# GAUGE THEORY AND GEOMETRIC LANGLANDS

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The original reference is [3]. Witten then continued with [4] and [5]. Frenkel wrote a nice survey [2], and then there is [1].

## 1. What is geometric Langlands

The following is an overview of geometric Langlands. First we start with a reductive algebraic group $G$ such as $\mathrm{SL}_n$, $\mathrm{GL}_n$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, etc. Then $G$ has Langland dual $G^\vee$ (also written ${}^LG$) (given by taking the dual weight data) e.g. $\mathrm{PGL}_n$, $\mathrm{GL}_n$, $\mathrm{SO}(2n)$, $\mathrm{Sp}(2n)$. Now fix $C$ an algebraic curve. Geometric Langlands is an equivalence of derived categories:

$$\mathcal{O}\text{-}\mathbf{Mod}\left(\mathbf{Loc}_{{}^LG}(C)\right)\, D\text{-}\mathbf{Mod}\left(\mathrm{Bun}_G(C)\right) \tag{1}$$

such that skyscraper sheaves go to "Hecke eigensheaves".

**Example 1.** For $G$ abelian, this equivalence is given by the Fourier-Mukai transform.

Now we have the following notions from physics.

- GL has been linked to two-dimensional conformal field theories (CFT's).
- ${}^LG$ is known as the GNO dual group.
- LHS of geometric Langlands: $\mathbf{Loc}_{{}^LG}(C)$ is the collection of flat[1] connections on $C$
- RHS of geometric Langlands: automorphic representation correspond to conformal blocks.

## 2. Kapustin-Witten

There are six points:

(1) There is a topological twist of four-dimensional $\mathcal{N}=4$ super Yang-Mills (SYM) which gives a $\mathbb{CP}^1$-family of TQFT's.

*Digression* 1. Let's try to understand this by a top down approach. In 10-dimensional SYM is basically unique, we don't even need any symmetry condition, all we have is some coupling parameters. The fields are :

- connections $A$ (a 1-form) in a principal bundle $\begin{matrix} P \\ \downarrow \\ \mathbb{R}^{10} \end{matrix}$ ;
- fermions $\lambda \in \Gamma\left(S^+ \otimes \mathrm{ad}(P)\right)$, $\Gamma_I$, $(I \in \{0,\ldots,9\})$ $S^+ \to S^-$.

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Notes by: Jackson Van Dyke, all errors introduced are my own.

[1]We can think of this as minimizing energy.

The Poincaré group $P$:

$$0 \to \mathbb{R}^d \to P \to \mathrm{Spin}\,(d) \to 0 \tag{2}$$

acts on the base. Now we can do dimensional reduction to get a 4-dimensional theory by declaring fields to be independent of 6 of the coordinates.

$$\mathrm{Spin}\,(10) \supset \underbrace{\mathrm{Spin}\,(4)}_{=\mathrm{SU}_2\times\mathrm{SU}_2} \times \underbrace{\mathrm{Spin}\,(6)}_{\mathrm{SU}(4)_R} . \tag{3}$$

Now diagonally embed:

$$\mathrm{Spin}\,(4) \hookrightarrow \mathrm{Spin}\,(4)\times\mathrm{Spin}\,(6) \tag{4}$$

$$\mathrm{SU}\,(2)_\ell \times \mathrm{SU}\,(2)_r \to \begin{pmatrix} \mathrm{SU}\,(2)_\ell & \\ & \mathrm{SU}\,(2)_r \end{pmatrix} \tag{5}$$

$$\text{extra }\ \mathrm{U}\,(1) \to i\begin{pmatrix} \mathrm{id} & \\ & -\mathrm{id} \end{pmatrix} . \tag{6}$$

