LECTURE 6
MATH 261A

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The midterm will cover structure theory, and will be multiple choice. We will continue with geometric structure theory, and next time we will move on to representation theory of Lie algebras.

1. Adjoint representations

Recall that for every $g \in G$ we got the Ad_g map in the following way: We know $G \circlearrowright G$ by conjugation, and then consider the induced action $G \circlearrowright TG$, and in particular the action $G \circlearrowright T_e G \simeq \mathfrak{g}$ which we call Ad.

Write the left action $\alpha : G \times G \to G$. Now α is invariant under another G-action, in particular, the diagram

$$\begin{array}{ccc} G \times (G \times G) & \xrightarrow{\mathrm{id} \times \alpha} & G \times G \\ \big\downarrow{\scriptstyle \mathrm{conj.} \times \alpha} & & \big\downarrow{\scriptstyle \alpha} \\ G \times G & \xrightarrow{\quad \alpha \quad} & G \end{array}$$

commutes. The point is, the vertical arrows are an additional invariance.

Lemma 1. *For $g \in G$, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra morphism. This means that*

$$[\mathrm{Ad}_g v, \mathrm{Ad}_g w] = \mathrm{Ad}_g ([v, w])$$

Proof. Consider $\mathrm{Ad}_g ([v, w])$. We can rewrite this as:

$$\begin{aligned} \mathrm{Ad}_g ([v, w]) &= \mathrm{Ad}_g \left([\tilde{v}, \tilde{w}]_e \right) \\ &= \left([\mathrm{Ad}_g \tilde{v}, \mathrm{Ad}_g \tilde{w}] \,|_e \right) \\ &= \left(\widetilde{\mathrm{Ad}_g (v)}, \widetilde{\mathrm{Ad}_g (w)} \right) |_e \end{aligned}$$

where we have used the fact that Ad_g is a diffeomorphism to get to the second line. □

Recall that if we differentiate again, only this time wrt the first G, we get $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. More formally,

$$\mathrm{ad}_v (w) = \frac{d}{dt} \left(\mathrm{Ad}_{\gamma(t)} (w) \right) |_{t=0}$$

where γ is some path such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then we have the following theorem:

Theorem 1. $\mathrm{ad}_v (w) = [v, w]$

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Proof. We know

$$[v, w] = [\tilde{v}, \tilde{w}]\,|_e = \left(\frac{d}{dt} L_{\gamma(t)}(\tilde{w})\right)|_{t=0}|_e$$

and since $\tilde{w}$ is right-invariant, we are done. □

2. Geometric structure theory

Assume everything is finite dimensional. Recall that we have the functor:

$$\textbf{Lie-Gp} \longrightarrow \textbf{Lie-Alg}$$

$$G \longrightarrow T_e G = \mathfrak{g}$$

Then we have the following:

Theorem 2. *This functor is an equivalence when restricted to connected, simply-connected Lie groups.*

Example 1. Consider $\mathfrak{g}$ abelian, so it is isomorphic to $\mathbb{R}^n$ (or just $\mathbb{C}^n$.) Since $\mathfrak{g}$ is abelian, $[\cdot,\cdot]$ is just 0. Which Lie groups have this algebra? The theorem tells us there is a unique simply connected one, namely $\mathbb{R}^n$ with addition. But then there is a whole tower of things covered by this such as $\left(S^1\right)^n$, $\left(\mathbb{C}^\times\right)^n$, $\left(\mathbb{C}^n/\mathbb{Z}^{2n}\right)$ and many more.

Example 2. Consider $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. Let's come up all the potential Lie groups which give rise to this algebra. Of course $\mathrm{SL}(2,\mathbb{C})$ gives rise to this, but this might not be the unique one we are looking for if it is not simply-connected, so we are instead looking for the universal cover of this.

What is the fundamental group of $\mathrm{SL}(2,\mathbb{C})$? We know $\mathrm{SL}(2,\mathbb{C}) \circlearrowright \mathbb{C}^2$. This has two orbits, i.e. when $v = 0$ and $v \neq 0$. The stabilizer of the first is everything, and

$$\mathrm{Stab}\begin{pmatrix}1\\0\end{pmatrix} = \left\langle \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} \right\rangle$$

This means

$$\mathrm{SL}(2,\mathbb{C})\,/\,\mathrm{Stab} \simeq \mathrm{SL}(2,\mathbb{C})\,/\mathbb{C} \simeq \mathbb{C}^2 \setminus \{0\}$$

This means $\pi_1(\mathrm{SL}(2,\mathbb{C}))$ is the same as π_1 of the complement of 0 in $\mathbb{C}^2$. But this is homotopy equivalent to S^3, which is simply connected (it has trivial π_1) so the unique simply connected Lie group is just $\mathrm{SL}(2,\mathbb{C})$.

But what other Lie groups might give rise to this algebra? This is really just considering things that $\mathrm{SL}(2,\mathbb{C})$ covers. If we consider the center

$$Z = Z(\mathrm{SL}(2,\mathbb{C})) \simeq \mathbb{Z}/2 = \left\langle \begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix} \right\rangle$$

and mod out by this, we get $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})\,/Z$ which has $\pi_1 \simeq \mathbb{Z}/2$. These turn out to be the only two. Equivalently, $\langle 1\rangle$ and $Z = \mathbb{Z}/2$ are the only two discrete normal subgroups of $\mathrm{SL}(2,\mathbb{C})$.

