LECTURE 8
MATH 261A

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The next midterm will probably be take home.

1. The last midterm question

Recall if we have a Lie group acting $G \circlearrowleft X$ we get an infinitesimal action, which is a map $\mathfrak{g} \to \mathrm{Vect}(X)$ which is a map of Lie algebras, so it is linear. The moment map is effectively the transpose to this map:

$$\mu : T^*X \to \mathfrak{g}^*$$

which is somehow no more or less information than $\mathfrak{g} \to \mathrm{Vect}(X)$. Explicitly, for $x \in X$ and $\xi \in T_x^*X$,

$$\mu(x, \xi)(v) = \xi(\widetilde{v}_x)$$

for $v \in \mathfrak{g}$.

Exercise 1. We know $d\mu$ is a $\mathfrak{g}^*$ valued 1-form on T^*X. Then $\omega^{-1}(d\mu)$ is now a $\mathfrak{g}^*$-valued vector field on T^*X, and now this can be evaluated at $v \in \mathfrak{g}$, so we get $\widehat{v} = \omega^{-1}(d\mu)(v)$ which is now a vector field on T^*X. Show that this vector field is tangent to the zero-section, and gives us $\tilde{v}$. I.e. show $\widehat{v}|_X = \widetilde{v}$. This is somehow recovering the infinitesimal action from the moment map and symplectic structure.

So now we want to calculate this explicitly in the examples from the midterm.

Example 1. Let $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times \circlearrowleft \mathbb{R}$ by $r \cdot x = rx$. This action generates the vector field $\tilde{v} = x\partial_x$, so $\mu(x, \xi) = \xi(x\partial_x) = x\xi$.

Example 2. Let $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times \circlearrowleft \mathbb{R}^2$ by $r \cdot (x_1, x_2) = \left(rx_1, r^{-1}x_2\right)$. Then the vector field is $x_1\partial_{x_1} - x_2\partial_{x_2}$. The moment map is just $\mu = x_1\xi_1 - x_2\xi_2$.

Example 3. Now let $G \circlearrowleft X = G$. In this case $T^*X = T^*G$ is parallelizable, so $T^*G = G \times \mathfrak{g}$, since $G \times \mathfrak{g} \xrightarrow{\sim} TG$ is just right-invariant vector fields, so it's just mapping $(g, v) \mapsto (g, \tilde{v}_g)$. This means function $\mu : T^*G \to \mathfrak{g}^*$ are just functions $G \times \mathfrak{g}^* \to \mathfrak{g}^*$.

If the action is trivial, the vector field is 0. This means $\mu = 0$.

If the action is left multiplication, then $\mu_l(g, \xi) = \xi$.

If the action is right multiplication, then

$$\mu_r(g, \xi)(v) = \mu_l(g, \xi)(\mathrm{Ad}_g v) = \mathrm{Ad}_g(\xi)(v)$$

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2. Lie algebras

2.1. **Ignoring groups.** We have now developed enough theory to see that the theory of simply connected Lie groups is the same theory as finite dimensional lie algebras. Therefore we will now ignore Lie groups and focus on Lie algebras. Not because we don't care about them, but because we understand they are equivalent.

Professor Nadler doesn't know how to answer the following:

Exercise 2. For $\mathfrak{g}$ a Lie algebra, then we can associate it to G a connected, simply connected Lie group. What is the center of G?

Solution. This solution doesn't make sense until after lecture 13 at the earliest. We claim the following:

Claim 1. The center of the simply connected compact group G associated to a Lie algebra $\mathfrak{g}$ can be identified with the dual of the finite group $\Lambda/\mathbb{Z}R$ where Λ is the weight lattice and $\mathbb{Z}R$ is the root lattice.

