OVERVIEW

ROK GREGORIC

Contents

The seminar website is here. We will follow Lurie's notes.

The idea is to start with some collection of spaces, and attach some kind of algebraic invariant to them. Then we might hope that not too much information is lost so we can understand these spaces via the easier-to-understand algebraic counterparts. One example of this is a cohomology theory, which is a functor

$$E^* \colon \mathbf{Spaces}^{\mathrm{op}} \to \mathbf{Ab} \tag{1}$$

satisfying the Eilenberg-Steenrod axioms. To a cohomology theory, we can consider $E^*(\mathrm{pt})$, the *coefficients* of E.

Theorem 1 (Eilenberg-Steenrod)**.** *If*

$$E^i(pt) = \begin{cases} A & i = 0 \\ 0 & i \neq 0 \end{cases} \tag{2}$$

then E is ordinary *cohomology:*

$$E^*(X) \cong H^*(X; A) \ . \tag{3}$$

Otherwise we say E^* is *extraordinary*.

Example 1.

$$KU^0(X) = \{\mathbb{C}\text{-vector bundles on } X\} / \tag{4}$$

which canonically extends to a cohomology theory KU, called *complex K-theory*, with coefficients:

$$KU^*(\mathrm{pt}) = \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases} , \tag{5}$$

so this is an extraordinary cohomology theory.

As it turns out, cohomology theories are equivalent to spectra, which can be thought of as "jazzed up" spaces.

We will assume E is *complex-oriented*, i.e.

$$E^*(\mathbb{CP}^\infty) \cong E^*(\mathrm{pt})[\![t]\!] \tag{6}$$

where t has degree $|t| = 2$.

Date: January 28, 2021.

Notes by: Jackson Van Dyke. Please email me at jacksontvandyke@utexas.edu with any corrections or concerns.

Example 2. Ordinary cohomology, which as a spectrum we will denote $E = H\mathbb{Z}$, is complex oriented since:

$$H^*\left(\mathbb{CP}^\infty; \mathbb{Z}\right) \simeq \mathbb{Z}\left[\!\left[t\right]\!\right] \ . \tag{7}$$

One reason to assume this is that this allows us to develop a theory of characteristic classes. The idea is that for $\mathcal{L}$ a line bundle on X, we would like to attach a class $c_1(\mathcal{L}) \in H^2(X;\mathbb{Z})$. A line is characterized by a function $f\colon X \to \mathbb{CP}^\infty$. Explicitly

$$\mathcal{L} \cong f^*\mathcal{O}(1) \ . \tag{8}$$

Using (6), we can define

$$c_1(\mathcal{L}) \coloneqq f^*(t) \in H^2(X;\mathbb{Z}) \ . \tag{9}$$

So for general complex oriented E we can define

$$c_1^E(\mathcal{L}) \in E^2(X) \ . \tag{10}$$

In the integral case, we have

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}') \ , \tag{11}$$

but this fails for the generalized classes c_1^E. There is, however, a weaker version:

$$c_1^E(\mathcal{L} \otimes \mathcal{L}') = f\left(c_1^E(\mathcal{L}), c_1^E(\mathcal{L}')\right) \tag{12}$$

for some power series:

$$f \in E^*(\mathrm{pt})\left[\!\left[u,v\right]\!\right] \cong E^*\left(\mathbb{CP}^\infty \times \mathbb{CP}^\infty\right) \ . \tag{13}$$

f satisfies some properties, which ultimately come from properties which line bundles satisfy. One thing which line bundles satisfy the property that

$$\mathcal{L} \otimes \text{trivial} \simeq \mathcal{L} \simeq \text{trivial} \otimes \mathcal{L} \tag{14}$$

which implies

$$f(u,0) = u = f(0,u) \ . \tag{15}$$

It also satisfies

$$\mathcal{L} \otimes \mathcal{L}' \simeq \mathcal{L}' \otimes \mathcal{L} \tag{16}$$

which implies

$$f(u,v) = f(v,u) \ . \tag{17}$$

Finally it satisfies

$$(\mathcal{L} \otimes \mathcal{L}') \otimes \mathcal{L}'' \simeq \mathcal{L} \otimes (\mathcal{L}' \otimes \mathcal{L}'') \tag{18}$$

which implies

$$f(f(u,v), w) = f(u, f(v,w)) \ . \tag{19}$$

When f satisfies properties (15), (17), and (19) we say that f is a *formal group law*. This defines a group operation

$$u + v \coloneqq f(u,v) \tag{20}$$

on $E^*(\mathrm{pt})\left[\!\left[t\right]\!\right]$. This can be thought of as functions on the formal affine line $\widehat{\mathbb{A}^1}$. The function f depends on t, and changes of t correspond to changes of orientation, or a change of coordinates on the formal affine line. But the group structure and the formal affine line itself do not depend on t.

The upshot is that we can pass from a complex oriented cohomology theory E to a formal group

$$E^*(\mathbb{CP}^\infty) \simeq E^*(\mathrm{pt})[\![t]\!] \ . \tag{21}$$

The punchline will eventually be that this is a very good invariant.

Given ring R, we can consider the collection of formal group laws $\mathrm{FGL}(R)$. So elements of this look like

$$f(u,v) \in R[\![u,v]\!] \ . \tag{22}$$

Example 3. For $E = H\mathbb{Z}$, $f(u,v) = u + v$ is a formal group law.

Example 4. For $E = KU$, $f(u,v) = e + v - uv$ is a formal group law.

As it turns out, the functor FGL is representable. I.e. there is a ring L, the Lazard ring, and a formal group law $f_{\mathrm{un}} \in \mathrm{FGL}(L)$ such that

$$\mathrm{FGL}(R) \leftrightarrow \{\varphi \in \mathrm{Hom}_{\mathbf{Ring}}(L,R) \,|\, f = \varphi^* f_{\mathrm{un}}\} \ . \tag{23}$$

Then

$$\mathrm{FGL} = \mathrm{Spec}(L) \cong \mathbb{A}^\infty \ . \tag{24}$$

Consider complex cohomology MU. Quillen's theorem says that

$$f_{MU}^{\mathrm{FGL}} = f^{\mathrm{un}} \ . \tag{25}$$

Then

$$MU^*(\mathrm{pt}) \cong L \ . \tag{26}$$

Since $f \in \mathrm{FGL}(R)$ is the same as a map $L \to R$, we can try to create a cohomology theory

$$E^*(X) = MU^*(X) \otimes_L R \ , \tag{27}$$

but this doesn't necessarily satisfy all of the Mayer-Vietoris axioms.

As it turns out, moduli of cohomology theories are encoded by the moduli stack $\mathcal{M}_{FG}$:

$$\text{moduli of coh. theories} \rightsquigarrow \mathcal{M}_{FG} \ . \tag{28}$$

$\mathcal{M}_{FG}$ has a canonical filtration by something called "height" and this has a corresponding filtration on the LHS, which turns out to be a helpful realization for calculating the homotopy groups of spheres.