# LECTURE 5

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The first midterm will be on Tuesday September 18.

## 1. Right-invariant vector fields

Recall that we have a functor

$$\mathbf{Lie\text{-}Gp} \longrightarrow \mathbf{Lie\text{-}Alg}$$

$$G \longrightarrow T_e G = \mathfrak{g} = \mathrm{Vect}\,(G)^r$$

where $\mathrm{Vect}\,(G)^r$ is the collection of right-invariant vector fields. This means for every morphism $h : H \to G$, we have a morphism $T\,(h) : \mathfrak{h} \to \mathfrak{g}$. A priori this is a map between these things, but potentially not a morphism of Lie algebras, though this will follow from the following.

We also have an association

$$\mathbf{LieGroupActions} \rightsquigarrow \mathbf{LieAlgActions}$$

where something on the left is of the form $G \to \mathrm{Diffeo}\,(X)$, which is then associated to $\mathfrak{g} \to \mathrm{Vect}\,(X)$.

**Example 1.** Consider the right-invariant vector fields on $\mathrm{GL}\,(n, \mathbb{C})$. This is an open subset of $\mathbb{R}^{n^2}$. How do we construct right-invariant vector fields? Well we looked at the left action of $G$ on $G$, differentiated it, and then this gave us these vector fields.

Recall $T_e G = \mathfrak{g} = \mathfrak{gl}\,(n, \mathbb{C}) = M\,(n, \mathbb{C})$. Now let's take some tangent vector, and think about extending it to a right-invariant vector field. Recall we have an isomorphism:

**Lemma 1.** *In this example,*

$$\mathfrak{g} \xrightarrow{\sim} \mathrm{Vect}\,(G)^r$$

$$v \longmapsto \tilde{v}$$

Now pick a path $\gamma : \mathbb{R} \to G$, a one-parameter subgroup. This just means $\gamma$ is a group homomorphism between $\mathbb{R}$ and $G$. For $G = \mathrm{GL}\,(n, \mathbb{C})$, take

$$\gamma\,(t) = e^{tv} = I + tv + \frac{t^2 v^2}{2} + \cdots$$

In general, for $v \in \mathfrak{g}$, we extend this to a right-invariant vector field $\tilde{v} \in \mathrm{Vect}\,(G)^r$. Now recall:

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**Theorem 1.** *There exists unique local solutions to ODEs.*

In this situation, this means that given any vector field in a small enough ball, we can find a "motion" of the space that integrates this vector field for small enough time. I.e. vector fields integrate locally in space and time. Sometimes they integrate globally, but then you have to worry that you might "fall" off your space. So now look for an integral curve $\gamma$ (check existence globally) of $\tilde{v}$ such that $\gamma(0) = e$.

**Exercise 1.** Check that $\gamma$ extends uniquely to a 1-parameter subgroup. I.e. check that this construction not only gives us a map $(-\epsilon, \epsilon) \to G$, but in fact $\mathbb{R} \to G$ and that it is a homomorphism. [Hint: The fact that it's a homomorphism is almost obvious, and then you can use this fact to get the extension.]

So in the case of $\mathrm{GL}(n, \mathbb{C})$, this is the unique such 1-parameter subgroup. So now what we want to do is take some $n \times n$ matrix and get a right-invariant vector field from it. To construct $\tilde{v}$, consider $\gamma$ acting on the left and differentiate with respect to $t$. Let $g \in \mathrm{GL}(n, \mathbb{C})$. Then we send

$$g \mapsto \gamma(t) \cdot g = e^{tv} g$$

Then differentiate to get

$$ve^{tv} g + e^{tv} g'|_{t=0} = vg$$

Note that $v$ is an $n \times n$ matrix, which we want to picture as a tangent vector at the identity. So now the question is, if you stand at $g$, what is the $n \times n$ matrix which is telling you the value of $\tilde{v}$ at $g$, and the answer is $vg$. This all takes advantage of the fact that we are working in an ambient $\mathbb{R}^{n^2}$. In general, we can only write the following:

$$\tilde{v}|_g = R_g \cdot v$$

which is just the statement that it is right-invariant.

## 2. Right-invariant vs. left-invariant

Now we might wonder, why this is right invariant rather than left invariant. It's clear that $G$ acting on the right gives a map from $G$ to the $G$-equivariant automorphisms of $G$, $\mathrm{Aut}^G(G)$, and this map is also clearly injective. But any such automorphism is determined by the image of the identity, which allows us to show that this is in fact surjective as well.