So then

$$\overline{4} = (2,1)^{-1} \oplus (1,2)^{1} \tag{7}$$

$$4 = (2,1)^{1} \oplus (1,2)^{-1} \tag{8}$$

and we get

$$\left(2,1,\overline{4}\right) = (2,1)^0 \otimes \left((2,1)^{-1} \oplus (1,2)^{1}\right) \tag{9}$$

$$= (1,2)^{-1} \oplus (3,1)^{-1} \oplus (2,2)^{1} \tag{10}$$

$$(1,2,4) = (2,2)^{1} \oplus (1,3)^{-1} \oplus (1,1)^{-1} \tag{11}$$

so we have a 1-dimensional representation popping up. So this will be our supercharge. One is left-handed, one is right-handed:

$$\epsilon = u\epsilon_\ell + v\epsilon_r \tag{12}$$

and

$$Q = uQ_\ell + vQ_r \tag{13}$$

such that $Q^2 = 0$. Then the twisting parameter is:

$$t = \frac{u}{v} \in \mathbb{CP}^1 \tag{14}$$

so we get a bunch of TQFT's.

Going down to four-dimensions we get:

$$I_4 = \frac{1}{e^2} = \int \mathrm{Tr}\,(F \wedge \star F) + \dots \tag{15}$$

and then we get a topological term

$$I_\theta = -\frac{\theta}{8\pi^2}\int \mathrm{Tr}\,(F\wedge F) . \tag{16}$$

Define the coupling constant to be:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} . \tag{17}$$

If $H_2(M) = 0$, then

$$\frac{1}{8\pi^2}\int \mathrm{Tr}\,(F\wedge F)\in\mathbb{Z} \tag{18}$$

so there is a $\tau \to \tau + 1$ symmetry (even classically). Now there is an interesting hidden symmetry, called $S$-duality, which says that there is also a symmetry under

$$\tau \mapsto \frac{1}{n_{\mathfrak{g}}\tau} \tag{19}$$

where $n_{\mathfrak{g}}$ is the lacing number. This has to be accompanied by $G$ being replaced by ${}^LG$. The matrix which does this is

$$\begin{bmatrix} 0 & -1/\sqrt{n_{\mathfrak{g}}} \\ \sqrt{n_{\mathfrak{g}}} & 0 \end{bmatrix}. \tag{20}$$

(2) Put this theory on $\Sigma \times C$ (where these are both surfaces). But there is some notion of volume on these, and the volume on $\Sigma$ is much larger than the volume of $C$. So we get a two-dimensional theory with fields are now maps:

$$\sigma : \Sigma \to \mathcal{M}_G(C) \tag{21}$$

where $\mathcal{M}_G(C)$ is the Hitchin moduli space. This has a complex structure depending on $t \in \mathbb{CP}^1$ from above. So this is a $\mathbb{CP}^1$ family of $A$-models.

(3) These four-dimensional theories have two different kinds of line operators: the Wilson and t'Hooft lines. The wilson lines take some loop, take the gauge connection, take the holonomy and take the trace so we get a number. The t'Hooft lines are more complicated. These descend to things in two-dimensions which act on branes. In particular, the electric/magnetic eigenbranes are mapped to multiples of themselves by these line operators. $S$ duality exchanges Wilson and t'Hooft lines, and likewise exchange the magnetic eigenbranes with electric ones.

(4) Electric eigenbranes are representations:

$$\mathbf{Rep}\left(\pi_1(C), {}^LG\right) \tag{22}$$

which sit on the LHS of the geometric Langlands equivalence.

(5) t'Hooft operators are called the Hecke operators.

(6) Hence, Magnetic eigenbranes correspond to $\mathcal{D}$-modules. So this is the RHS of geometric Langlands.

So the link between the LHS and RHS is $S$-duality. Since this is a statement about categories, maybe this should really happen in codimension 2. And indeed, in [4] the claim is that there is a six-dimensional theory which we can put on an elliptic curve, and then since it only depends on $\tau$ up to $\mathrm{SL}(2,\mathbb{Z})$, and then the $S$-duality, from the 6D perspective, is just that there is some redundancy in the description, and then at least the abelian case is described very nicely, where now we start with a self-dual 3-form, and we impose the equations of motion that it's closed, so now it says that it is the curvature for the connection on a gerbe. So the abelian case appears very naturally. And then for the nonabelian case, we need to understand what a gerbe should be, but supersymmetry somehow comes to save the day.

## References


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