THE FLOER HOMOLOGY OF A DEHN TWIST

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The full dictionary between Morse homology and Floer homology is in table 1. We now define the entries. Let Σ be a closed orientable surface of genus 2. Let $\varphi \in \mathrm{Aut}(\Sigma, \omega)$. Define

$$\Omega_\varphi \Sigma = \{\gamma \in C^\infty(\mathbb{R}, \Sigma) \mid \gamma(t+1) = \varphi(\gamma(t))\} \ . \tag{1}$$

A tangent vector on this is a collection of tangent vectors along the path. We take d of the equation $\gamma(t+1) = \varphi(\gamma(t))$ to get this.

Now define

$$da_\gamma(\xi) = \int_{[0,1]} \omega\left(\frac{d\gamma}{dt}, \xi(t)\right) dt \tag{2}$$

This won't always be exact, but we can get it to be exact by moving to a universal cover, called the Novikov cover, where H^1 is 0, so every form is exact.

In Morse theory, we care about critical points. In Floer homology the analogous thing to ask for is

$$\int \omega\left(\frac{d\gamma}{dt}\right) = 0 \ . \tag{3}$$

But ω is nondegenerate, so this is equivalent to

$$\frac{d\gamma}{dt} = 0 \ . \tag{4}$$

This corresponds to a fixed point.

Recall the Morse complex is defined as

$$\mathrm{CM}_k = \mathbb{Z}\langle x_{1_k}, \ldots, x_{n_k}\rangle \tag{5}$$

where the x_{k_i} are critical points of index[1] k. The differential is defined by:

$$d(x_k) = \sum_{y_{k-1}} n(x_k, y_{k-1}) / \mathbb{R} y_{k-1} \tag{6}$$

where $n(x_k, y_{k-1})$ counts the trajectories between the critical points.

Now in Floer theory, the chain complex will be generated by the fixed points of φ with Conley-Zehnder index k:

$$\mathrm{CF}_k = \mathbb{Z}\langle x_{1_k}, \ldots, x_{n_k}\rangle \tag{7}$$

Then the differential is counting index 1 trajectories between these fixed points. So now we want to understand what it means to be a trajectory between fixed points,

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[1] The index of a critical point is the number of negative eigenvalues of the Hessian matrix at this point.

i.e. a trajectory between paths. Let J be an almost-complex structure. I.e. an endomorphism of the tangent space which squares to $-\mathrm{id}$. Note that

$$\operatorname{grad} a(\gamma) = -J\frac{d\gamma}{dt} \tag{8}$$

so for a path of paths $\gamma_t : \mathbb{R}_s \to \Omega_\varphi\Sigma$ the Floer equation is the ODE

$$\frac{d\gamma_t}{ds} + \frac{J\,d\gamma_t}{dt} = 0 \tag{9}$$

which looks like the Cauchy-Riemann equation. But this is very badly behaved. It is an ODE on an infinite-dimensional manifold, so we instead consider a PDE on a finite-dimensional manifold.

So the idea is to look for maps $u : \mathbb{R}_s \times \mathbb{R}_t \to \Sigma$ and the condition becomes

$$\partial_t u + J\partial_s u = 0\ . \tag{10}$$

So we want maps u such that this is true, and that

$$u(s,t+1) = \varphi u(s,t) \qquad \lim_{s\to\pm\infty} u(s,t) = x_\pm \tag{11}$$

for some fixed points $x_\pm$. So the space of these maps turns out to be finite-dimensional.

Table 1. The dictionary between Morse homology and Floer homology.

Morse homology	Floer homology for φ
Σ	$\Omega_\varphi\Sigma$
$h : \Sigma \to \mathbb{R}$	a
dh	da
Critical point	γ such that $\frac{d\gamma}{dt} = 0$
Morse-Smale condition	$\det(1 - d\varphi) \neq 0$
CM_k	CF_k

Given $V \simeq S^1$ in Σ, we can perform a Dehn twist along this curve by cutting, twisting by π, then gluing back. Call this automorphism τ_V. Note this definitely doesn't satisfy the condition $\det(1 - d\varphi) \neq 0$, so we have to perturb.

Theorem 1. *Assume* $\pi_2(\Sigma) = 0$. $\mathrm{HF}(\tau_V) = H_*(\Sigma, V)$.

Sketch of proof. Consider any Morse function H on $\Sigma \setminus V$. Consider the Hamiltonian vector field X_H. Recall this means

$$\omega(X_H, \cdot) = dH\ . \tag{12}$$

Write the time 1 flow to be φ_H^1. Then extend this to Σ by setting it to be 0 on V. This is not Morse anymore. For H small enough we cannot have periodic orbits. So fixed points correspond to critical points $dH = 0$. Therefore the Floer homology is just the Morse homology except near V. Now we stretch the neck. Since this is a Dehn twist, there are no fixed points introduced when we stretch. But now no matter how long we make the neck, the energy of a trajectory which connects a critical point from one side to another is always the same. So assume we have some such trajectory. This means it must have bubbling, but this cannot be the case since we assumed $\pi_2(\Sigma) = 0$. $\square$

Exercise 1. For any continuous map $f : T^2 \to \Sigma_g$ (for $g \geq 2$) prove that

$$f_* \left[T^2\right] = 0 \in H_2\left(\Sigma_g, \mathbb{Z}\right) \ . \tag{13}$$

Solution. We know $f_* \left[T^2\right] = n\left[\Sigma_g\right]$. Assume $n \neq 0$. View this as a fibration by passing to the mapping cone, then a generic fiber is 0-dimensional. So this is, up to homotopy, a covering, but this cannot be because of homological rigidity (e.g. calculate the Euler characteristic). So the degree of the map is 0.