Between electric-magnetic duality and the Langlands program

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Chapter 1

Overview

The geometric Langlands program is some kind of middle-ground between number theory and physics. Another point of view is that we will be navigating the narrow passage between the whirlpool Charybdis (physics) and the six-headed monster Scylla (number theory), as in Odysseus' travels.[1]

Lecture 1; January 19, 2021

The inspiration for much of this course comes from [Mac78], which provides a historical account of harmonic analysis, focusing on the idea that function spaces can be decomposed using symmetry. This theme has long-standing connections to physics and number theory.

The spirit of what we will try to do is some kind of harmonic analysis (fancy version of Fourier theory) which will appear in different guises in both physics and number theory.

1.1 Modular/automorphic forms

1.1.1 Rough idea

The theory of modular forms is a kind of harmonic analysis/quantum mechanics on arithmetic locally symmetric spaces. The canonical example of a locally symmetric space is given by the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})\,/\,\mathrm{SO}_2$. I.e. we are considering the quotient

$$\mathcal{M}_{\mathrm{SL}_2\mathbb{R}} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \mathrm{SL}_2(\mathbb{Z})\,\backslash\,\mathrm{SL}_2(\mathbb{R})\,/\,\mathrm{SO}_2 \tag{1.1}$$

as in fig. 1.1.

For a general reductive algebraic group G we can consider the space

$$\mathcal{M}_G = \Gamma\backslash G/K \tag{1.2}$$

where Γ is an arithmetic lattice, and K is a maximal compact subgroup. For now we restrict to

$$G = \mathrm{SL}_2(\mathbb{R}) \qquad \Gamma = \mathrm{SL}_2(\mathbb{Z}) \qquad K = \mathrm{SO}_2\ .$$

We want to do harmonic analysis on this space, i.e. we want to decompose spaces of functions on this in a meaningful way. In the case of quantum mechanics we're primarily interested

[1] One can expand this analogy. Calypso's island is probably derived algebraic geometry (DAG), etc.

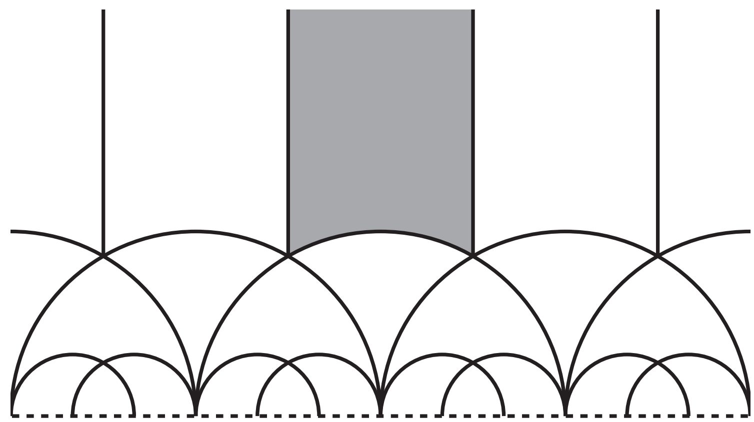

Figure 1.1: Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

in L^2 functions:

$$L^2(\Gamma\backslash G/K) \ , \tag{1.3}$$

and on this we have an action of the hyperbolic Laplace operator. I.e. we want to study the spectral theory of this operator.

The same information, possibly in a more accessible form, is given by getting rid of the K. That is, we can just study L^2 functions on

$$\Gamma\backslash G = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{1.4}$$

which is the unit tangent bundle, a circle bundle over the space we had before. Instead of studying the Laplacian, this is a homogeneous space so we can study the action of all of $\mathrm{SL}_2(\mathbb{R})$.

One can expand this to include differentials and pluri-differentials, i.e. sections of (powers of) the canonical bundle:

$$\Gamma\left(\Gamma\backslash\mathbb{H}, \omega^{k/2}\right) \ . \tag{1.5}$$

Definition 1. The Δ-eigenfunctions in $L^2(\Gamma\backslash G/K)$ are called *Maass forms*. *Modular forms* of weight k are holomorphic sections of $\omega^{k/2}$.

