THE TORIC CODE

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Let X be a CW complex, G a finite group. Write X^i for the i-skeleton. Write $\mathrm{Bun}_G\left(X^1, X^0\right)$ for the collection of principal G bundles:

$$\begin{array}{ccc} & & P \\ & \overset{s}{\nearrow} & \downarrow \\ X^0 & \hookrightarrow & X' \end{array} .$$

This forms a groupoid. For (P, s) and $(P', s') \in \mathrm{Bun}_g\left(X^1, X^0\right)$, a morphism $f : (P, s) \to (P', s')$ is a map $P \to P'$ such that the following diagram commutes:

$$\begin{array}{cccc} & & P & \xrightarrow{f} & P' \\ & \overset{s}{\nearrow} & \downarrow & \swarrow & \\ X^0 & \hookrightarrow & X' & & \end{array} .$$

Note that

$$\pi_0 \mathrm{Bun}_G\left(X^1, X^0\right) \simeq \prod_{\text{edge}} G .$$

Our Hilbert space will be

$$\mathcal{H} = \mathrm{Fun}\left(\pi_0 \mathrm{Bun}_G\left(X^1, X^0\right), \mathbb{C}\right)$$

which, as a vector space, is given by

$$\bigotimes_{\text{edges}} \mathbb{C}[G] .$$

Now we define a bunch of local operators.

(1) For $v \in X^0$, define

$$\mathrm{Bun}_G\left(X^1, X^0\right) \xrightarrow{\varphi_v^g} \mathrm{Bun}_G\left(X^1, X^0\right)$$

$$(P, s) \longmapsto (P, s')$$

where

$$s'(v) = s(v) \cdot g^{-1} .$$

Now define

$$A_v^g := \left(\varphi_v^g\right)^* : \mathcal{H} \to \mathcal{H}$$

and then define

$$A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g .$$

Date: February 11, 2020.

Notes by: Jackson Van Dyke; all errors introduced are my own.

This can be interpreted as the projection onto state space that is gauge invariant at v.

(2) Let f be a 2-cell. Then define $B_g^h \in \mathrm{End}\,(\mathcal{H})$ to send

$$\psi \mapsto \delta_{h,\mathrm{hol}_f(h)}\psi$$

and define $B_f \coloneqq B_f^{\mathrm{id}}$.

Claim 1. The for $v \in \partial f$, $\{A_v, B_f\}$ pairwise commute.

Now define $H \in \mathrm{End}\,(\mathcal{H})$ by

$$H = \sum_v (1 - A_v) + \sum_f (1 - B_f) \tag{1}$$

The ground state is:

$$GS \coloneqq \ker H = \left(\bigcap_v \ker(1 - A_v)\right) \cap \left(\bigcap_f \ker(1 - B_f)\right) . \tag{2}$$

$\psi \in GS$ implies it factors as:

$$\begin{array}{ccc} \mathrm{Bun}_G\left(X^1, X^0\right) & \xrightarrow{\qquad\qquad} & \mathbb{C} \\ & \searrow \qquad \nearrow & \\ & \mathrm{Bun}_G\left(X^1\right) & \end{array}$$

so it doesn't depend on the trivialization at any point in X^0, it is really a function on principal bundles over X^1.

Now it doesn't have any holonomy at anything in the 2-skeleton, so it can be extended, and there are no obstructions to extending in higher dimensions since G is a finite group. Therefore ψ is nonzero only on $P \to X^1$ which extend to $P \to X$. Therefore the ground state is

$$GS = \mathrm{Map}\,(\mathrm{Bun}_g(X), \mathbb{C}) = \mathrm{Map}\left(H^1(X; G), \mathbb{C}\right) . \tag{3}$$

since $\mathrm{Bun}_g(X) \simeq H^1(X; G)$.

Example 1. On Σ_g, for $G = \mathbb{Z}/2$,

$$GS = \mathrm{Map}\left((\mathbb{Z}/2)^{2g}, \mathbb{C}\right)$$

which has dimension 4^g.

