LECTURE 9

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1. Comments and corrections from last time

Recall semi-simple Lie algebras are direct sums of simple Lie algebras. Equivalently, $\mathfrak{rad}(\mathfrak{g})$, the sol-radical (maximal solvable ideal) is 0. Similarly $\mathfrak{g}$ is reductive iff $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}$ is equal to its center.

We also said something about a Lie algebra being split by its center, which is not true in general. To see this, consider the following example:

Example 1. Let $\mathfrak{g} = \mathbb{C}\langle x, y, \kappa\rangle$ such that $[x, y] = \kappa$, $[x, \kappa] = [y, \kappa] = 0$. Clearly $\mathfrak{z} = \mathbb{C}\langle\kappa\rangle$, but there is no complement to the center which is closed under the bracket.

2. Representations of $\mathfrak{sl}(2, \mathbb{C})$

2.1. **Motivation.** Recall we found that all dimension 1 Lie algebras are abelian, or just $\mathbb{C}$, and for dimension 2, we have either $\mathfrak{g} \cong \mathbb{C}^2$, or

$$\mathfrak{g} \cong \left\langle \begin{pmatrix} a & u \\ 0 & -a \end{pmatrix} \mid a, u \in \mathbb{C} \right\rangle$$

Now we move on to three dimensions.

We could play a similar game in dimension 3, but the interesting thing about 3 dimensions is that we get our first simple Lie algebra: $\mathfrak{sl}(2, \mathbb{C})$.

2.2. **Preliminaries.** We will think of $\mathfrak{sl}(2, \mathbb{C})$ as having the following basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where the brackets are:

$$[H, X] = 2X \qquad [H, Y] = 2Y \qquad [X, Y] = H$$

Exercise 1. Check these by hand.

Note $\mathfrak{sl}(2, \mathbb{C})$ by definition comes as 2×2 traceless matrices. Our generic goal here is to classify the matrix representations of Lie algebras such as $\mathfrak{sl}(2, \mathbb{C})$.

Recall:

Definition 1. A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for a vector space $V/\mathbb{C}$. Recall $\mathfrak{gl}(V) = \mathrm{End}(V)$.

We will write the category of such representations as $\mathbf{Rep}(\mathfrak{g})$. This is an abelian category, which basically means we can do all of our friendly vector space operations to these things.

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Example 2. We always have the trivial one which is just $V = \mathbb{C}$ 1-dimensional and ρ the zero map.

Example 3. We also always have the adjoint representation where $V = \mathfrak{g}$, and $\rho = \mathrm{ad}$.

We will focus on the theory of finite dimensional representations, the category of which we write as $\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})$. Note that this just means $\dim V$ is finite. This doesn't mean infinite dimensional ones aren't worth considering, but they have their own beautiful[1] story.

2.3. **Producing some representations.** We could start deductively, but we will instead start with some examples, and then see why we happen to end up with everything.

Example 4. Let $V_0 = \mathbb{C}$ be the trivial representation, so $\rho_0 = 0$. Let $V_1 = \mathbb{C}^2$, and ρ_1 be the inclusion $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{gl}(2,\mathbb{C})$.

Now we will "generate" more representations using linear algebra. One thing we can always do, is take direct sums of representations. We will write $(V_1, \rho_1) \oplus (V_2, \rho_2) = (V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ where $\rho_1 \oplus \rho_2$ acts via block matrices. This isn't really so interesting though.

We can also take the tensor product, which is very very interesting.[2]

$$(V_1, \rho_1) \otimes (V_2, \rho_2) = (V_1 \otimes V_2, \rho_1 \otimes \rho_2)$$

The definition of this map is as follows:

$$\rho_1 \otimes \rho_2 (x) = \rho_1 (x) \otimes \mathrm{id}_{V_2} + \mathrm{id}_{V_1} \otimes \rho_2 (x)$$

This definition effectively results from the Leibniz rule for differentiating the natural Lie group action on the tensor product. In particular, if we replace x with some $\gamma(t)$, we get

$$\rho_1 \otimes \rho_2 (\gamma(t)) (v_1 \otimes v_2) = \rho_1 (\gamma(t)) v_1 \otimes \rho_2 (\gamma(t)) v_2$$

and differentiating gives us the above definition.

Now let's calculate some tensor products.

Exercise 2. Tensoring with the trivial representation is the identity functor on **Rep**.

