COUNTEREXAMPLES TO HKR IN POSITIVE CHARACTERISTIC

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1. Hochschild homology

Fix a field k (not necessarily of characteristic 0) and R a k-algebra (automatically flat for fields). Define:

$$\mathrm{HH}\left(R/k\right) \coloneqq R \overset{L}{\otimes}_{R\otimes_k R} R \tag{1}$$

Let $M \in \mathbf{Mod}\text{-}A$ and $N \in A\text{-}\mathbf{Mod}$. Recall the ordinary tensor product is defined as the co-equalizer of:

$$M \otimes A \otimes N \rightrightarrows M \otimes N \tag{2}$$

where

$$\begin{gathered} x \otimes a \otimes y \overset{d_0}{\mapsto} xa \otimes y \\ x \otimes a \otimes y \overset{d_1}{\mapsto} x \otimes ay \end{gathered} \quad . \tag{3}$$

This is equivalent to

$$M \otimes_A N = \operatorname{coker}\left(M \otimes A \otimes N \xrightarrow{d_0 - d_1} M \otimes N\right) \ . \tag{4}$$

Now we can define the derived tensor product in the analogous way:

$$M \overset{L}{\otimes}_R N \simeq \left(\dots \overset{d}{\to} M \otimes R \otimes R \to N \overset{d}{\to} M \otimes R \otimes N \overset{d}{\to} M \otimes N\right) \tag{5}$$

where

$$x \otimes a \otimes b \otimes y \mapsto xa \otimes b \otimes y - x \otimes ab \otimes y + x \otimes a \otimes by \ . \tag{6}$$

Note that

$$H_i\left(M \overset{L}{\otimes} N\right) = \mathrm{Tor}_i^R\left(M, N\right) \ . \tag{7}$$

To get to HH, set $M = N = R$, $A = R \otimes R$. Then the Hochschild complex looks like:

$$\dots \to R \otimes_R (R \otimes R) \otimes_R (R \otimes R) \otimes_R R \to R \otimes_R (R \otimes R) \otimes_R R \to R \otimes_R R \tag{8}$$

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Notes by: Jackson Van Dyke, all errors introduced are my own.

which can be rewritten as:

$$\begin{array}{ccccc} \ldots \longrightarrow R\otimes R\otimes R & \longrightarrow & R\otimes R & \xrightarrow{0} & R \\ x\otimes y\otimes z \longmapsto xy\otimes z - x\otimes yz + zx\otimes y & & & & \\ x\otimes y \mapsto xy - yx = 0 & & & & \end{array} \quad . \tag{9}$$

Now we can read off that

$$\mathrm{HH}_0(R) \simeq R \tag{10}$$

$$\mathrm{HH}_1(R) \simeq R\otimes R/\left(xy\otimes z - x\otimes yz + zx\otimes y\right) \ . \tag{11}$$

So this relation says that:

$$x\otimes yz = xy\otimes z + xz\otimes y \ . \tag{12}$$

Now recall Kähler differentials are given by:

$$\Omega^1_{Rk} \simeq \{x\,dy \mid x\,d(yz) = xy\,dz + xz\,dy\} \ . \tag{13}$$

Now these look awfully familiar, since they are the same. We get an isomorphism by sending $x\otimes y \mapsto x\,dy$.

This required no assumptions on the ring, or on the characteristic. If R/k is smooth, we get more:

$$\mathrm{HH}_i(R/k) \cong \Omega^i_{R/k} \ . \tag{14}$$

Now let k have characteristic 0. Then we can define a homomorphism

$$\begin{array}{ccc} R^{\otimes(i+1)} & \xrightarrow{\otimes\mapsto d} & \Omega^i_R \\ x_0\otimes x_1\otimes\ldots\otimes x_i & \longmapsto & \frac{1}{i!}x_0\,dx_1\wedge dX_2\wedge\ldots\wedge dx_i \end{array} \ . \tag{15}$$

If $\mathrm{ch}\,k = p \gneq 0$, and $i \geq p$ this doesn't make any sense. In characteristic 0:

$$\mathrm{HH}(R/k) = \bigoplus_i \Omega^i_R[i] \ . \tag{16}$$

2. Derived Hochschild homology

Why do we want to move to the derived world? We want to remove this smoothness hypothesis. Kähler differentials $\Omega^1_{-/k}$ over k suffer from a defect. If we have a morphism of schemes $X \xrightarrow{f} Y$ (over S) then this induces an exact sequence

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \ . \tag{17}$$

But this isn't short-exact, i.e. the first map isn't necessarily injective, unless we are in the smooth case. So let's "upgrade" Kähler differentials to the cotangent complex L_X. This is a complex of quasi-coherent sheaves on X. This is a complex such that

- $H^0\left(L_{X/S}\right) \simeq \Omega^1_{X/S}$;
- for a smooth resolution of X, there is a compatible one of L_X;
- when X/S is smooth, $L_{X/S} \simeq \Omega^1_{X/S}$;

- some kind of exactness, i.e. for $X \xrightarrow{f} Y \to S$,

(18) $$f^* L_{Y/S} \to L_{X/S} \to L_{X/Y}$$

  is a fiber sequence (derived analogue of short exact sequence); and
- it satisfies the base change property, so for a cartesian square

(19) $$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

  we have

(20) $$L_{X/Y} \simeq f^* \left(L_{X'/Y'} \right) \ .$$

The point is, that in derived algebraic geometry, trying to form Ω^1 leads to L.