In conclusion, the map $\mathfrak{g} \to \mathbf{Vect}(G)$ lands in the right invariant vector fields.

**Exercise 2.** Suppose this whole theory was developed with right actions instead of left actions. So we sent $v \to \tilde{v}^r$, and someone else sends $v \mapsto \tilde{v}^l$. Write a formula relating $\tilde{v}^r|_g$ and $\tilde{v}^l|_g$.

**Solution.** For $G = \mathrm{GL}(n, \mathbb{C})$ we just have $\tilde{v}^r|_g = vg$ and $\tilde{v}^l|_g = gv$. Then

$$\tilde{v}^l|_g = \widetilde{gvg^{-1}}^r|_g$$

So we just have to conjugate it.

Note also that $\tilde{v}^l|_\gamma = \tilde{v}^r|_\gamma$ since $\tilde{v}^l|_\gamma = \gamma \cdot v = e^{tv} \cdot v$ and $\tilde{v}^r|_\gamma = v \cdot \gamma = v \cdot e^{tv} = e^{tv} \cdot v$. So they certainly agree at the identity, but in general they will disagree elsewhere. The point here, is that the image of $\mathbb{R}$ in $G$ is abelian, since $\mathbb{R}$ is abelian, so anywhere in this image, conjugation doesn't do anything, and they agree everywhere on $\gamma$.

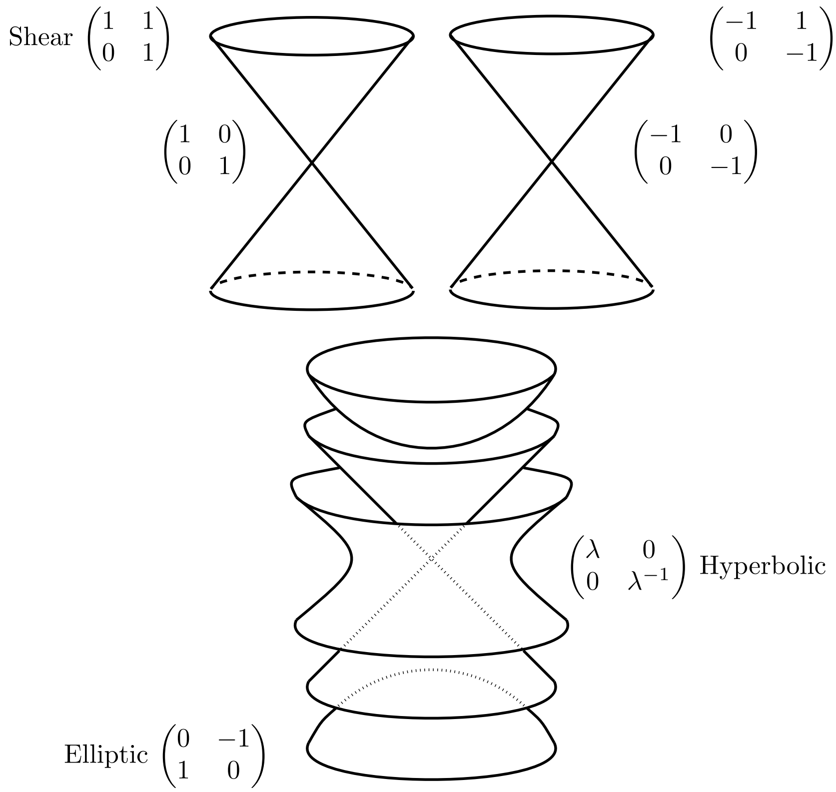


FIGURE 1. The cones which make up the orbits of the action of $\mathrm{SL}(2,\mathbb{R})$ on itself under conjugation. Note that for $\mathrm{SL}(2,\mathbb{C})$, the hyperboloid of two-sheets is not present since these matrices are then diagonalizable.

## 3. Conjugation and adjoint representations

We now consider conjugation in general as an additional natural action of $G$ on itself.

**Warning 1.** Action by conjugation is neither free nor transitive.

**Example 2.** Consider the action of $\mathrm{GL}(n,\mathbb{C})$ on itself under conjugation. The orbits of this action are indexed by the so-called Jordan forms with nonzero eigenvalues.

