# FULLY DUALIZABLE OBJECTS

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Contents



## 1. Dualizable objects

1.1. **Vector spaces.** Let

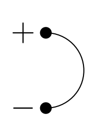
$$Z\colon \mathbf{Cob}\,(1) \to \mathbf{Vect}_k$$

be a 1-dimensional topological field theory. This sends every closed, oriented 0-manifold to a vector space. Every such manifold is a disjoint union of points, either positively or negatively oriented,[1] written $+$ and $-$ respectively.

Therefore, for some $k$ vector space $V$, we have:

$$Z\,(+) = V\ .$$

Since the following map:

$Z$ ( 

) $: Z\,(+) \otimes Z\,(-) \to k$

is a perfect pairing, the assignment to $Q$ is isomorphic to the linear dual $V^\vee$:

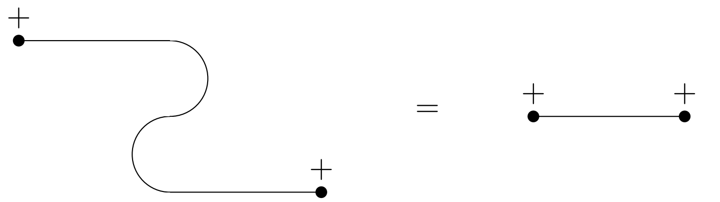
$$Z\,(-) = V^\vee = Z\,(+)^\vee\ .$$

*Date*: September 23, 2020.

[1]There are only two oriented manifolds which consist of only a single point up to orientation-preserving diffeomorphism.

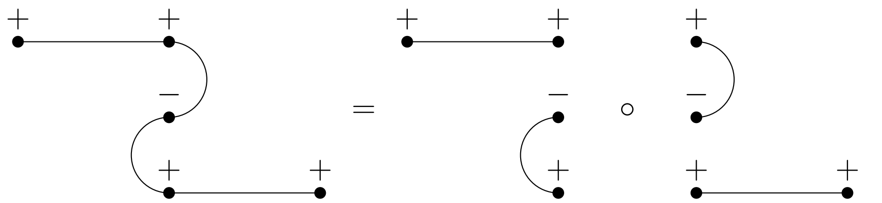
Recall specifying finite-dimensional $V$ is sufficient to determine $Z$. The main point being that $Z(S^1) = \dim V$.

1.2. **Rewriting finite dimensionality.** We want to rephrase the notion of a vector space having finite dimension in categorical terms. Consider the "Mark of Zorro" as a bordism between $+$ and itself:

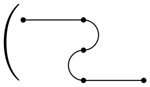



Notice we can "decompose" this bordism as follows:

(1)




$Z$ sends this to a morphism:

$Z\Bigg($  $\Bigg) : V \to V$

which, because of the above decomposition, factors as:

(2)
$$\begin{tikzcd} V \simeq V \otimes k \arrow[r, "\mathrm{id}_V \otimes \mathrm{coev}_V"] \arrow[rr, bend right, "\mathrm{id}_V"'] & V \otimes V^\vee \otimes V \arrow[r, "\mathrm{ev}_V \otimes \mathrm{id}_V"] & k \otimes V \simeq V \end{tikzcd} .$$

Similarly, reversing the orientation of (1) gives us

(3)
$$\begin{tikzcd} V^\vee \simeq k \otimes V^\vee \arrow[r, "\mathrm{coev}_V \otimes \mathrm{id}_{V^\vee}"] \arrow[rr, bend right, "\mathrm{id}_{V^\vee}"'] & V^\vee \otimes V \otimes V^\vee \arrow[r, "\mathrm{id}_{V^\vee} \otimes \mathrm{ev}_V"] & V^\vee \otimes k \simeq V^\vee \end{tikzcd} .$$

**Proposition 1.** *A $k$-vector space $V$ is finite-dimensional if and only if there is some other $k$-vector space $V^\vee$ and linear maps*

$$\mathrm{ev}\colon V \otimes V^\vee \to k \qquad\qquad \mathrm{coev}\colon k \to V^\vee \otimes V$$

*such that (2) and (3) commute.*

*Remark* 1. This is really a special case of Proposition 2.

