QUANTUM GROUPS AND THEIR REPRESENTATIONS IN THE CONTEXT OF $3-2-1$ TQFTS.

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This is based on this talk and its sequels.

Let $\mathbf{Bord}_1^3$ denote the following 2-category of bordisms. This is a symmetric monoidal category under the operation of disjoint union. This has 1-manifolds as objects, morphisms between two objects are 2-manifolds with the objects as its boundary. Then 2-morphisms between 1-morphisms are given by 3-manifolds with the surfaces as its boundary.

Let **LinCat** denote the category of linear categories. The objects are categories such that the Hom sets in the category are complex vector spaces, composition is linear, and there are coproducts The morphisms are linear functors, and 2-morphisms are natural transformations. The unit in **LinCat** is $\mathbf{Vect}_{\mathbb{C}}$. The 2-category **LinCat** categorifies $\mathbf{Vect}_{\mathbb{C}}$ in the sense that

$$\operatorname{End}_{\mathbf{LinCat}}\left(\mathbf{Vect}_{\mathbb{C}}\right) \simeq \mathbf{Vect}_{\mathbb{C}} \tag{1}$$

where $F \mapsto F(\mathbb{C})$.

A $3-2-1$ topological quantum field theory (TQFT) is a functor

$$\mathbf{Bord}_1^3 \xrightarrow{Z} \mathbf{LinCat}\ . \tag{2}$$

Since the image of the empty 1-manifold $Z(\emptyset_1) = \mathbf{Vect}_{\mathbb{C}}$, we have that for a closed surface Σ, $Z(\Sigma) \in \operatorname{End}_{\mathbf{LinCat}}\left(\mathbf{Vect}_{\mathbb{C}}\right) \simeq \mathbf{Vect}_{\mathbb{C}}$ so we get the classical notion of a TQFT.

For A a $\mathbb{C}$ algebra, a prototype for an element of **LinCat** is A-**Mod**.

Question 1.
1. Which linear categories can we assign to S^1?
2. Which A can be chosen for $Z\left(S^1\right) = A$-**Mod**?

Write $\mathcal{C} = Z\left(S^1\right)$. Then the following are true:

(i) This is monoidal.
(ii) All objects of $\mathcal{C}$ have duals:

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & x \otimes x^\vee \arrow[r] & 1 \end{tikzcd}\ . \tag{3}$$

(iii) $\mathcal{C}$ is semisimple (all x are of the form $\bigoplus_{i=1}^n y_i^{k_i} = x$).
(iv) $\mathcal{C}$ has finitely many simple objects.

This is monoidal because of the pair of pants. Semisimplicity follows from the mark of Zorro stuff.

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Notes by: Jackson Van Dyke, all errors introduced are my own.

Now start with a $\mathbb{C}$-algebra A, set $\mathcal{C} = A$-**Mod**. The monoidal structure tells us that we have a map

$$A\text{-}\mathbf{Mod} \otimes A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \ . \tag{4}$$

Because the following diagram commutes:

$$\begin{tikzcd} A\text{-}\mathbf{Mod} \otimes A\text{-}\mathbf{Mod} \arrow[d] & A\text{-}\mathbf{Mod} \arrow[d] \\ \mathbf{Vect}_{\mathbb{C}} \otimes \mathbf{Vect}_{\mathbb{C}} \arrow[r] & \mathbf{Vect}_{\mathbb{C}} \end{tikzcd} \ . \tag{5}$$

We need an A-**Mod** structure on $M \otimes_{\mathbb{C}} N$. This is encoded by a map

$$A \xrightarrow{\Delta} A \otimes A \ . \tag{6}$$

Now we have the assignment of an A-module M to its dual $M^{\vee}$, which had underlying vector space $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$. The action of A is given by a map $s : A \to A$, called the antipode. Explicitly, the action is given by:

$$(a \cdot f)(m) = f(s(a)m) \ . \tag{7}$$

Associativity of the action tells us that:

$$s(ab) = s(b)s(a) \ . \tag{8}$$

Eventually, if we continued in this way, we would find that this is a Hopf algebra.

Our first guess for how to find these might be that we should start with a Lie algebra and form the universal enveloping algebra:

$$\mathcal{U}(\mathfrak{g}) = T\mathfrak{g}/([x,y] = xy - yx) \ . \tag{9}$$

The above maps are given by:

$$\begin{tikzcd} \mathcal{U}\mathfrak{g} \arrow[r, "\Delta"] & \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} \\ x \arrow[r, mapsto] & x \otimes 1 + 1 \otimes x \\ \mathcal{U}\mathfrak{g} \arrow[r, "s"] & \mathcal{U}\mathfrak{g} \\ x \arrow[r, mapsto] & -x \\ \mathcal{U}\mathfrak{g} \arrow[r] & \mathbb{C} \\ x \arrow[r, mapsto] & 0 \\ 1 \arrow[r, mapsto] & 1 \end{tikzcd} \ . \tag{10}$$

The issue is that this has infinitely many simple objects.

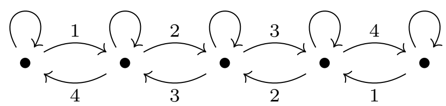
Example 1. For example, we can take $\mathfrak{g} = \mathfrak{sl}_2$. Recall this is generated by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

so a presentation is given by

$$\mathfrak{g} = \langle E, F, H \mid [H, E] = 2E, [H, F] = -2F, [E, F] = H \rangle \ . \tag{12}$$

For all $n \geq 0$, we get L_n an $n+1$ complex dimensional irreducible representation of $\mathfrak{g}$, which looks like

(13) 

.

