LECTURE 7
MATH 261A

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Recall last time we were about to prove:[1]

Theorem 1. *The following functor*

$$\textbf{Lie-Gp} \longrightarrow \textbf{Lie-Alg}$$

$$G \longmapsto \mathfrak{g}$$

is an equivalence when restricted to connected, simply-connected groups.

Why is this a functor? I.e. why does $\varphi : H \to G$ induce a Lie-algebra homomorphism $d\varphi : \mathfrak{h} \to \mathfrak{g}$. Consider $H \circlearrowright G$ on the left via φ. Then

$$\begin{array}{ccc} \operatorname{Vect}^r(H) & \longrightarrow & \operatorname{Vect}(G) \\ & \searrow & \uparrow \\ & & \operatorname{Vect}^r(G) \end{array}$$

Definition 1. Let H, G be Lie groups. Then they are said to be locally isomorphic if there is some neighborhood $U_H \subset G$ of $e \in H$ and some neighborhood $U_G \subset G$ of $e \in G$ and a diffeomorphism $\varphi : U_H \to U_G$ mapping $e \mapsto e$ such that for any $h_1, h_2 \in U_H$, $h_1 h_2 \in U_H$ iff $\varphi(h_1)\varphi(h_2) \in U_G$ and in this case,

$$\varphi(h_1, h_2) = \varphi(h_1)\varphi(h_2)$$

1. Examples

We will consider

Example 1. First of all $\mathbb{C}^n$ is the universal cover of $(\mathbb{C}^\times)^n = \mathbb{C}^n/\mathbb{Z}^n$ and so they are locally isomorphic.

Define $\operatorname{Spin}(n, \mathbb{C})$ to be the double cover of $\operatorname{SO}(n, \mathbb{C})$. For $n > 2$, $\operatorname{Spin}(n, \mathbb{C})$ is simply connected, so it is also the universal cover of $\operatorname{SO}(n, \mathbb{C})$.

Example 2. $\operatorname{SO}(1, \mathbb{C})$ is a single point, so $\operatorname{Spin}(1) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 3. $\operatorname{SO}(2, \mathbb{C}) \cong \mathbb{C}^\times$, so $\operatorname{Spin}(2) = \mathbb{C}^\times$ as a double cover.

Example 4. $\operatorname{Spin}(3, \mathbb{C}) = \operatorname{SL}(2, \mathbb{C})$ and $\operatorname{SO}(3, \mathbb{C})$ are locally isomorphic. Recall we have the short exact sequence

$$\mathbb{Z}/2 \to \operatorname{SL}(2, \mathbb{C}) \to \operatorname{SO}(3, \mathbb{C})$$

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[1] According to professor Nadler, we will at least prove this by December...

Example 5. We seek to show that

$$\mathrm{Spin}\,(4,\mathbb{C}) = \widetilde{\mathrm{SO}}\,(4,\mathbb{C}) = \mathrm{SL}\,(2,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C})$$

The first thing to find is $\dim \mathrm{SO}\,(n,\mathbb{C})$, which is of course $\dim \mathfrak{so}\,(n,\mathbb{C})$. Now differentiating $AA^T = I$ we get

$$X\left(A^T|_{t=0}\right) + AX^T|_{t=0} = 0$$

so $X + X^T = 0$. This means the dimension of this is $n(n-1)/2$, so it makes sense that $\mathrm{Spin}\,(4,\mathbb{C}) = \mathrm{SL}\,(2,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C})$ To see this identification, we can consider $\mathrm{SL}\,(2,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C}) \circlearrowright M_{2\times 2}(\mathbb{C}) \simeq \mathbb{C}^4$. Then we can take $Q = \det : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ to be our quadratic form.