*Proof sketch.* Both are equivalent to ev determining a map

$$\mathrm{Hom}(W, W' \otimes V) \longrightarrow \mathrm{Hom}(W \otimes V^\vee, W' \otimes V \otimes V^\vee) \xrightarrow{\circ\, \mathrm{ev}} \mathrm{Hom}(W \otimes V^\vee, W')$$

(for any $W, W' \in \mathbf{Vect}_k$) which has inverse induced by coev:

$$\mathrm{Hom}(W \to V^\vee, W') \xrightarrow{\mathrm{coev}} \mathrm{Hom}(W \otimes V^\vee \otimes V, W' \otimes V) \xrightarrow{\mathrm{coev}\,\circ} \mathrm{Hom}(W, W' \otimes V) .$$

$\square$

### 1.3. 0-dualizability.

Proposition 1 motivates the following definition.

**Definition 1.** Let $\mathbf{C}$ be a monoidal category.[2] Let $V \in \mathbf{C}$. An object $V^\vee \in \mathbf{C}$ is a *right dual of* $V$ if there are maps

$$\mathrm{ev}_V : V \otimes V^\vee \to \mathbf{1} \qquad \qquad \mathrm{coev}_V : \mathbf{1} \to V^\vee \otimes V$$

such that the zig-zag identities (2) and (3) hold. In this case, $V$ is the *left dual* of $V^\vee$.

*Remark* 2. If $\mathbf{C}$ is a symmetric monoidal category then the notions of left and right dual coincide. If $\mathbf{C}$ also has an internal hom, then the dual of $V$ can always be taken to be $V^\vee = \hom(V, \mathbf{1})$. Many of the usual facts about dual vector spaces hold in this general setting.

**Fact 1.** *If a dual $V^\vee$ and the corresponding morphisms* ev *and* coev *exist, then they are unique up to unique isomorphism.*

*Remark* 3. This is really a special case of fact 2.

*Proof sketch.* Let $D_1$ and $D_2$ be two different duals of and object $V$. Write the (co)evaluation maps as $\mathrm{ev}_i$ and $\mathrm{coev}_i$ for $i = 1, 2$. Now consider the maps

$$D_1 \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_2} D_1 \otimes V \otimes D_2 \xrightarrow{\mathrm{ev}_1 \otimes \mathrm{id}} D_2$$

$$D_2 \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_1} D_2 \otimes V \otimes D_1 \xrightarrow{\mathrm{ev}_2 \otimes \mathrm{id}} D_1 \ .$$

The fact that these are mutually inverse follows from the definition of a dual object. See [EGNO15, Proposition 2.10.5] for a full proof. ☐

1.3.1. *Towards higher dualizability.* 1-dualizability turns out to be 0-dualizability plus an extra condition asking the maps ev and coev to have "duals". So we need to answer the following question.

**Question 1.** What is the correct notion of dualizability for a morphism?

## 2. Dualizable 1-morphisms

The correct notion of a "dual pair" of 1-morphisms turns out to be that of an adjoint pair.

### 2.1. Cat.

We recall the usual definition of an adjunction between functors. We will eventually generalize it to a relationship between 1-morphisms in any 2-category.

The 2-category of (small) categories, **Cat**, has

- objects given by (small) categories,
- 1-morphisms given by Functors, and
- 2-morphisms given by natural transformations.

[2]A category equipped with a tensor product $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is unital and associative up to coherent isomorphism.

2.2. **Rewriting adjunction.** Let $\mathbf{X}, \mathbf{Y} \in \mathbf{Cat}$. Consider two functors $f\colon \mathbf{X} \to \mathbf{Y}$ and $g\colon \mathbf{Y} \to \mathbf{X}$. An *adjunction* between $f$ and $g$ is a collection of bijections:

$$\varphi_{a,b}\colon \operatorname{Hom}_{\mathbf{Y}}(f(a), b) \simeq \operatorname{Hom}_{\mathbf{X}}(a, g(b)) \ ,$$

which depend functorially on $a \in \mathbf{X}$ and $b \in \mathbf{Y}$. Then we say $f$ is *left adjoint* to $g$ and that $g$ is *right adjoint* to $f$. This is sometimes written simply as $f \dashv g$.

