The Drinfeld center and topological symmetries

Jackson Van Dyke
UT Austin

April 24, 2023
Consider a 3-dimensional (framed) TQFT:

\[
\text{Bord}_3^{fr} \xrightarrow{F} \text{Alg}_1(\text{Cat})
\]

The point goes to some monoidal category:

\[
F(\bullet) = (\mathcal{C}, \ast)
\]

The interval goes to the identity bimodule:

\[
F(\bullet \rightarrow \bullet) = c\mathcal{C}_\mathcal{C}
\]

The circle will be sent to some category:

\[
F\left(\begin{array}{c}
\bullet \\
\quad \\
\end{array}\right) = ? \in \text{Cat} \cong \text{End}_{\text{Alg}_1(\text{Cat})}(1)
\]
Example of a 3-dimensional TQFT

- Before we identify $F(S^1)$, let’s consider an example.
- Consider the category of vector spaces graded by a finite abelian group $G$:

$$\mathcal{C} = \textbf{Vect}[G].$$  \hspace{1cm} (1)

The simple objects are given by ‘skyscrapers’ $\mathbb{C}_g$ for $g \in G$.
- This has a tensor product given by convolution $\ast$. On simple objects $\mathbb{C}_g$ it is simply:

$$\mathbb{C}_g \ast \mathbb{C}_h = \mathbb{C}_{gh}. \hspace{1cm} (2)$$

- The TQFT $F$ associated to this particular fusion category is finite gauge theory with gauge group $G$. If $\tau$ is a cocycle for a class in $H^3(BG, \mathbb{C}^\times)$, we can define a nontrivial associator for $\textbf{Vect}[G]$ using $\tau$, resulting in Dijkgraaf-Witten theory for $(G, \tau)$. 

The assignment to the circle

From this picture, we have an action:

\[ F ( \bigcirc ) \sim \text{End}_{\text{\textit{C-bimod}}} (cCc) \].

This map turns out to be an equivalence. [DSS20, Section 3.2.2]

This has more structure, e.g. a product map given by composition, which we will discuss in a couple of slides.

But first, let’s notice: this is the Drinfeld center!
The Drinfeld center

- The *Drinfeld center* of a tensor category \((\mathcal{C}, \star)\) is:

\[
\mathcal{Z}(\mathcal{C}) = \text{End}_{\mathcal{C}\text{-bimod}}(\mathcal{C}\mathcal{C}) = \text{End}_{\mathcal{C} \otimes \mathcal{C}^{\text{op}}}(\mathcal{C}) .
\]

- So the upshot of the previous slide is:

\[
F(\bigcirc) \cong \mathcal{Z}(F(\bullet) = \mathcal{C}) .
\]

- The Drinfeld center has a more concrete description: consider the category with objects given by pairs \((X, \sigma_X)\), where \(X\) is an object of \(\mathcal{C}\), and \(\sigma_X\) is a natural transformation:

\[
\sigma_X : X \otimes (-) \to (-) \otimes X .
\]

The morphisms are (appropriately compatible) morphisms in \(\mathcal{C}\). See [Eti+15, Prop. 7.13.8] for the equivalence between the two definitions.
We have seen that if $F(\bullet) = \mathcal{C}$, then:

$$F(\bigcirc) \cong \mathcal{Z}(\mathcal{C}) = \text{End}_{\mathcal{C} \otimes \mathcal{C}^{\text{op}}}(\mathcal{C}) .$$

This is naturally a monoidal category: composition of endomorphisms is the same as the multiplication map induced by the pair of pants bordism:

$$F \left( \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \right) : F(\bigcirc) \otimes F(\bigcirc) \to F(\bigcirc) .$$

In fact there is even a braiding, induced by moving one of the “legs” around the other.
Recall our finite abelian gauge theory example: \( F(\bullet) = \text{Vect}[G] \).

The Drinfeld center of this fusion category turns out to be:

\[
\mathcal{Z}(\text{Vect}[G]) \cong \text{Vect}[G \oplus G^\vee],
\]

where \( G^\vee = \text{Hom}(G, \mathbb{C}^\times) \) is the character dual.

The monoidal structure is still convolution, and the braiding is given on simple objects by:

\[
\mathbb{C}(g,\chi) \ast \mathbb{C}(h,\omega) \xrightarrow{\chi(h)\omega(g) \text{id}} \mathbb{C}(h,\omega) \ast \mathbb{C}(g,\chi). \tag{3}
\]
What does any of this have to do with symmetry?