E sends me to the right, and F sends me to the left, and H sends me on the loops.

Now the solution is that we need to go quantum. Dynkin diagrams classify all finite dimensional semisimple Lie algebras. In fact, they give us a general presentation in terms of (a_{ij}), the Cartan matrices, and d_i the edge count.

For $q \in \mathbb{C}^{\times}$ we will deform the universal enveloping algebra to get a new algebra over $\mathbb{C}$, $\mathcal{U}_q(\mathfrak{g})$. The adjoint action of the E_i on E_j becomes

$$\sum_{r=0}^{|a_{i,j}|+1} (-1)^r \binom{|a_{ij}|+1}{r} E_i^{|a_{ij}|+1-r} E_j E_i^r \tag{14}$$

and likewise for the F_i. The relation $[H_i, E_i] = a_{ij} E_j$ becomes

$$H_i E_j = E_i (H_i + a_{ij} I) \ . \tag{15}$$

The point here is that we want to identify the H_i's as functionals on the weights, and read the above as saying that

$$H_i E_j = E_i \tau_j (H_i) \tag{16}$$

where τ_j sends α to $H_i \alpha + \alpha_j$.

Now introduce quantum integers:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \ldots + q^{n-3} + q^{n-1} \ . \tag{17}$$

Then define the quantum factorial to be

$$[n_q]! = [n]_q \ldots [1]_q \tag{18}$$

and the quantum binomial coefficient to be:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q}{[k]_q [n-k]_q} \ . \tag{19}$$

Then this gives us all of the ingredients to fill the dictionary table 1.

Now this is still a Hopf algebra. In particular,

$$\Delta : \mathcal{U}_q\mathfrak{g} \otimes \mathcal{U}_q\mathfrak{g} \otimes \mathcal{U}_q\mathfrak{g} \qquad s : \mathcal{U}_q\mathfrak{g} \to \mathcal{U}_q\mathfrak{g} \qquad \epsilon : \mathcal{U}_q\mathfrak{g} \to \mathbb{C} \tag{20}$$

$$\Delta(K_i) = K_i \otimes K_i \qquad s(K_i) = K_i^{-1} \qquad s(F_i) = -K_i F_i \tag{21}$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad s(E_i) = -E_i K_i^{-1} \qquad \epsilon(E_i) = 0 \tag{22}$$

$$\Delta(F_i) = F_i \otimes 1 + K_i \qquad s(F_i) = -K_i F_i \qquad \epsilon(F_i) = 0 \ . \tag{23}$$

TABLE 1. The dictionary between Morse homology and Floer homology.

Classical $\mathfrak{g}$	Quantum $\mathcal{U}_q(\mathfrak{g})$
Generators $E_i,\ F_i,\ H_i$.	$E_i,\ F_i,\ K_i = q^{d_i H_i},\ K_i^{-1}$
$[H_i, H_j] = 0$, $[H_i, E_i] = a_{ij} E_j$, $[H_i, F_i] = -a_{ij} F_j$, $[H_i, F_i] = \delta_{ij} H_i$	$K_i K_j = K_j K_i$ $K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j$ $K_i F_j K_i^{-1} = q^{-d_i a_{ij}} F_j$
$\mathrm{ad}\,(E_i)^{\lvert a_{ij}\rvert+1}(E_j) = 0$, $\mathrm{ad}\,(F_i)^{\lvert a_{ij}\rvert+1}(F_j) = 0$	same but with quantum numbers

Example 2. For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{U}_q\mathfrak{sl}_2$ admits analogous representations to the L_ns. Now these look like

$$\begin{tikzcd} \bullet \arrow[loop above, "q^{-4}"] \arrow[r, bend left, "{[1]_q}"] & \bullet \arrow[loop above, "q^{-2}"] \arrow[r, bend left, "{[2]_q}"] \arrow[l, bend left, "{[4]_q}"] & \bullet \arrow[loop above, "q^{0}"] \arrow[r, bend left, "{[3]_q}"] \arrow[l, bend left, "{[3]_q}"] & \bullet \arrow[loop above, "q^{2}"] \arrow[r, bend left, "{[4]_q}"] \arrow[l, bend left, "{[2]_q}"] & \bullet \arrow[loop above, "q^{4}"] \arrow[l, bend left, "{[1]_q}"] \end{tikzcd} \tag{24}$$

where F still moves to the left, E moves to the right, and H brings us on the loops.

In the classical picture, we look at the weight lattice Λ and we pick a dominant weight $\lambda \in \Lambda_+$ and take the quotient of the associated Verma module

$$M_\lambda = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda \tag{25}$$

where $\mathfrak{b} = \mathrm{Span}\,\{E_i, H_i\}$, to get L_λ.

In the quantum picture, we can still define $M_\lambda = \mathcal{U}_q\mathfrak{g} \otimes_{\mathcal{U}_q\mathfrak{b}} \mathbb{C}_\lambda$, but its quotient may no longer all be irreducible. Call these quotient Weyl modules W_λ. These lead to the study of tilting modules.