVar}}\left(\mathbb{P}^{n-1}\right)$.

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Assume a Lie group $G \circlearrowright X$ simply transitively, i.e. transitive and free. Then we might hope that X is also a group, or that they're canonically isomorphic. But the point is, they are not until you choose a point in $x \in X$. This point is then the identity. The way this is an isomorphism, is $G \to X$ where $g \mapsto g \cdot x$. In this situation, we call this a principal G bundle over a point, or a G-torsor.

Example 1. One example is V a vector space and X an affine space modeled on V. This doesn't have an origin a priori.

2. More manifold review

Now we start differentiating. Let's review tangent and cotangent bundles. Recall the category **Mfd** where manifolds are objects, and the morphisms are smooth maps between them. Then we have a tangent bundle functor $T : \mathbf{Mfd} \to \mathbf{Mfd}$. Dually, there is a cotangent bundle which maps $T^* : \mathbf{Mfd} \to \mathbf{SymplMfd}$.

2.1. **Tangent bundles.**

Recall for M a smooth n-manifold, we have a rank n vector bundle $\pi : TM \to M$. If we regard $M \subset \mathbb{R}^N$ as living in an ambient $\mathbb{R}^N$, there is a copy of $\mathbb{R}^N$ living at every point $x \in M$, and then we can consider the subspace of this space which consists of vectors tangent to this point on the manifold. Furthermore, if $M = \{F = 0\}$, for $F : \mathbb{R}^N \to \mathbb{R}^{N-n}$, we have $M = F^{-1}(0)$ for 0 a regular value. In this space, the tangent space is the kernel of dF_x.

Observe that $TM \subset \mathbb{R}^N \times \mathbb{R}^N$ where the first copy consists of the points x, and the second copy consists of tangent vectors. So this whole thing is cut out by $F = 0$ and $dF = 0$.

2.2. **Cotangent bundles.**

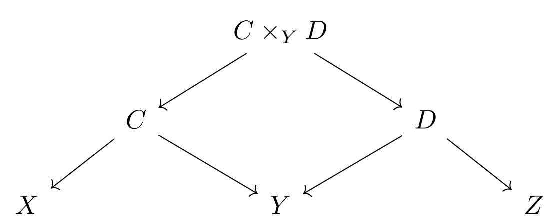
A cotangent bundle $\pi : T^*M \to M$ consists of the dual space of these tangent space at each point. Note that T^* is a functor into symplectic manifolds rather than ordinary manifolds.

Consider a smooth map $f : M \to N$. If we want we can pass to the tangent bundles and get df. What we get here is a correspondence:

$$T^*M \xleftarrow[df^*]{} T^*N|_M = T^*N \times_N M \longrightarrow T^*N$$

Exercise 3. T^* is a functor.

Recall the composition of correspondences goes like this: If we have spaces X and Y and some correspondence C between them, along with a correspondence D between Y and Z. Then we form the fiber product $C \times_Y D$.



In fact not only are the manifolds that we end up with symplectic, but the correspondences are Lagrangian correspondences. Recall a symplectic manifold consists of a pair (X, ω) where ω is a 2-form which is closed and nondegenerate.

Alternatively we can view this as a section $\omega \in \Gamma\left(X, \Lambda^2 T^* X\right)$. All symplectic forms locally look like

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Any time we consider a correspondence, we can think of $T^*N|_M$, which can be thought of as a subset of the actual product (it is a fiber product after all). In fact it is even a submanifold:

$$L = T^*N|_M \hookrightarrow T^*M \times T^*N$$

Even better than this, it is Lagrangian, so $\omega|_L = 0$. Note that the symplectic form on $T^*M \times T^*N$ is $-\omega_M + \omega_N$.

3. Lie algebras

Consider the following question: what algebraic structure do vector fields on a manifold have? The whole point is that everything we will end up doing will be analogous to this. Write

$$\mathrm{Vect}\,(M) = \Gamma\,(M, TM)$$

The first thing we learn in a manifolds course is that this has a Lie bracket defined as follows: for two vector fields v, w, we can apply the following to a function:

$$[v, w]\, f = (vw - wv)\, f$$

Exercise 4. Show $[v, w]$ is also a vector field.

Solution. We can write:

$$v = \sum g_i \partial_{x_i} \qquad\qquad w = \sum_{i=1}^{n} h_i \partial_{x_i}$$

in some local coordinates. Then

$$[v, w]\, f = \left(\sum_i g_i \partial_{x_i}\right)\left(\sum_j h_j \partial_{x_j}\right) f - \left(\sum_j h_j \partial_{x_j}\right)\left(\sum_i g_i \partial_{x_i}\right) f$$

Then we have the identity that

$$\partial_{x_i} p = p \partial_{x_i} + p_i$$

for any function p. Now we want to move every derivative to the right, and the only remaining terms are first order.