- A boundary theory \(1 \to F\) should be thought of as a “2-dimensional theory with a \(C\)-action”: usually a 2d theory sends the point to a category, and now it is sent to a \(C\)-module category.

- In the finite gauge theory example, where \(C = \text{Vect}[G]\), a \(C\)-module structure on a category can be thought of as a categorical action of \(G\) itself.
Boundary theories for the 3-dimensional theory

- \( F : \bullet \mapsto \mathcal{C} \) is the *Turaev-Viro (TV) theory associated to* \( \mathcal{C} \).
- The theory sending the circle to a particular braided category \( \mathcal{B} \) is the *Reshetikhin-Turaev (RT) theory associated to* \( \mathcal{B} \).
- So the RT theory for \( \mathcal{Z}(\mathcal{C}) \) agrees with the TV theory for \( \mathcal{C} \), but not all RT theories may be of this form.

**Theorem ([FT21])**

*An RT theory admits a nonzero boundary theory if and only if it is a TV theory.*
The Drinfeld center is a braided category, and turns out to be sufficiently dualizable \cite{BJS21} to define a 4-dimensional TQFT:

\[ \alpha : \text{Bord}_4 \ni \bullet \mapsto \mathcal{Z}(\mathcal{C}) \in \text{Alg}_2(\text{Cat}) . \]

This is the \textit{Crane-Yetter (CY) theory} associated to the braided category \( \mathcal{Z}(\mathcal{C}) \).
The Drinfeld center of a tensor category manifestly acts on the original tensor category, since we have a forgetful functor

\[ \mathcal{Z}(\mathcal{C}) \to \mathcal{C}. \]

In terms of the theories, this means that \( F \) can be upgraded to a boundary condition:

\[ \tilde{F} : 1 \to \alpha. \]

The value of \( \tilde{F} \) on the point is \( \mathcal{C} \) as a \( \mathcal{Z}(\mathcal{C}) \)-module.

More specifically there is a \( (\alpha, \rho) \)-module structure on \( F \), in the sense of [FMT22].
Back to the example

- Consider our running example: if $F$ is $G$-gauge theory, then the theory
  \[ \alpha : \text{pt} \mapsto Z(\text{Vect}[G]) \cong \text{Vect}[G \oplus G^\vee], \]
  can be described as the quantization (in the sense of [Fre+10]) of the groupoid $B^2(G \oplus G^\vee)$, twisted by a cocycle for the class
  \[ \text{ev} \in \text{Hom}(G \oplus G^\vee, \mathbb{C}^\times) \cong H^4\left(B^2(G \oplus G^\vee), \mathbb{C}^\times\right). \]

- Lagrangian subgroups $L$ of $(G \oplus G^\vee, \text{ev})$ now give rise to boundary theories $1 \to \alpha$, by quantizing the correspondence (as in [FMT22]):
  \[ \bullet \leftarrow B^2L \to B^2(G \oplus G^\vee). \]

- The boundary theories corresponding to $L = G$ and $G^\vee$ are related by an “integral transform”.\(^1\)

\(^1\)This is studied in my upcoming work “A twice-categorified finite Fourier transform”. 

Jackson Van Dyke  UT Austin  Drinfeld center and top'l symmetries  April 24, 2023  12 / 15
We might wonder if the theory $F$ can be upgraded to have an action of the automorphisms of the center or, more concretely, of the group $O(G \oplus G^\vee)$.

At the level of the fusion category itself, this is answered by [ENO10]: this group acts if and only if specific obstructions are trivializable, and an action is determined by a trivialization.

Passing the obstruction theory from [ENO10] through the “quantization of groupoids” formalism developed in [Fre+10] yields a collection of anomaly theories and symmetry theories for the theory associated to the fusion category we started with.

This is spelled out in my upcoming work.²

²“Equivariance and anomalies of finite topological gauge theory”
Summary:

TV for $\mathcal{C}$:

1. governs $\mathcal{C}$-symmetry,
2. is equivalent to RT for $\mathcal{Z}(\mathcal{C})$, and
3. is (can be upgraded to) a boundary theory for CY for $\mathcal{Z}(\mathcal{C})$. 
References