Let's tensor the standard representation (V_1, ρ_1) with itself. First let $V_1 = \mathbb{C}\langle e_1, e_2 \rangle$ where e_i is the usual basis $(1,0)$ $(0,1)$. Then

$$V_1 \otimes V_1 = \mathbb{C}\langle e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2 \rangle \ .$$

Now we calculate the action:

$$H (e_1 \otimes e_1) = (He_1) \otimes e_1 + e_1 \otimes (He_1) = e_1 \otimes e_1 + e_1 \otimes e_1 = 2 (e_1 \otimes e_1)$$

and similarly:

$$H (e_2 \otimes e_2) = -2 (e_2 \otimes e_2) \qquad H (e_1 \otimes e_2) = H (e_2 \otimes e_1) = 0$$

[1] And combinatorially complicated.

[2] Professor Nadler says that this if we remember only one thing, this should maybe be it.

Now since X annihilates e_1, we can calculate

$$X(e_1 \otimes e_1) = 0 \qquad X(e_1 \otimes e_2) = e_1 \otimes e_1$$
$$X(e_2 \otimes e_1) = e_1 \otimes e_1 \qquad X(e_2 \otimes e_2) = e_1 \otimes e_2 + e_2 \otimes e_1$$

and finally for Y, we have

$$Y(e_1 \otimes e_1) = e_2 \otimes e_1 + e_1 \otimes e_2 \qquad Y(e_1 \otimes e_2) = e_2 \otimes e_2$$
$$Y(e_2 \otimes e_1) = e_2 \otimes e_2 \qquad Y(e_2 \otimes e_2) = 0$$

If we order our basis as follows:

$$e_1 \otimes e_1 \qquad e_1 \otimes e_2 \qquad e_2 \otimes e_1 \qquad e_2 \otimes e_2$$

we can explicitly write:

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Now we want to see if $V_1 \otimes V_1$ has any nontrivial proper subrepresentation. We can see that:

$$\mathbb{C}\langle e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1 \rangle = \operatorname{Sym}^2(V_1)$$

is such a subrepresentation.

Remark 1. Recall that:

$$\operatorname{Sym}^2(V) := V \otimes V / (v \otimes w - w \otimes v)$$

for any vector space V. In general this is defined as:

$$\operatorname{Sym}^n(V) = V^{\otimes n} / (\cdots \otimes v_i \otimes v_{i+1} \otimes \cdots - \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots)$$

Does $\operatorname{Sym}^2(V_1)$ have a complement? I.e. the following sequence is exact, but is it split?

$$0 \to \operatorname{Sym}^2(V_1) \to V_1 \otimes V_1 \to V_1 \otimes V_1 / \operatorname{Sym}^2(V_1) \to 0$$

Remark 2. Recall that (over $\mathbb{C}$) we always have the following splitting:

$$V^{\otimes 2} = \operatorname{Sym}^2(V) \oplus \wedge^2 V \tag{1}$$

which consists of the symmetric tensors, and the skew-symmetric tensors.

Exercise 3. Show that the splitting in (1) respects the bracket structure for any V.

Solution. Take an arbitrary $\mathfrak{g}$ representation (V, ρ) and consider the representation $(V^{\otimes 2}, \rho^{\otimes 2})$. First consider $v \otimes w \in \operatorname{Sym}^2 V$. For any $X \in \mathfrak{g}$, we have:

$$(\rho \otimes \rho)(X)(v \otimes w) = \rho(v) \otimes w + v \otimes \rho(w)$$
$$(\rho \otimes \rho)(X)(w \otimes v) = \rho(w) \otimes v + w \otimes \rho(v)$$

But since $v \otimes w = w \otimes v$, these are actually equal, so this is in $\operatorname{Sym}^2 V$ as well. Effectively the same argument holds for $\wedge^2 V$ by linearity.

So this does indeed have a complement, and this gives us another subalgebra

$$\wedge^2(V) = \mathbb{C}\langle e_1 \otimes e_2 - e_2 \otimes e_1 \rangle$$

Now the question is, do we already know these by another name?

We know $\mathrm{Sym}^2(V_1)$ is 3-dimensional, and $\wedge^2(V_1)$ is 1-dimensional. As it turns out, we can check manually, that $\wedge^2(V_1) = V_0$ is the trivial representation, but we also have the following:

Exercise 4. Any 1-dimensional representation of a semi-simple Lie algebra is trivial.