Recall a Lie bracket satisfies several properties:

(1) Bilinearity over $\mathbb{R}$
(2) Skew-symmetric: $[v, w] = -[w, v]$
(3) Jacobi identity: $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$

We can think of the Jacobi identity as a sort of Leibniz rule.

Fix a field k.

Definition 1. A Lie algebra over k is a k-vector space equipped with a bracket $[\cdot, \cdot] : V \otimes V \to V$ satisfying the three properties from above.

A morphism of Lie algebras

$$\varphi : (V_1, [\cdot,\cdot]_1) \to (V_2, [\cdot,\cdot]_2)$$

is a linear map such that $[\varphi v, \varphi w] = \varphi([v, w])$. In a certain imprecise sense, these can all be thought of as vector fields. There are two “standard” sources of Lie algebras.

Any time we have A such that A/k is an associative algebra, we can consider derivations of A, written $\operatorname{Der}(A)$. This consists of maps $d : A \to A$ which are k-linear and satisfy the Leibniz rule:

$$\partial(ab) = \partial a \cdot b + a \cdot \partial b$$

Exercise 5. Show that the composition of two derivations is not a derivation, but the bracket of two derivations is a derivation.

Exercise 6.
a. Show that $\operatorname{Der}(A)$ is a Lie algebra with a natural Lie bracket.
b. Check that $A = \mathcal{C}^\infty(M)$ is a commutative algebra, and

$$\operatorname{Vect}(M) = \operatorname{Der}(\mathcal{C}^\infty(M))$$

The other source is the following. Whenever D/k is an associative algebra, we get a Lie algebra D, where the bracket of two elements is just

$$[d_1, d_2] = d_1 d_2 - d_2 d_1$$

Exercise 7.
a. Show that D is a Lie algebra.
b. Consider $D = \operatorname{Diff}(M)$, the associative algebra of differential operators. This can be viewed as living inside $\operatorname{End}(\mathcal{C}^\infty(M))$. There is a natural filtration, where $\operatorname{Diff}^0(M) = \mathcal{C}^\infty(M)$. I.e. multiplying by any function is a differential operator. The elements p are characterized by satisfying $[p, f] = 0$ for any $f \in \mathcal{C}^\infty(M)$. Then $\operatorname{Diff}^{\leq 1}(M) \supseteq \operatorname{Diff}^0(M)$ which are characterized by $[p, f] \in \operatorname{Diff}^0(M)$ and so-on.

Then we have a map of Lie algebras

$$\operatorname{Vect}(M) \to \operatorname{Diff}(M)$$

and the image lands in $\operatorname{Diff}^{\leq 1}(M)$. In fact $\operatorname{Diff}^{\leq 1}(M)$ forms a Lie algebra with quotient and sub-algebra:

$$0 \longrightarrow \mathcal{C}^\infty(M) \longrightarrow \operatorname{Diff}^{\leq 1}(M) \longrightarrow \operatorname{Vect}(M) \longrightarrow 0$$

Check that this is all the case.

4. Relationship between associative algebras and Lie algebras

There is a pair of adjoint functors:

$$\mathbf{Lie}_k \underset{\text{Forget}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathbf{Alg}_k$$

where the left adjoint to $F = \text{Forget}$ brings an algebra to the universal enveloping algebra. This means

$$\operatorname{Hom}_{\mathbf{Alg}}(\mathcal{U}V, A) = \operatorname{Hom}_{\mathbf{Lie}}(V, F(A))$$

Warning 1. One might hope that now it is the case that differential operators are the enveloping algebra of vector fields but this is not true in general.

Exercise 8. What is the relationship between $\mathrm{Diff}\,(M)$ and $\mathcal{U}\,(\mathrm{Vect}\,M)$.

5. Preview of relationship between Lie algebras and groups

Next time we will return to Lie groups, and in particular their relationship to Lie algebras. Lie algebras should roughly be viewed as derivations, e.g. vector fields, and similarly Lie groups should roughly be seen as morphisms, e.g. diffeomorphisms. Then groups give us big symmetries, and vector fields give us infinitesimal symmetries. In particular, the Lie algebra $\mathfrak{g}$ associated to a Lie group G is somehow a tangent plane to G at the origin.