If $\Lambda = \mathrm{Hom}\left(H, \mathbb{C}^\times\right)$ is the weight lattice and $\mathbb{Z}R$ is the root lattice, which is given by $\mathbb{Z}$-linear combinations of the nonzero eigenvalues of the adjoint representation ad, then we can write down the dual of these things to get:

$$\Lambda^* = \{X \in \mathfrak{h} \,|\, \forall L \in \Lambda, LX \in \mathbb{Z}\}$$
$$(\mathbb{Z}R)^* = \{X \in \mathfrak{h} \,|\, \forall \alpha \in \mathbb{Z}R, \alpha X \in \mathbb{Z}\}$$

now under the exponential map, $(\mathbb{Z}R)^*$ maps onto the center of H, which is the center of G, so we just need to quotient out by the kernel of the exponential, but this is exactly Λ^*.

Recall this is important because if $G \to G/\Gamma$ is some covering, then $\Gamma \subseteq Z(G)$. So knowing the center lets us calculate the types of covers and therefore all of the groups G which might give rise to $\mathfrak{g}$.

2.2. **Fields.** From now on we will focus on representation theory of Lie algebras. We can consider Lie algebras over any field.[1] We will usually let this be $\mathbb{C}$, but first we make some comments about the general setting. For any $\mathfrak{g}/k$, we can pass to $\mathfrak{g} \otimes_k \overline{k}/\overline{k}$ where we have extended all of the bracket operations linearly.

First note that in general this operation somehow loses information. That is, many different $\mathfrak{g}/k$ might go to the same $\mathfrak{g} \otimes_k \overline{k}/\overline{k}$.

Example 4. Consider $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{so}(3,\mathbb{R}) \to \mathfrak{so}(3,\mathbb{C})$. We already saw that $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{so}(3,\mathbb{C})$. But the point is that $\mathfrak{sl}(2,\mathbb{R}) \not\simeq \mathfrak{so}(3,\mathbb{R})$. One way to see this, is that the universal cover of $\mathrm{SL}(2,\mathbb{R})$ is contractible and noncompact. Whereas the universal cover of $\mathrm{SO}(3,\mathbb{R})$ is $\mathrm{Spin}(3)$, which is compact.

In what follows, we will start with $\mathfrak{g}/\mathbb{C}$ finite dimensional. If time permits, one thing we could do is talk about what happens when k is not algebraically closed.

3. Rough classification

We won't worry too much about the details of these definitions or their relationship right now. We're more worried about getting a rough idea of what we are looking at.[2] Recall Ado's theorem, which says that $\mathfrak{g} \hookrightarrow \mathfrak{gl}(n,\mathbb{C})$ for some n. The

[1] Or rings, but we won't worry too much about this.

[2] Professor Nadler compared this to going to the Zoo. It is nice to have some idea what the big cat house is and what the reptile house is.

point being, that these always somehow come as matrices. This will be a theme throughout.

3.1. **Abelian Lie algebras.** First we might study $\mathfrak{g}$ abelian, so $[v, w] = 0$ for all $v, w \in \mathfrak{g}$. Therefore these are just complex vector spaces of some finite dimension.

Example 5. The classic example of this is just diagonal matrices $\mathbb{C}^n \subseteq \mathfrak{gl}(n, \mathbb{C})$.

For arbitrary $\mathfrak{g}$, we can always associate a certain abelian Lie algebra to $\mathfrak{g}$, called its center which is defined as

$$\{v \in \mathfrak{g} \mid \forall w \in \mathfrak{g}, [v, w] = 0\}$$

Example 6. If we consider $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, then the center is $\mathfrak{z}(\mathfrak{g}) = \{zI_n \mid z \in \mathbb{C}\}$.

3.2. **Nilpotent Lie algebras.** For any $\mathfrak{g}$ we can define the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ which consists of all linear combinations of commutators of elements of $\mathfrak{g}$. Then we can continue to take the commutator of this object with $\mathfrak{g}$ to get a series:

$$[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \qquad [[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] \qquad \cdots$$

If this process ever reaches 0, we say $\mathfrak{g}$ is nilpotent.