Solution. Consider the case of a simple Lie algebra. The kernel of ρ must contain the bracket, but if $\mathfrak{g}$ is simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so ρ must be trivial. But for semi-simple, we also have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ so it holds for this case as well.

Now notice that $\mathrm{Sym}^2(V_1)$ is just the adjoint representation. In fact, we can write down an isomorphism explicitly:

$$H \mapsto -(e_1 \otimes e_2 + e_2 \otimes e_1) \qquad X \mapsto e_1 \otimes e_1 \qquad Y \mapsto -e_2 \otimes e_2$$

Now we just have to check this map respects the action of the basis elements H, X, and Y. This map clearly respects the H action since the eigenvalues match. For the X action we can calculate:

$$\begin{aligned}
\mathrm{ad}_X X &= 0 = X(-e_1 \otimes e_1) \\
\mathrm{ad}_X Y &= H \mapsto -(e_1 \otimes e_2 - e_2 \otimes e_1) = X(-e_2 \otimes e_2) \\
\mathrm{ad}_X H &= -2X \mapsto -2e_1 \otimes e_1 = X(-(e_1 \otimes e_2 + e_2 \otimes e_1))
\end{aligned}$$

Finally, we have to check the action of Y:

$$\begin{aligned}
\mathrm{ad}_Y(Y) &= 0 = Y(-e_2 \otimes e_2) \\
\mathrm{ad}_Y(X) &= -\mathrm{ad}_X(Y) = -H \mapsto e_2 \otimes e_2 + e_1 \otimes e_2 = Y(e_1 \otimes e_1) \\
\mathrm{ad}_Y(H) &= -\mathrm{ad}_H(Y) = 2Y \mapsto -2e_2 \otimes e_2 = Y(-(e_1 \otimes e_2 + e_2 \otimes e_1))
\end{aligned}$$

So this does indeed preserve the action of the basis of $\mathfrak{sl}(2, \mathbb{C})$.

2.4. General story.

Definition 2. A semi-simple category is a category such that all objects are a direct sum of irreducible objects.

Here irreducible means there are no nontrivial proper subrepresentations.

Theorem 1. *The category of finite dimensional representations, $\mathbf{Rep}_{fd}(\mathfrak{sl}(2, \mathbb{C}))$, is a semi-simple category. The irreducible representations are all of the form $V_n = \mathrm{Sym}^n(V_1)$ for $n \in \mathbb{N}$.*

Note that V_0 is trivial, V_1 is standard, V_2 is adjoint, and the rest don't have names.

Lemma 1 (Schur)**.** *Let V_1 and V_2 be irreducible representations of some Lie algebras $\mathfrak{g}$, then*

$$\mathrm{Hom}_{\mathbf{Rep}(\mathfrak{g})}(V_1, V_2) = \begin{cases} \langle 0 \rangle & V_1 \ncong V_2 \\ \mathbb{C} & V_1 \cong V_2 \end{cases}$$

Exercise 5. Prove this. This doesn't have much to do with Lie algebras and is more related to abelian categories.

Remark 3. Some aspects of this theorem generalize, for example $\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{g})$ is a semisimple category iff $\mathfrak{g}$ is semisimple.

We now explain some structure we will use in the proof next time. Our strategy for understanding all representations, is to first hope and pray it is abelian, and if not we can just look at the diagonals and build up from there. Accordingly we first focus on a subalgebra $\mathfrak{h} = \mathbb{C}\langle H\rangle \subseteq \mathfrak{sl}(2,\mathbb{C})$. This is a 1-dimensional abelian Lie algebra, and

$$\mathbf{Rep}_{\mathrm{fd}}(\mathfrak{h}) = \mathbb{C}[H]\text{-}\mathbf{Mod}_{\mathrm{fd}}$$

so every such representation is just a choice of a vector space, and a choice of endomorphism $H \circlearrowright V$.

Recall the classification of such things uses Jordan forms, so block matrices with a generalized eigenvalue along the diagonal, and 1 along the super diagonal. We can picture this as a complex plane, where we have attached a generalized eigenspace at each λ_i:

$$V = \bigoplus_{\lambda_i} V_{\lambda_i}$$

Next time, we will take $\mathfrak{sl}(2,\mathbb{C})$ and see how the other operators interact with this picture.