**Proposition 2.** *$f$ and $g$ form an adjunction if and only if there are natural transformations*

$$u\colon \mathrm{id}_{\mathbf{X}} \to g \circ f \qquad\qquad v\colon f \circ g \to \mathrm{id}_{\mathbf{Y}}$$

*such that the following diagrams commute:*

$$\begin{tikzcd} f \simeq f \circ \mathrm{id}_{\mathbf{X}} \arrow[r, "\mathrm{id} \times u"] \arrow[rr, bend right, "\mathrm{id}_f"'] & f \circ g \circ f \arrow[r, "v \times \mathrm{id}"] & \mathrm{id}_{\mathbf{Y}} \circ f \simeq f \end{tikzcd}$$

$$\begin{tikzcd} g \simeq \mathrm{id}_{\mathbf{X}} \circ g \arrow[r, "u \times \mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}_g"'] & g \circ f \circ g \arrow[r, "v \times \mathrm{id}"] & g \circ \mathrm{id}_{\mathbf{Y}} \simeq g \end{tikzcd} \tag{4}$$

.

*Proof sketch.* ($\Longleftarrow$): Suppose we have such natural transformations $u$ and $v$. Then for any $a \in \mathbf{X}$ and $b \in \mathbf{Y}$ we get the bijections:

$$\operatorname{Hom}(f(a), b) \to \operatorname{Hom}(g \circ f(a), g(b)) \xrightarrow{\circ u_x} \operatorname{Hom}(a, g(b))$$
$$\operatorname{Hom}(a, g(b)) \to \operatorname{Hom}(f(a), f \circ g(b)) \xrightarrow{v_y \circ} \operatorname{Hom}(f(a), b) \ .$$

They are mutually inverse because they satisfy (4).

($\implies$): Let $\{\varphi_{a,b}\}_{a \in \mathbf{X}, y \in \mathbf{Y}}$ be an adjunction between $F$ and $G$. Setting $y = f(a)$, we get an isomorphism

$$\varphi_{a,f(a)}\colon \operatorname{Hom}(f(a), f(a)) \xrightarrow{\sim} \operatorname{Hom}(a, g \circ f(a))$$

which sends $\mathrm{id}_{f(a)}$ to some morphism

$$u_a\colon a \to g \circ f(a)$$

which together form the natural transformation $u$.

Setting $a = g(b)$, we get an isomorphism

$$\varphi_{g(b),b}\colon \operatorname{Hom}(f \circ g(b), b) \to \operatorname{Hom}(g(b), g(b))$$

so there is some morphism

$$v_b\colon f \circ g(b) \to b$$

which goes to $\mathrm{id}_{g(b)}$ under this isomorphism. These comprise the natural transformation $v$.

$u$ and $v$ satisfy (4) because the morphisms $\varphi_{x,y}$ defining the adjunction are isomorphisms. $\square$

2.3. **Adjunction in a 2-category.** Proposition 2 motivates the following definition.

**Definition 2.** Let $\mathbf{C}$ be an arbitrary 2-category. Suppose we are given a pair of objects $X, Y \in \mathbf{C}$ and a pair of 1-morphisms $f\colon X \to Y$ and $g\colon Y \to X$. A 2-morphism

$$u\colon \mathrm{id}_X \to g \circ f$$

is the *unit of an adjunction* between $f$ and $g$ if there exists another 2-morphism

$$v\colon f \circ g \to \mathrm{id}_Y$$

such that the diagrams in (4) commute.

*Remark* 4. Notice that (4) is effectively the same condition as (2) and (3). The difference being that we replaced $V$ with $f$, and $V^\vee$ with $g$. Furthermore, the 2-morphisms $u$ and $v$ are playing the role that the 1-morphism ev and coev were playing before.