Exercise 1. Show $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SO}(3,\mathbb{C})$

Solution. Recall $\mathrm{SO}(3)$ consists of M such that $M^T = M^{-1}$ and $\det M = 1$. We want to send any A and $-A$ to the same $B \in \mathrm{SO}(3)$.

Recall $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, so for $g \in G$, $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$. So apply this to $G = \mathrm{SL}(2,\mathbb{C})$. Given $A \in \mathrm{SL}(2,\mathbb{C})$ and $v \in \mathfrak{sl}(2,\mathbb{C})$, we send this to $AvA^{-1} \in \mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{sl}(2,\mathbb{C})$ is 3-dimensional, so this makes Ad_A into a 3×3 matrix. So the question is, what 3×3 matrices do we obtain? To find out, we consider the inner product $\langle v, w\rangle = \mathrm{Tr}(vw)$. So if

$$v = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \qquad\qquad w = \begin{pmatrix} r & s \\ t & -r \end{pmatrix}$$

the inner product is

$$\langle v, w\rangle = xr + yt + zs + xr$$

and in particular,

$$\langle v, v\rangle = x^2 + yz + zy + x^2 = 2\left(x^2 + yz\right)$$

which is a nondegenerate quadratic form. But there is only one nondegenerate quadratic form on a complex vector space, i.e. in a different basis, this is just the sum of the squares. Now this inner product is clearly invariant under the $\mathrm{SL}(2,\mathbb{C})$ action, since

$$\mathrm{Tr}\left(AvA^{-1}AwA^{-1}\right) = \mathrm{Tr}\left(AvwA^{-1}\right) = \mathrm{Tr}(vw)$$

so the matrices preserve this quadratic form. So these 3×3 matrices land in the orthogonal group of this quadratic form, now we just have to check it has determinant 1. To do this, we consider the following basis for $\mathfrak{sl}(2,\mathbb{C})$:

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Now we calculate the action of some arbitrary $A \in \mathrm{SL}(2,\mathbb{C})$ as

$$\begin{aligned} Av_1A^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \begin{pmatrix} ad+bc & -2ab \\ 2dc & -(ad+bc) \end{pmatrix} \end{aligned}$$

Completing the same calculation for the other basis elements, we can express Ad_A as the following matrix:

$$\begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2dc & -c^2 & d^2 \end{pmatrix} \tag{1}$$

then we can calculate:

$$\begin{aligned} &(ad+bc)\left(a^2d^2 - b^2c^2\right) + ac\left(-2abd^2 + 2b^2cd\right) + bd\left(2abc^2 - 2a^2cd\right) \\ &= a^3d^3 - ab^2c^2d + a^2bcd^2 - b^3c^3 - 2a^2bcd^2 + 2ab^2c^2d + 2ab^2c^2d - 2a^2bcd^2 \\ &= (ad-bc)^3 = 1 \end{aligned} \tag{2}$$

since $A \in \mathrm{SL}(2,\mathbb{C})$.

This means the image lands in $\mathrm{SO}(3,\mathbb{C})$, where this is the orthogonal group with respect to the above inner product, which is fine since all such inner products are the same. So now we just need to show the kernel of this map is the center. But from the matrix in (1) we can see this directly, since if A is in the kernel, $a = d = \pm 1$ and therefore $c = d = 0$ as desired.

Example 3. What about $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. If we try the same game as in the complex case, we find that $\mathrm{SL}(2, \mathbb{R}) / \mathrm{Stab} \simeq \mathbb{R}^2 \setminus \{0\}$ is the orbit, which is homotopy equivalent to S^1, which has $\pi_1 = \mathbb{Z}$. The universal cover, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, has a map to $\mathrm{SL}(2, \mathbb{R})$ with fibers $\mathbb{Z}$.

Exercise 2. If we have a group which is not simply-connected, then the universal cover is naturally a Lie group.

Solution. The universal cover is space of homotopy classes of paths from a base point. Then we can multiply two paths pointwise to get a group, and the projection is homomorphism.

Warning 1. This universal cover does not have any finite dimensional representations, so it cannot be viewed as consisting of matrices.

We have just been assuming this so far, but for $G = \mathrm{GL}(n, \mathbb{C})$, the fact that $\mathrm{ad}_v(w) = [v, w]$ means that $[\tilde{v}, \tilde{w}] = vw - wv$, so the bracket on $\mathfrak{gl}(n, \mathbb{C})$ is truly the commutator of the matrices since

$$\frac{d}{dt}\left(\mathrm{Ad}_{\gamma(t)}\, w \gamma(t)^{-1}\right) = vw + w(-v)$$

Theorem 3 (Ado)**.** *Any finite dimensional Lie algebra is a subalgebra of* $\mathfrak{gl}(n, \mathbb{R})$ *for some* n.

Partial proof. Assume $Z(\mathfrak{g}) = \langle 0 \rangle$, so nothing has bracket 0 with everything. We know $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which we can view as a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(n, \mathbb{R})$ where $n = \dim \mathfrak{g}$. Since the center is trivial, the kernel is trivial, so this is an injection. □

2.1. **Killing form.** The Killing form is an inner product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ (or $\mathbb{R}$) where we take

$$\langle v, w \rangle_k = \mathrm{Tr}(\mathrm{ad}_v \, \mathrm{ad}_w)$$

which is a bilinear pairing.

Warning 2. This is not always nondegenerate.

For example, if $\mathfrak{g}$ is abelian, ad_v and ad_w are 0. Note that all of my matrices preserve this inner product. Now write $Q_k(v) = \langle v, v \rangle$, and then $\mathrm{Ad} : G \to \mathrm{O}(Q_k)$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{o}(Q_k)$.