Notice that $\widetilde{\mathrm{SO}\,(4,\mathbb{C})}$ is a $\mathbb{Z}/2$ cover of $\mathrm{SL}\,(4,\mathbb{C})$, but this still has nontrivial center, so this is a $\mathbb{Z}/2$ cover of some sort of $PSO\,(4,\mathbb{C})$ which turns out to be. $\mathrm{SO}\,(3,\mathbb{C}) \times \mathrm{SO}\,(3,\mathbb{C})$ As it turns out, we can do the other possible $\mathbb{Z}/2$ quotients to get the diagram:

$$\begin{array}{ccccc} & & \mathrm{SL}\,(2,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C}) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{SO}\,(3,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C}) & & \mathrm{SO}\,(4,\mathbb{C}) & & \mathrm{SO}\,(3,\mathbb{C}) \times \mathrm{SL}\,(2,\mathbb{C}) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{SO}\,(3,\mathbb{C}) \times \mathrm{SO}\,(3,\mathbb{C}) & & \end{array}$$

This is the Galois covering diagram for the Galois group $\mathbb{Z}/2$.

Example 6. For $n = 5$ we have $\mathrm{Spin}\,(5,\mathbb{C}) = \mathrm{Sp}\,(4,\mathbb{C})$ and the diagram is just:

$$\begin{array}{c} \mathrm{Sp}\,(4) \\ \downarrow \\ \mathrm{SO}\,(5,\mathbb{C}) \end{array}$$

Example 7. For $n = 6$ we have $\mathrm{Spin}\,(6,\mathbb{C}) = \mathrm{SL}\,(4,\mathbb{C})$. This has the diagram:

$$\begin{array}{c} \mathrm{SL}\,(4) \\ \downarrow \\ \mathrm{SO}\,(6) \\ \downarrow \\ PSO\,(6) = \mathrm{PSL}\,(4,\mathbb{C}) \end{array}$$

where the arrows represent quotienting by $\mathbb{Z}/2$, even though $Z\,(\mathrm{SL}\,(4)) = \mathbb{Z}/4$. So we are quotienting by subgroups of the center to move down this tower.

This is the end of the spin group coincidences.

2. Lie's fundamental theorems

Theorem 2. *For H and G locally isomorphic, $\mathfrak{h}$ and $\mathfrak{g}$ are locally isomorphic.*

Theorem 3. *If $\mathfrak{g}$ and $\mathfrak{h}$ are locally isomorphic, then any Lie groups G and H which give rise to them are locally isomorphic, so they have the same universal cover.*

