GEOMETRIC QUANTIZATION

LECTURE BY ALI SHEHPER

1. Classical physics

Let Q be our configuration manifold. Write $M = T^*Q$. This is our phase space. We will write $(q, p) \in T^*Q$ for $q \in Q$ and p in the fiber. This has canonical symplectic form locally given by $\omega = \sum dp_i \wedge dq_i$.

The space of classical observables is

$$\mathcal{A}_{\mathrm{cl}} = \{f : T^*Q \to \mathbb{R}\} \ . \tag{1}$$

Note that there exists $H \in \mathcal{A}$ such that $H : T^*Q \to \mathbb{R}_{\geq 0}$, and for $f \in \mathcal{A}$,

$$\frac{df}{dt} = \{f, H\} = \omega\left(X_f, X_H\right)$$

where X_f is a vector field defined as

$$\omega\left(X_f, -\right) = df\ (-) \ .$$

The canonical transformation (or symplectomorphism) is generated by ξ:

$$\mathcal{L}_\xi \omega = 0 = \cancel{\iota_\xi\left(d\omega\right)} + d\left(\iota_\xi \omega\right) \qquad \rightsquigarrow \qquad d\left(\iota_\xi \omega\right) = 0 \ .$$

(Note $\mathcal{L}$ denotes the Lie derivative.) Locally $\omega = dA$, and $\iota_\xi \omega = df$ for some $f \in \mathcal{A}$. Conversely, for any f, define ξ such that $\iota_\xi \omega = df$, so each $f \in \mathcal{A}$ generates a symplectomorphism. We call A the symplectic potential. Note that $\mathcal{L}_\xi A = d\Lambda$, where $\Lambda = \iota_\xi A - f$.

2. Quantization

The idea is to take the data $\left(\left(M, \omega\right), \mathcal{A}_{\mathrm{cl}}\right)$ and get an irreducible representation $\mathcal{H} : \mathcal{A}_q \circlearrowleft$ such that

$$\left[\hat{f}, \hat{g}\right] = i\hat{\mathcal{O}} \tag{2}$$

for $\mathcal{O} = \{fg\}$. This might be too strict for everything in $\mathcal{A}_q$, but you at least want it for some. This relation comes from the uncertainty principle.

Example 1. If $M = T^*Q$, $\mathcal{H} = L^2\left(Q\right)$, then $\hat{q}_i$ acts by multiplication, and $\hat{p}_i$ by $-i\nabla_{q_i}$. For example, $f = qp$ goes to

$$\hat{f} = \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \ .$$

Notes by: Jackson Van Dyke; all errors introduced are my own.

But what if $M \neq T^*Q$? Geometric quantization will give us

$$\begin{array}{c} E \\ {\scriptstyle\psi}\circlearrowright\downarrow \\ M \end{array}$$

such that A is a connection on E with curvature ω.

Then

$$\mathcal{L}_\xi \psi = \left(\xi^\mu D_\mu + if\right)\psi$$

and $\mathcal{H}$ is the space of sections of E. Then a candidate for $\hat{f}$ is

$$-i\left(\xi^\mu D_\mu + if\right)$$

and then the inner product between sections is given by

$$\langle\psi|\xi\rangle = \int d\,\mathrm{Vol}\;\psi^*\xi\ .$$

Let $\dim_{\mathbb{R}} M = 2n$, and assume ω^n is a nowhere vanishing $2n$-form proportional to $d\,\mathrm{Vol}$.

In one-dimensional quantum mechanics, $Q = \mathbb{R}$, so $M = T^*\mathbb{R}$ with $\omega = dp \wedge dq$, $A = p\,dx$. Then

$$\hat{f} = -i\left(\xi^\mu D_\mu + if\right)$$

and

$$\hat{q} = i\frac{\partial}{\partial p} + x \qquad\qquad \hat{p} = -i\frac{\partial}{\partial x}\ .$$

Restrict to ψ such that

$$\frac{\partial\psi}{\partial p} = 0\ .$$

We could have made a different choice for f, but we would have gotten isomorphic representations.

