# LECTURE 4
# MATH 261A

LECTURES BY: PROFESSOR DAVID NADLER
NOTES BY: JACKSON VAN DYKE

Office hours are officially Thursday 12:30-2 in Evans 740 or maybe Evans 814. Midterms will be short, potentially multiple choice. Then you can probably just drop the bad one. We are nearing the end of the basic intro part of the class, and will soon be moving to representation theory and structure theory, so there will likely be a midterm soon.

## 1. Sources of Lie algebras

Recall from last time, we defined Lie algebras, and we talked about where they come from. Recall the sources are:

(1) Whenever you have an associative algebra $A$, you can consider the derivations $\mathrm{Der}(A)$, and this is a Lie algebra.
(2) For $A$ an associative algebra, we can just forget the fact that it's an algebra, and just remember the $[\cdot,\cdot]$ structure.

**Example 1.** The key example of the first one is the algebra $\mathcal{C}^\infty(M)$, and then $\mathrm{Vect}(M) = \mathrm{Der}(\mathcal{C}^\infty(M))$.

**Example 2.** The key example of the second is $A = \mathrm{Diff}(M)$ where we just think of this as a Lie algebra directly.

### 1.1. Enveloping algebras.

In the case of the examples above, $\mathrm{Vect}(M) \to \mathrm{Diff}(M)$. One might hope that the following is the case, though it is not.

**Warning 1.** $\mathrm{Diff}(M) \neq \mathcal{U}\,\mathrm{Vect}(M)$

The functor $\mathcal{U} : \textbf{Lie-Alg} \to k\text{-}\textbf{Alg}_{ass}$ is the adjoint functor to the forgetful functor. Explicitly, for $\mathfrak{g}$ a Lie algebra,

$$\mathcal{U}\mathfrak{g} = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} / (x \otimes y - y \otimes x = [x, y])$$

Before modding out, this is just sums of words of the elements of $\mathfrak{g}$.

*Remark* 1. So why is the above warning true? Well $\mathfrak{g}^0$ is just $k$, but the zeroth portion of $\mathrm{Diff}(M)$ is smooth functions. So the corrected relationship is that the sheaf of differential operators is the universal enveloping algebroid of the tangent sheaf. The sheaf of differential operators is somehow a universal construction of this sheaf of vector fields.

*Date*: September 4, 2018.

## 2. Associating a Lie algebra to a Lie group

For $G$ a Lie group, then $T_eG$, the tangent space at $e$ is a Lie algebra.

**Example 3** (Meta-example)**.** For $G = \text{Diffeo}(M)$, the group of diffeomorphisms of $M$, what is $\mathfrak{g} = T_e \text{Diffeo}(M)$? It is $\text{Vect}(M)$. In any sense that one might conceive of, this consists of infinitesimal diffeomorphisms, or basically vector fields. $G$ consists of the symmetries of something, and the identity is a god-given symmetry, and then we are looking for symmetries nearby. Open neighborhoods are already too complicated, so we just want to consider the linearization.

**Example 4.** Let $G = \text{GL}(n, \mathbb{C}) \circlearrowright V = \mathbb{C}^n$. We won't use it, but it just so happens that it is acting on a vector space. So we have a map $G = \text{GL}(n, \mathbb{C}) \xrightarrow{\sim} \text{Aut}(V)$.

So now we want to look at the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and understand what the bracket is all about. $\text{GL}(n, \mathbb{C})$ is open inside all $n \times n$ matrices, $M(n, \mathbb{C})$ so the tangent space at any point is also just $M(n, \mathbb{C}) = \text{End}(V)$.

**Exercise 1.** Show that the Lie algebra structure on $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ is just the usual commutator $[A, B] = AB - BA$.

2.1. **Vector fields.** Now we return to considering a general Lie group $G \circlearrowright X$ for some manifold $X$. Then we can differentiate this action, i.e. we have a map $\alpha : G \times X \to X$ and we can differentiate to get a map $T(\alpha) : TG \times TX \to TX$. Now we can restrict to $T_eG \times X \to TX$ where this copy of $X$ is regarded as the zero-section of $X$.

Then we have a map of vector bundles $A : \mathfrak{g} \times X \to TX$ so this is a moment where we have used the fact that we are taking the tangent space at the identity in particular. Now we can pass to global sections, so for each $x \in X$, we obtain a linear map $A_x : \mathfrak{g} \to T_xX$, called the infinitesimal action map at $x \in X$.

All together, we obtain a linear map $\mathfrak{g} \to \text{Vect}(X)$ which maps any vector field $v \mapsto \tilde{v}$ such that $\tilde{v}_x = A_x v$.

**Example 5.** If $G$ consists of diffeomorphisms, then the tangent space at the identity consists of vector fields, so this construction gives us the identity.

The idea is that for $G \circlearrowright X$, we can say we are looking at a map of pairs $G \to \text{Diffeo}(X)$, which is a group homomorphism, and from this, we saw $T_eG \to T_e \text{Diffeo}(X) = \text{Vect}(X)$, which will be a Lie algebra homomorphism when we understand the Lie structure on $\mathfrak{g}$. So basically the goal is to find a Lie algebra structure on $\mathfrak{g}$ such that this map is always a Lie algebra homomorphism.

**Example 6.** Consider $\text{GL}(1, \mathbb{C}) = \mathbb{C}^\times \circlearrowright V = \mathbb{C}$. The action map is $\alpha : \mathbb{C}^\times \times \mathbb{C} \to \mathbb{C}$ which takes $\alpha(z, x) \to zx$. Now we want to unwind the definitions in order to see what this map $\mathfrak{g} \to \text{Vect}(X)$ looks like in this case.