For $G = \mathrm{SL}(2,\mathbb{C})$, the picture here is as in fig. 1. The stabilizer at $\mathbb{1}$ and $-\mathbb{1}$ is just $G$, and then

$$\mathrm{Stab}\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\rangle \qquad \mathrm{Stab}\left\langle \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\rangle$$

Finally we have:

$$\mathrm{Stab}\left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\rangle$$

for $\alpha \in \mathbb{C}^\times$.

If we quotient by conjugation, then only the trace is well defined, because for any matrix, we either get $\lambda$ or $\lambda^{-1}$ under the map $\mathrm{tr} : G \to \mathbb{C}$. At any point we see $\lambda + \lambda^{-1}$, and then there are two special points $\pm 2$, each of which somehow has these cones living above it, so they have a sort of fuzzy piece above it corresponding to the open piece. So this is some sort of strange non-Hausdorff space. The main takeaway is that the quotient looks like a line given by the trace.

We can see that the shear matrices form a cone as follows. Notice that they must have $\operatorname{Tr} A = 2$, and $\det A = 1$. So if we write the matrix as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

this means $a + d = 2$ and $ad - bc = 1$, so $d(2-d) - bc = 1$. Now sending $d \mapsto d - 1$ we get $(d+1)(d-1) + bc = 1$ or

$$d^2 + bc = 0$$

now the quadratic form associated to this is

$$\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The eigenvalues of this matrix are $\pm 1/2$ and 1, so this corresponds to the equation

$$\frac{1}{2}x^2 - \frac{1}{2}y^2 + z^2 = 0$$

which is the equation for a cone.

Now, as usual, we want to differentiate this and see what sort of structures we get. So let's restrict $TG \times TG \to TG$ to $G \times T_e G \to T_e G$ where the first copy of $TG$ has been restricted to the zero section. This is a homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, called the adjoint representation. All we've done here is take the conjugation action of $G$ on itself, and then look at what this does to tangent vectors at the identity.

**Example 3.** For $G = \mathrm{GL}(n, \mathbb{C}) \circlearrowright M(n, \mathbb{C})$, the adjoint action is just $(g, h) \mapsto ghg^{-1}$, and we are differentiating with respect to $h$, and we get $(g, h') \mapsto gh'g^{-1}$. So the adjoint action of $\mathrm{GL}(n, \mathbb{C})$ is just given by this formula.

So the first thing we get from considering the conjugation, is we get a linearization of it, which is a conjugation of $G$ on its Lie algebra. The following claim shows that it doesn't just act by any old matrices, but in fact by matrices which preserve the Lie algebra structure:

**Claim 1.** $G \to \mathrm{GL}(\mathfrak{g})$ acts by Lie algebra isomorphisms. I.e. $[\mathrm{Ad}_g v, \mathrm{Ad}_g w] = \mathrm{Ad}_g([v, w])$.

*Proof.* Recall the left multiplication action $L : G \times G \to G$. Now we can act on everything by conjugation:

$$\begin{tikzcd} (g,h) \arrow[r, mapsto, "L"] \arrow[d, mapsto] & gh \arrow[d, mapsto] \\ (kgk^{-1}, khk^{-1}) \arrow[r, mapsto, "L"] & kghk^{-1} \end{tikzcd}$$

Recall that $[v, w] = [\tilde{v}, \tilde{w}]\,|_e$. Now we can act by $\mathrm{Ad}_g$ on this expression to get:

$$\mathrm{Ad}_g\,[v, w] = \mathrm{Ad}_g\,([\tilde{v}, \tilde{w}]\,|_e) = \mathrm{Ad}_g\,[\tilde{v}, \tilde{w}]\,|_e = [\mathrm{Ad}_g\,\tilde{v}, \mathrm{Ad}_g\,\tilde{w}]\,|_e = \left[\widetilde{\mathrm{Ad}_g\,v}, \widetilde{\mathrm{Ad}_g\,w}\right]|_e$$

by right invariance. □

Now we want to differentiate this with respect to the first variable as well. So instead restrict to $\mathrm{ad} : T_eG \times T_eG \to T_eG$, so this is a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which maps $(v, w) \mapsto \mathrm{ad}_v\,w$.

**Theorem 2.** $\mathrm{ad}_v\,(w) = [v, w]$