**Definition 3.** Let $\mathbf{C}$ be a 2-category. $\mathbf{C}$ *has adjoints for* 1*-morphisms* if the following conditions are satisfied:

(1) For every 1-morphism $f\colon X \to Y$ in $\mathbf{C}$, there exists another 1-morphism $g\colon Y \to X$ and a 2-morphism $u\colon \mathrm{id}_X \to g \circ f$ which is the unit of an adjunction.
(2) For every 1-morphism $g\colon Y \to X$ in $\mathbf{C}$, there exists another 1-morphism $f\colon X \to Y$ and a 2-morphism $u\colon \mathrm{id}_X \to g \circ f$ which is the unit of an adjunction.

**Fact 2.** *If it exists, a left or right adjoint is unique up to unique isomorphism.*

*Proof sketch.* The proof is effectively the same as fact 1. If $g_1$ and $g_2$ are two right adjoints of $f$, then the isomorphisms are given by the following mutually inverse morphisms:

$$\begin{gathered} g_1 \xrightarrow{u_2 \times \mathrm{id}} g_2 \circ f \circ g_1 \xrightarrow{\mathrm{id} \times v_1} g_2 \\ . \\ g_2 \xrightarrow{u_1 \times \mathrm{id}} g_1 \circ f \circ g_2 \xrightarrow{\mathrm{id} \times v_2} g_1 \end{gathered}$$

□

## 3. Full dualizability

### 3.1. Definition.

**Definition 4.** Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category. An object $X \in \mathcal{C}$ is 0*-dualizable* if it admits a dual as an object of the homotopy category $h\mathcal{C}$, defined as follows.

- The objects are the objects of $\mathcal{C}$.
- The morphisms are isomorphism classes of objects of $\mathrm{Map}_{\mathcal{C}}(X, Y)$.

**Definition 5.** Let $\mathcal{C}$ be an $(\infty, n)$-category for $n \geq 2$. The *homotopy* 2*-category* for $\mathcal{C}$, written $h_2\mathcal{C}$, is defined as the following category.

- The objects of $h_2\mathcal{C}$ are the objects of $\mathcal{C}$.
- The 1-morphisms of $h_2\mathcal{C}$ are the 1-morphisms of $\mathcal{C}$.
- For two objects $X, Y \in \mathcal{C}$ and a pair of 1-morphisms $f, g\colon X \to Y$, we define a 2-morphism from $f$ to $g$ in $h_2\mathcal{C}$ to be an isomorphism class of 2-morphisms from $f$ to $g$ in $\mathcal{C}$.
- $\mathcal{C}$ *admits adjoints for* 1*-morphisms* if $h_2\mathcal{C}$ admits adjoints for 1-morphisms.

- $\mathcal{C}$ *admits adjoints for $k$-morphisms* if, for all $X, Y \in \mathcal{C}$, the $(\infty, n-1)$-category $\mathrm{Map}_{\mathcal{C}}(X, Y)$ admits adjoints for $(k-1)$-morphisms.
- $\mathcal{C}$ *has adjoints* if $\mathcal{C}$ admits adjoints for $k$-morphisms for all $0 < k < n$.

**Definition 6.** Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category. $\mathcal{C}$ *has duals* if $\mathcal{C}$ has duals for objects, and $\mathcal{C}$ has adjoints.

**Claim 1.** Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category. There exists another monoidal $(\infty, n)$-category $\mathcal{C}^{\mathrm{fd}}$ and a monoidal functor $i \colon \mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$ with the following properties.

(1) $\mathcal{C}^{\mathrm{fd}}$ has duals.
(2) For any monoidal $(\infty, n)$-category $\mathcal{D}$ with duals and any monoidal functor $F \colon \mathcal{D} \to \mathcal{C}$, there exists a monoidal functor $f \colon \mathcal{D} \to \mathcal{C}^{\mathrm{fd}}$ and an isomorphism $F \simeq i \circ f$; moreover, $f$ is uniquely determined up to isomorphism.

**Definition 7.** We say that $X \in \mathcal{C}$ is *fully-dualizable* if it is in the image of $\mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$.

*Remark* 5. Passing from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{fd}}$ amounts to removing any $k$-morphisms without left and right adjoints (and all objects without duals).