First of all, $\mathfrak{g} = \mathbb{C}$. Now we want to understand how to differentiate $\alpha$. So for an element $v \in \mathfrak{g}$, we should get vector field on $\mathbb{C}$.

*Remark* 2. The general construction is as follows: Consider a map $F : M \to N$. If we have a point $x \in M$ and a vector $v \in T_xM$, and we want to somehow transport $v$ to $T_{f(x)}N$, then we take an arbitrary path $\gamma : \mathbb{R} \to M$ such that the tangent line to $\gamma$ at $x$ is $v$, then we can map this to $F(\gamma)$, and take $F(\gamma)'(0)$, to get our $TF(v) \in T_{f(x)}N$.

Let's take $v = \partial_z$, and a path which has $v$ as its tangent at 1. Take $\gamma(t) = e^t$, so this is a path from $\mathbb{R} \to G = \mathbb{C}^\times$, such that $\gamma(0) = e$, and $\gamma'(0) = (1,0) = \partial_z$. Acting by $\gamma(t)$ for small $t$ gives a small motion of $\mathbb{C}$.

$$\alpha(\gamma(t), z) = \gamma(t)\, z = e^t z$$

So this is the image of the path, and we just need to differentiate with respect to $t$, and find $\tilde{v} = z\partial_z$.

**Example 7.** Consider $\mathrm{SL}(2,\mathbb{C}) \circlearrowright X = \mathbb{CP}^1$. The goal is again to calculate the map $\mathfrak{g} \to \mathrm{Vect}\left(\mathbb{CP}^1\right)$. For any vector field on $\mathbb{CP}^1$, we can restrict to the same vector field on $\mathbb{CP}^1 \setminus \infty$,

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{Vect}\left(\mathbb{CP}^1\right) \\ & \searrow & \downarrow \\ & & \mathrm{Vect}\left(\mathbb{CP}^1 \setminus \{\infty\}\right) \end{array}$$

Recall $\mathrm{SL}(2,\mathbb{C}) = \{\det g = 1\}$. So this is sitting inside of $\mathbb{C}^4$, and the tangent space is

$$\mathfrak{sl}(2,\mathbb{C}) = \{x \mid \operatorname{tr} x = 0\}\ .$$

(The reason is, if we start out with the identity matrix, and want to add an extra matrix up to some multiple of $\epsilon$, and maintain that the determinant is 0, we have:

$$\begin{aligned} \det\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \epsilon\begin{pmatrix}1 & b\\ c & d\end{pmatrix}\right) &= \det\begin{pmatrix}1+\epsilon a & \epsilon b\\ \epsilon c & 1+\epsilon d\end{pmatrix} \\ &= (1+\epsilon a)(1-\epsilon d) - \epsilon^2 bc = 1 + \epsilon(a+d) + \epsilon^2 ad \end{aligned}$$

so we just require that the additional matrices have trace 0.)

Now for each $x \in \mathfrak{sl}(2,\mathbb{C})$, we want some $f(s)\,\partial_s \in \mathrm{Vect}(\mathbb{C})$. So choose our favorite basis

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad E = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad F = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

Now define $\gamma_M(t) = e^{tM}$ to get:

$$\gamma_H(t) = e^{tH} = \begin{pmatrix}e^t & 0\\ 0 & e^{-t}\end{pmatrix} \quad \gamma_E(t) = e^{tE} = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \quad \gamma_F(t) = e^{tF} = \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$$

now we just apply these to our elements of $\mathbb{CP}^1$.

$$\gamma_F(t)\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a\\ at+b\end{pmatrix}$$

so it took a point $s = b/a$ and transformed it into $s_t = (at+b)/a$. Now we take the derivative, and evaluate at $t = 0$ to get 1, so under this map

$$F \mapsto \partial_s\ .$$

Similarly, we can calculate

$$\gamma_H(t)\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}e^t a\\ e^{-t} b\end{pmatrix}$$

so $s = b/a \mapsto s_t = e^{-2t} b/a$, and again we differentiate, to discover that

$$H \mapsto -2s\partial_s\ .$$

Finally, we get

$$\gamma_E(t)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+bt \\ b \end{pmatrix}$$

so $s = b/a \mapsto b/(a+bt)$, and now differentiating, we get $-b^2/(a+bt)^2$, so finally

$$E \mapsto -s^2 \partial_s \ .$$

Let's now consider $G \circlearrowright X = G$ acting by left multiplication. In this case, we get a map $\mathfrak{g} \to \operatorname{Vect}(G)$.

Let $H \circlearrowright Y$, then we can talk about $\operatorname{Vect}(Y)^H \subseteq \operatorname{Vect}(Y)$ the $H$-invariant vector fields.

**Exercise 2.** $\operatorname{Vect}(Y)^H \subseteq \operatorname{Vect}(Y)$ is a Lie subalgebra.

**Lemma 1.** *The image of this map is precisely the collection of right-invariant vector fields* $\operatorname{Vect}(G)^r$.

*Proof.* The claim here is that $\mathfrak{g} \xrightarrow{A} \operatorname{Vect}(G)^r$ is an isomorphism of Lie algebras. The fact that the image is contained in the right-invariant vector fields follows from commuting the right action with the left action.

The fact that it is an isomorphism follows from the fact that $A|_e = \mathrm{id} : \mathfrak{g} \to \mathfrak{g}$. □

**Definition 1.** The Lie algebra structure on $\mathfrak{g}$ is transported via $\mathfrak{g} \xrightarrow{\sim} \operatorname{Vect}(G)^r \subset \operatorname{Vect}(G)$ which is a sub Lie-algebra.

**Theorem 1.** *If $G$ acts on arbitrary $X$, then the map $\mathfrak{g} \to \operatorname{Vect}(X)$ is a Lie algebra homomorphism.*