But it is unclear how to deal with not being a cotangent bundle. If M is Kähler, we can deal with this since we have a notion of holomorphicity. Being Kähler means we have a metric g and an (almost) complex structure J such that

$$\omega\left(u,v\right) = g\left(Ju,v\right)\ .$$

As a $(1,1)$ form

$$\begin{aligned}\omega &= ig_{a\bar{b}}\,dz^a \wedge d\bar{z}^{\bar{b}} \\ &= i\partial\bar{\partial}K\end{aligned}$$

where K is the (real) Kähler potential. Note that $\omega = dA$ where

$$A = -\frac{i}{2}\partial_A K\,dz^a + \frac{i}{2}\partial_{\bar{a}}K\,d\bar{z}^a\ .$$

We want solutions of

$$D_{\bar{a}}\psi = 0 \qquad\qquad \left(\partial_{\bar{a}} + \frac{1}{2}\partial_{\bar{a}}K\right)\psi = 0\ .$$

These all look like

$$\psi = e^{-K/2}f\left(z\right)$$

where f is some locally holomorphic function. Then we have that the inner product of two sections is:

$$\langle 1|2\rangle = \int d\,\mathrm{Vol}\; e^{-K} \ .$$

Example 2. For $X = \mathbb{R}$, $\omega = i\,dz \wedge d\bar{z}$, and $K = z\bar{z}$. Then

$$\psi = e^{-|z|^2} f(z) \ .$$

$\hat{z}$ is multiplication by z, and $\widehat{\bar{z}} = \partial_z$. Note that $[\bar{z}, z] = 1$, and $\mathrm{Hol}(\mathbb{C}) \simeq L^2(\mathbb{R})$.

In usual quantum mechanics, $\bar{z}$ would be written as the annihilation operator a, and z would be written as the creation operator $a^\dagger$. These names can be motivated by considering how they act on a polynomial of a given degree.

Example 3. Let $M = S^2$ with symplectic form

$$\omega = \frac{i\,dz \wedge d\bar{z}}{\left(1 + |z|^2\right)^2} \ .$$

Note that

$$\int_{S^2} \omega = 2\pi$$

so

$$K = \frac{n}{2} \log\left(1 + |z|^2\right)$$

and ψ is always of the form

$$\psi = e^{-K/2} f(z) \ . \tag{3}$$

The inner product is given by

$$\langle \psi|2\rangle = \frac{i(n+1)}{2\pi} \int \frac{dz \wedge d\bar{z}}{\left(1 + |z|^2\right)^{n+2}} f_1^* f_2 \ . \tag{4}$$

This integral only converges[1] for $f(z) \sim z^k$ where $k \in \{0, \ldots, n\}$ which means $\dim \mathcal{H} = n + 1$. Recall $\mathfrak{so}(3)$ acts on $\mathbb{R}^3 \supset S^2$. As usual, write

$$J_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$J_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$J_3 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

Then this action is given by

$$\hat{J}_+ = z^2 \partial_z - nz$$

$$\hat{J}_- = -\partial_z$$

$$\hat{J}_3 = z\partial_Z - \frac{n}{2}$$

So $\mathcal{H}$ is a representation of $\mathrm{SU}(2)$.

In the basis of $\mathcal{H}$ with basis vectors z^k ($k \in \{0, \ldots, n\}$) a J_3 eigenvalue of z^k is $k - n/2$ for $k \in \{0, \ldots, n\}$. This is an $(n+1)$-dimensional representation of $\mathfrak{su}(2)$.

[1] We could get this dimension by just calculating the dimension of the holomorphic sections for the sphere.

3. Chern-Simons theory

Consider Chern-Simons theory for $G = \mathrm{SU}(2)$ on $M_3 = \mathbb{R}_+ \times \Sigma_g$. Then

$$\delta = \frac{k}{2\pi} \int A\, dA + \frac{2}{3} A \wedge A \wedge A$$

for $k \in \mathbb{Z}$. So we get that the phase space consists of the flat $\mathrm{SU}(2)$ connections on Σ_g. The form is given by

$$\omega = \frac{k}{4\pi} \int_\Sigma \delta A \wedge \delta A \ . \tag{5}$$

Note that $\dim \mathcal{H} = \chi\left(M, \mathcal{L}^{\otimes k}\right)$. χ is the index of $\bar{\partial}_{\mathcal{L}^{\otimes k}}$, which is the integral of the wedge of the Todd class with $e^{c_1(\mathcal{L})}$. Then this should be equal to the following:

$$\left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin\left(\frac{\pi j}{k+2}\right)\right)$$

This is the *Verlinde formula.*