2.1. **HKR for non-smooth schemes.** Now it is important that k is of characteristic 0.

(21) $$\mathrm{HH}(X) \simeq \bigoplus_{i \geq 0} \wedge^i_{\mathcal{O}_X} L_X [i]$$

(22) $$\simeq \mathrm{Sym}^*_{\mathcal{O}_X} (L_X [1]) \ .$$

Note that

(23) $$\mathrm{Sym}^i (M[1]) \simeq \left(\wedge^i M \right) [i] \ .$$

If we are in the affine case $X = \operatorname{Spec} A$ we have

(24) $$\mathrm{HH}(A) \simeq A \overset{L}{\otimes}_{A \otimes A} A$$

which in DAG corresponds geometrically to $X \times_{X \times X} X$. The multiplication $A \otimes A \to A$ corresponds to the diagonal $\Delta : X \to X \times X$, so we get a cartesian diagram:

(25) $$\begin{array}{ccc} X \times_{X \times X} X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array} \ .$$

So we have that

(26) $$\mathrm{HH}(X) \simeq \mathcal{O} (X \times_{X \times X} X)$$

is functions on the derived self-intersection of the diagonal.

Remark 1. This interpretation bears some resemblance to the Euler characteristic of a manifold. One way to calculate it is to count the self intersection of the diagonal with itself, maybe with signs.

Another way of viewing this is as follows. Consider the following homotopy pushout:

(27) $$\begin{array}{ccc} \mathrm{pt} \amalg \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & S^1 \simeq \mathrm{pt} \coprod_{\mathrm{pt} \coprod \mathrm{pt}} \mathrm{pt} \end{array} \ .$$

This is S^1 because a point is as good as an interval up to homotopy, so we can view this as gluing two intervals along their endpoints.

Now we can interpret HKR in this language. What we have figured out is that

$$\mathcal{O}(\mathcal{L}X) \simeq \mathrm{HH}(X) \ . \tag{28}$$

Then the loop space has an operation given by concatenation. So $\mathcal{L}X$ is some kind of (derived) group (scheme) over X. Now when we take the tangent complex and restrict to X, we get

$$T_{\mathcal{L}X}\big|_X \simeq T[-1]X \tag{29}$$

Now HKR is the statement that the exponential map

$$\mathrm{Spec}_X \mathrm{Sym}^*_{\mathcal{O}_X}(L_X[-1]) = T[-1]X \xrightarrow{\exp} \mathcal{L}X \tag{30}$$

is an isomorphism.

Sketch proof of HKR. Assume X is affine.[1] We have the free loop space which is the stack of maps:

$$\mathcal{L}X \simeq \underline{\mathrm{Map}}_{\mathbf{DSch}}(B\mathbb{Z}, X) \tag{31}$$
$$\simeq \underline{\mathrm{Map}}_{\mathbf{DSch}}(\mathrm{Spec}\,\mathcal{O}(B\mathbb{Z}), X) \tag{32}$$

and $\mathcal{O}(B\mathbb{Z}) \simeq C^*(S^1; k)$. In characteristic 0, these chains are formal, so

$$C^*(S^1; k) \simeq H^*(S^1; k) \tag{33}$$
$$\simeq k \oplus k[-1] \tag{34}$$
$$\simeq k[\epsilon]/(\epsilon^2) \tag{35}$$

where ϵ has degree 1, so

$$\mathcal{L}X \simeq \underline{\mathrm{Map}}_{\mathbf{DSch}}\left(\mathrm{Spec}\left(k[\epsilon]/(\epsilon^2)\right), X\right) \simeq T[-1]X \ . \tag{36}$$

If ϵ had degree 0, we would get the 0-shifted tangent complex. □

Remark 2 (Sam). Embed the additive group $\mathbb{Z}$ to the additive group scheme $\mathbb{G}_a$, then we have the map from the formal group (formal completion at the origin to $\mathbb{G}_a$), so we get:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{G}_a \\ & & \uparrow \\ & & \hat{\mathbb{G}}_a \end{array} \ . \tag{37}$$

Then we can take B of this to get

$$\begin{array}{ccc} B\mathbb{Z} & \longrightarrow & B\mathbb{G}_a \\ & & \uparrow \\ & & B\hat{\mathbb{G}}_a \end{array} \tag{38}$$

[1]There is a descent procedure which reduces the general question to this.

Claim 1. For X a variety over a field of characteristic 0 we have:

$$\begin{array}{ccc} \mathrm{Map}\left(B\mathbb{Z}, X\right) & \xleftarrow[\simeq]{} & \mathrm{Map}\left(B\mathbb{G}_a, X\right) \\ & & \downarrow \simeq \\ & & \mathrm{Map}\left(B\hat{\mathbb{G}}_a, X\right) \end{array} \tag{39}$$

So we have shown that

$$X^{S^1} \simeq T\left[-1\right] X \tag{40}$$

by showing that they are both isomorphic to a third thing.[2]

The map $\mathbb{Z} \to \mathbb{G}_a$ would factor through $\mathbb{Z}/p$ in characteristic p, so it is unlikely to be true.

3. Hochschild homology in positive characteristic

To be continued...

[2]This was a lesson Dennis Gaitsgory taught Sam.