3.2. **Examples.**

**Example 1.** The fully dualizable objects of the $(\infty, 1)$-category $\mathcal{C} = \mathbf{Vect}(k)$ are exactly the finite-dimensional ones.

**Example 2.** For $\mathcal{C}$ a monoidal $(\infty, 1)$-category, $\mathcal{C}^{\mathrm{fd}}$ is the subcategory of $\mathcal{C}$ spanned by the 0-dualizable (in the sense of definition 4) objects of $\mathcal{C}$.

**Example 3.** The assignment to a circle should be Hochschild of the assignment to a point:

$$Z\left(S^1\right) = \mathrm{HH}^*\left(Z\left(+\right)\right) \ .$$

Therefore, in dimension 2, this is the Frobenius algebra from the usual classification of 2-dimensional TFT's by their assignment to $S^1$.

If we start with a Frobenius algebra assigned to $S^1$, the assignment to $+$ is the category of finitely-generated modules over the algebra. See [Hes18] (arXiv link) for more on this equivalence.

**Example 4.** For each $n \geq 0$, the symmetric monoidal $(\infty, n)$-category $\mathbf{Bord}_n$ has duals. Let $f \colon X \to Y$ be a $k$-morphism, i.e. an oriented $k$-manifold $M$, with boundary:

$$\partial M = \overline{X} \sqcup_{\partial X = \partial Y} Y \ .$$

Then $\overline{M}$ can be interpreted as a $k$-morphism $Y \to X$, which is both right and left adjoint to $f$.

3.3. **$k$-adjoints as 1-adjoints.** First we realize objects of a category as 1-morphisms in such a way that duals are 1-adjoints. Then we realize $k$-morphisms as 1-morphisms in such a way that $k$-adjoints are 1-adjoints. So there is a sense in which we were just asking for a 1-adjoint at every level.

0-*dualizability.* First we rephrase 0-dualizability as an adjunction. Let $(\mathcal{C}, \otimes)$ be a monoidal category.

**Definition 8** (Delooping category)**.** Define the *delooping category* $B\mathcal{C}$ to be the 2-category with:

- a single object $*$,
- $\mathrm{Map}_{B\mathcal{C}}(*, *) = \mathcal{C}$, and
- composition is given by the tensor product for $\mathcal{C}$:

$$\otimes\colon\ \mathrm{Map}_{B\mathcal{C}}(*, *) \times \mathrm{Map}_{B\mathcal{C}}(*, *) \to \mathrm{Map}_{B\mathcal{C}}(*, *)\ .$$

**Proposition 3.** *Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category. $\mathcal{C}$ has duals for objects (in the sense of definition 4) if and only if $B\mathcal{C}$ has adjoints for 1-morphisms (in the sense of definition 5).*

*Higher adjoints.* Let $\mathcal{C}$ be a monoidal $(\infty, n)$-category and consider two objects $X, Y \in \mathcal{C}$. Let $f\colon X \to Y$ be a $k$-morphism. If $k > 1$, then $X$ and $Y$ both have source $S$ and target $T$. Then we can form a 2-category $h_2(S, T)$ as follows.

- The objects are $(k-1)$-morphisms $S \to T$.
- The 1-morphisms are $k$-morphisms in $\mathcal{C}$ between these $(k-1)$-morphisms.
- Given two objects and two 1-morphisms between them, a 2-morphism is an isomorphism class of $(k+1)$-morphisms in $\mathcal{C}$ between the $k$-morphisms.

Note $f$ is a 1-morphism in $h_2(S, T)$, so we can ask if it has an adjoint.

**Proposition 4.** *$f$ has an adjoint as a $k$-morphism in $\mathcal{C}$ (in the sense of definition 5) if and only if $f$ has an adjoint as a 1-morphism in $h_2(S, T)$.*

## References


[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. 3

[Hes18] Jan Hesse. An equivalence between semisimple symmetric Frobenius algebras and Calabi-Yau categories. *J. Homotopy Relat. Struct.*, 13(1):251–272, 2018. 6