Theorem 4. *Every $\mathfrak{g}$ is the Lie algebra of some G.*

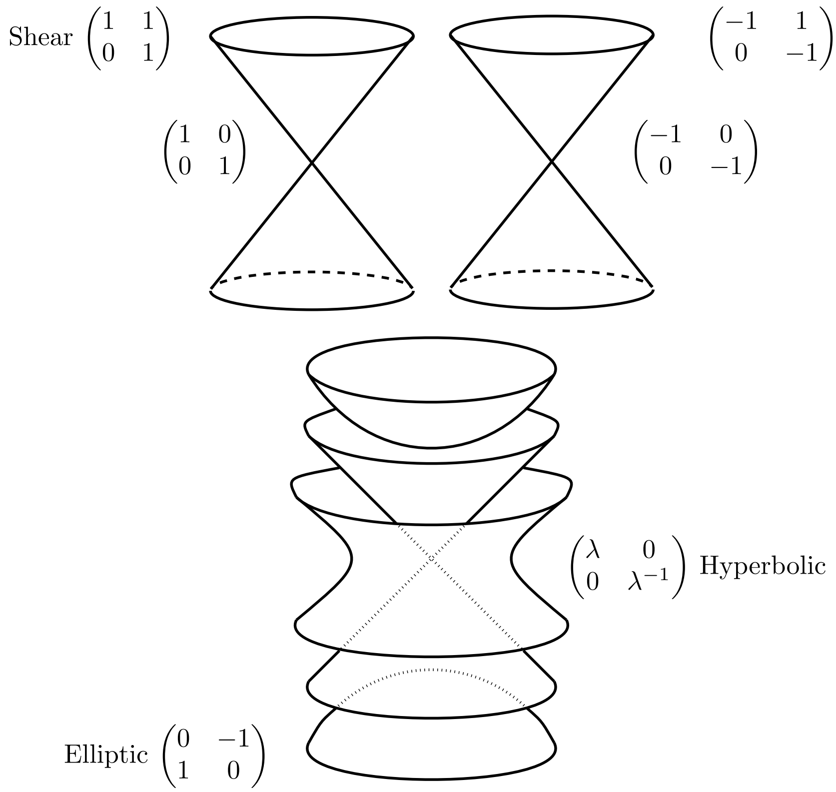

FIGURE 1. The cones which make up the orbits of the action of $\mathrm{SL}(2,\mathbb{R})$ on itself under conjugation. Note that for $\mathrm{SL}(2,\mathbb{C})$, the hyperboloid of two-sheets is not present since these matrices are then diagonalizable.

Proof. Recall Ado's theorem says that any finite dimensional Lie algebra is a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ for some n. I.e. it has a faithful representation. Note that if $Z(\mathfrak{g}) = \langle 0 \rangle$, then $\mathrm{ad} : \mathfrak{g} \hookrightarrow \mathrm{GL}(\mathfrak{g})$

Take $G \subseteq \mathrm{GL}(n,\mathbb{R})$ to be generated by all 1-parameter subgroups generated by $\gamma(t) : \mathbb{R} \to G$ with $\gamma'(0) \in \mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$.

Then there are lots of things to check.

Exercise 1. Not every element of, for example $\mathrm{SL}(2,\mathbb{R})$, is in the image of some 1-parameter subgroup.

Recall this cone picture from fig. 1 Recall $\gamma(t) = e^{tv}$, then we can take $v \in \mathfrak{sl}(2,\mathbb{R})$ and put it in Jordan form so we get matrices of the types:

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \qquad \begin{pmatrix}a & 0\\0 & -a\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & -b\\b & 0\end{pmatrix}$$

for $a \neq -a$ and $b \neq 0$. Now we go and write the exponentials of these things, and we get for example

$$\exp t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ct & -st \\ st & ct \end{pmatrix}$$

so we can get the negative identity, but we can't get the negative shears or negative hyperbolic elements.

Anyway, this is a sketch of a proof of the third theorem, or the essential surjectivity of the theorem from the beginning. □

For any Lie group G, we have the exponential map $\exp : \mathfrak{g} \to G$ defined as the map such that if $\exp(v) = \gamma(1)$ where $\gamma(T) : \mathbb{R} \to G$, then $\gamma(0) = v$.

Note for $G \subseteq \mathrm{GL}(n, \mathbb{R})$, exp is the exponential we already know.

Exercise 2. Take the differential $T(\exp) : T\mathfrak{g} \to TG$, and restrict this to $\{0\} \times \mathfrak{g}$, which gives us a map $\mathfrak{g} \to T_e G$. Show this is the identity.

Lemma 1. *The image of one-parameter subgroups contains an open neighborhood of e.*

Proof. By the exercise, $\exp : \mathfrak{g} \to G$ is a local diffeomorphism from a neighborhood of 0 to a neighborhood of the identity. □

Proof of the theorem. It remains to show the bijection on maps. We first show it is surjective. Consider a Lie algebra map $\varphi : \mathfrak{h} \to \mathfrak{g}$ and then we want a map $H \to G$ assuming H is simply connected.

Now we don't want to construct the actual map, but rather the graph of the map. We know a lot about subgroups, so we want to embed this problem in the context of constructing subgroups.

Consider the graph of φ as $\Gamma_\varphi \subseteq \mathfrak{h} \times \mathfrak{g}$. Since φ is a homomorphism, we can check that Γ_φ is a subalgebra, and now we can generate a subgroup of $H \times G$ whose Lie algebra will be this graph. Then this subgroup will be the graph of the desired map of Lie groups.

Then this is not a cover, since G is simply connected, so it's really a map, not a correspondence. □