Example 7. The classic example is strictly upper triangular $n \times n$ matrices written $\mathfrak{n}(n, \mathbb{C})$. If we take the commutator, we lose the super diagonal, and then each commutator after that we lose another diagonal.

Theorem 1. *If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g} \subseteq \mathfrak{n}(n, \mathbb{C})$ for some n.*

Fact 1. *All subalgebras of nilpotent Lie algebras are nilpotent.*

3.3. **Solvable Lie algebras.** There are many equivalent definitions for solvable Lie algebras, but we define this to be a Lie algebra $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. This doesn't mean $\mathfrak{g}$ is nilpotent, since this condition just says that:

$$[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \qquad [[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [\mathfrak{g}, \mathfrak{g}]] \qquad \cdots$$

eventually reaches 0.

Example 8. The classic example of a solvable Lie algebra is $\mathfrak{b}(n, \mathbb{C})$ consisting of upper triangular matrices. Note that the commutator subalgebra $[\mathfrak{b}, \mathfrak{b}]$ of course yields strictly upper triangular matrices $\mathfrak{n}$, which we already saw were nilpotent.

Theorem 2. *Any solvable Lie algebra $\mathfrak{g}$ is contained $\mathfrak{g} \subseteq \mathfrak{b}(n, \mathbb{C})$ for some n.*

3.4. **Simple.** There are many formulations of simple Lie algebras, but one is that $\mathfrak{g}$ is not abelian, and has no proper non-zero ideals. The non-abelian condition is basically just to omit $\mathbb{C}$.

Example 9. The classic example is $\mathfrak{sl}(n, \mathbb{C})$. Note that $\mathfrak{gl}(n, \mathbb{C})$ is not simple, since this looks like $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}$, and therefore has two non-zero proper ideals.

Theorem 3. *If $\mathfrak{g}$ is simple, it somehow sits inside $\mathfrak{sl}(n, \mathbb{C})$.*

3.5. **Semisimple.** $\mathfrak{g}$ is semi-simple if it is a direct sum of simple Lie algebras.

Example 10. The classic example is

$$\bigoplus_i \mathfrak{sl}(n_i, \mathbb{C}) \qquad \sum_i n_i = n$$

so this is just sort of $n \times n$ block diagonal matrices where each block has trace 0.

Fact 2. $\mathfrak{g}$ *is semi-simple iff the* radical[3]*, which is the maximal solvable ideal,* $\mathfrak{rad}(\mathfrak{g}) = \langle 0 \rangle$.

Note that this also means semisimple Lie algebras have no center.

3.6. **Reductive.** The idea here is that $\mathfrak{g}$ is reductive if it is the direct sum of a semi-simple Lie algebra and an abelian Lie algebra:

$$\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{z} \ .$$

This abelian Lie algebra will of course also be the center of $\mathfrak{g}$.

Fact 3. $\mathfrak{g}$ *is reductive iff the radical* $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}$ *is just the center.*

Example 11. A classic example is $\mathfrak{gl}(n, \mathbb{C})$. In some sense the reason we define this is, well, to include this, and also to contain

$$\bigoplus_{i=1}^{l} \mathfrak{gl}(n_i, \mathbb{C})$$

which consists of block matrices with no conditions on the blocks

Proposition 1. *This contains abelian Lie algebras as well as semi-simple Lie algebras.*

Fact 4. *The sum of any two nilpotent ideals is a nilpotent ideal.*

Example 12. One might be worried about strictly upper triangular matrices, and strictly lower triangular matrices. So we can add these and take their span, but why is this not violating that the sum of nilpotent Lie ideals is a nilpotent Lie ideal? Neither of these are nilpotent ideals. They are somehow nilpotent, but not normal.