A *site* is a pair (v, f) with $v \in \partial f$. An elementary excitation should be a state

$$\psi \in \left(\bigcap_{v' \neq v} \ker(1 - A_v)\right) \cap \left(\bigcap_{f' \neq f} \ker(1 - B_g)\right) .$$

Let $a = (v_1, f_1)$ and $b = (v_2, f_2)$ be two sites. We will write $\mathcal{L}(a, b)$ for the corresponding 2-particle states. Define this to be

$$\mathcal{L}(a, b) \coloneqq \left(\bigcap_{v \neq v_1, v_2} \ker(1 - A_v)\right) \cap \left(\bigcap_{f \neq f_1, f_2} \ker(1 - B_f)\right) . \tag{4}$$

Consider the algebra generated by $A_{v_1}^g, B_{f_1}^h$ where

$$A_{v^1}^g \cdot A_{v_1}^h = A_{v_1}^{gh} \qquad , \qquad B_{f_1}^g \cdot B_{f_1}^h = \delta_{g,h} B_f^h .$$

and

$$A^g_{v_1} \cdot B^h_{f_1} = B^{ghg^{-1}}_f A^g_v \ .$$

These are the relations of the *quantum double* $D(G)$. We start with the group algebra $\mathbb{C}[G]$. This is a Hopf algebra. Then we have the function algebra $\mathcal{F}(G)$, which is functions[1]on G. The Hopf algebra structure is given by:

	$\mathbb{C}[G]$	$\mathcal{F}(G)$	$D(G) \simeq \mathbb{C}[G] \otimes \mathcal{F}(G)$
mult.	$x \otimes y \mapsto xy$	$\delta_x \otimes \delta_y \mapsto \delta_{x,y}\delta_y$	$(\delta_g \otimes x) \otimes (\delta_h \otimes y)$ $\mapsto \delta_{g,xhx^{-1}} (\delta_g \otimes xy)$
unit	e	$1 \mapsto \sum_{g \in G} \delta_g$	$\dots$
comult.	$x \mapsto x \otimes x$	$\delta_x \mapsto \sum_{gh=x} \delta_g \otimes \delta_h$	
counit	$\epsilon : x \mapsto 1$	$\epsilon : \delta_G \mapsto \delta_{g,1}$	
antipode	$\gamma : x \mapsto x^{-1}$	$\gamma : \delta_g \mapsto \delta_{g^{-1}}$	

Now we send

$$\delta_h \otimes g \mapsto B^h_{f_1} \cdot A^g_{v_1} \ .$$

For a and b two sites, we have

$$D(G) \underset{\varphi_b}{\overset{\varphi_a}{\rightrightarrows}} \mathrm{End}(\mathcal{L}(a,b))$$

and

$$D(a) \coloneqq \mathrm{im}(\varphi_a) \qquad\qquad D(b) \coloneqq \mathrm{im}(\varphi_b) \ .$$

The takeaway is that irreducible representations of $D(a)$ are elementary particles (anyons) at the site a.

Claim 2. $\mathbf{Rep}_f(D(G))$ is semisimple.

Let $V \in \mathrm{Obj}\left(\mathbf{Rep}_f(D(G))\right)$, $v \in V$.

Fact 1. *The irreps of $D(G)$ are classified by $V_{\bar{g},\pi}$ where π is an irreducible representation of $Z(g)$.*

Example 2. For $G = \mathbb{Z}/2$, $\bar{g}$ can be $\{0\}$ or $\{1\}$, and π can be 0 or 1. So our irreducible representations are

$$\begin{array}{cccc} V_{0,0} & V_{0,1} & V_{1,0} & V_{1,1} \\ \emptyset & e & m & e \times m \end{array} \ .$$

This can be viewed as a field theory in the sense that it sends S^1 to $\mathbf{Rep}(D(G))$, which is equivalent to $\mathbf{Vect}(G/G)$.

[1] This is kind of the same as $\mathbb{C}[G]$ but different as a Hopf algebra.