3.7. **Containments.** Note that all abelian Lie algebras are trivially nilpotent, but we also have that all nilpotent Lie algebras are solvable. Also note that trivially all simple Lie algebras are semisimple, and all semisimple Lie algebras are reductive. So being abelian and being simple are somehow two forms of "good" behavior that are just being generalized to get the other four types. In fact we have the following:

Lemma 1. *The intersection of semi-simple and solvable Lie algebras is empty.*

Proof. Let $\mathfrak{g}$ be semi-simple. Then it must be the direct sum of some simple Lie algebras $\mathfrak{g}_i$. It follows from linearity of the bracket, that

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_i [\mathfrak{g}_i, \mathfrak{g}_i]$$

but since the $\mathfrak{g}_i$ are simple, they cannot have nonzero proper ideals, so the $[\mathfrak{g}_i, \mathfrak{g}_i]$ have to be trivial or the whole algebra, but if they were trivial then $\mathfrak{g}_i$ would be abelian, which is also not allowed. Therefore $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ so $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ which prevents $[\mathfrak{g}, \mathfrak{g}]$ from being nilpotent, and therefore prevents $\mathfrak{g}$ from being solvable. □

[3] This is also called the sol-radical, and is written $\mathcal{S}(\mathfrak{g})$.

Corollary 1. *The intersection of reductive and solvable Lie algebras consists of all abelian Lie algebras.*

Along a similar vein, we have the Levi decomposition, which says that the following sequence is split-exact:

$$0 \to \mathfrak{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}^{ss} \to 0$$

This sits in contrast with the following sequence which is always exact, but not necessarily split exact:

$$0 \to \mathfrak{nil}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}^{\text{red}} \to 0$$

where $\mathfrak{nil}(\mathfrak{g})$ is the nilradical of $\mathfrak{g}$ (the maximal nilpotent ideal) and $\mathfrak{g}^{\text{red}}$ is some reductive Lie algebra.

Proposition 2. *If* $\mathfrak{nil}\mathfrak{g} \not\subseteq \mathfrak{rad}\mathfrak{g}$ *then* $\mathfrak{g}$ *is solvable.*

Proof. Take $\mathfrak{nil}\mathfrak{g} \oplus \mathfrak{rad}\mathfrak{g}$, this is solvable and strictly contains $\mathfrak{nil}\mathfrak{g}$ so it must be the whole thing and therefore must be solvable. □

4. Classification by dimension

We will classify one and two dimensional Lie algebras, and then we will focus on simple Lie algebras. In dimension 1, we have abelian $\mathbb{C}$, but every Lie algebra of dimension 1 is abelian.

In dimension 2, this can just be written $\mathfrak{g} = \mathbb{C}\langle x, y\rangle$. By definition, we know $[x,x] = [y,y] = 0$, and then we want to consider $[x,y] = -[y,x] = ax + by = z$.

$$[x, ax+by] = b(ax+by) \qquad [y, ax+by] = -a(ax+by)$$

which means $\mathbb{C}\langle ax+by\rangle \subseteq \mathfrak{g}$ is a Lie ideal, so either $a = b = 0$ or one of $a, b \neq 0$. In the first case we just have $\mathfrak{g} = \mathbb{C} \oplus \mathfrak{g}'$, but then $\mathfrak{g}'$ is of dimension 1, which means $\mathfrak{g}$ is abelian. Otherwise this is just some line, and WLOG we let $b \neq 0$. Now for $\mathfrak{g} = \mathbb{C}\langle x, z\rangle$ we get $[x, z] = bz$ so setting $x' = x/b$, we get $[x', z] = z$. In other words, any two-dimensional Lie algebra is either abelian, or has a basis $\{x', z\}$ such that $[x', z] = z$. As it turns out, this case is just:

$$\left\langle \begin{pmatrix} s & u \\ 0 & -s \end{pmatrix} \mid s, u \in \mathbb{C} \right\rangle$$

and in particular,

$$z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad x' = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

In three dimensions we encounter our first simple Lie algebra, which is $\mathfrak{sl}(2, \mathbb{C})$.