Remark 1. For a topologist, one might instead want to study (topological) cohomology (instead of forms) with coefficients in some local system (twisted coefficients). Indeed, modular forms can also arise by looking at the (twisted) cohomology of $\Gamma\backslash\mathbb{H}$. This is known as Eichler-Shimura theory.

One might worry that this leaves the world of quantum mechanics, but after passing to cohomology we're doing what is called *topological* quantum mechanics. We will be more concerned with this than honest quantum mechanics.

The idea is that there are no dynamics in this setting. We're just looking at the ground states, so the Laplacian is 0, and we're just looking at harmonic things. And this really has to do with topology and cohomology. But modular forms are some kind of ground states.

Remark 2. If we take general G, K, and Γ then we get the more general theory of *automorphic forms.*

Example 1. If we start with $G = \mathrm{Sp}_{2n}(\mathbb{R})$ and take $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $K = \mathrm{SO}_n$ then we get *Siegel modular forms.*

1.1.2 Structure

There is a long history of thinking of this problem[2] as quantum mechanics on this locally symmetric space. But there is a lot more structure going on in the number theory than seems to be present in the quantum mechanics of a particle moving around on this locally symmetric space.

Restrict to the case $G = \mathrm{SL}_2(\mathbb{R})$.

Number field

The question of understanding

$$L^2\left(\mathrm{SL}_2 \mathbb{Z} \backslash \mathrm{SL}_2 \mathbb{R} / \mathrm{SO}_2\right) \tag{1.6}$$

has an analogue for any number field. We can think of $\mathbb{Z}$ as being the ring of integers in the rational numbers:

$$\mathbb{Z} = \mathcal{O}_{\mathbb{Q}} \tag{1.7}$$

and from this we get a lattice $\mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}})$. Writing it this way, we see that we can replace $\mathbb{Q}$ by any finite extension F, and $\mathbb{Z}$ becomes the ring of integers $\mathcal{O}_F$:

$$\begin{aligned} \mathbb{Q} &\rightsquigarrow F \\ \mathbb{Z} &\rightsquigarrow \mathcal{O}_F \ . \end{aligned} \tag{1.8}$$

The upshot is that when we replace Q with some other number field $F/\mathbb{Q}$, then the space $\mathcal{M}_{G,\mathbb{Q}}$ becomes some space $\mathcal{M}_{G,F}$. Then we linearize by taking either L^2 or H^* of $\mathcal{M}_{G,F}$.

Example 2. This holds for all reductive algebraic groups G, but let $G = \mathrm{PSL}_2 \mathbb{R}$. Then

$$\mathcal{M}_{G,\mathbb{Q}} = \mathrm{PSL}_2 \mathbb{Z} \backslash \mathrm{PSL}_2 \mathbb{R} / \mathrm{SO}_2 \tag{1.9}$$

is the locally symmetric space in fig. 1.1. If we replace $\mathbb{Q}$ with an arbitrary number field $F/\mathbb{Q}$, then we get

$$\mathcal{M}_{G,F} = \mathrm{PSL}_2(\mathcal{O}_F) \backslash \mathrm{PSL}_2(F \otimes_{\mathbb{Q}} \mathbb{R}) / K \ . \tag{1.10}$$

Note that

$$F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{\times r_1} \times \mathbb{C}^{\times r_2} \tag{1.11}$$

where r_1 is the number of real embeddings of F, and r_2 is the number of conjugate pairs of complex embeddings.

Example 3. Let $F = \mathbb{Q}\left(\sqrt{d}\right)$. If it is real ($d \geq 0$) then $r_1 = 2$ (corresponding to $\pm\sqrt{d}$) and $r_2 = 0$, so we get

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right) \otimes_{\mathbb{Q}} \mathbb{R}\right) = \mathrm{PSL}_2 \mathbb{R} \times \mathrm{PSL}_2 \mathbb{R} \ . \tag{1.12}$$

[2]The problem of understanding L^2 functions on a locally symmetric space.

This leads to what are called *Hilbert modular forms*.

If it is imaginary ($d < 0$) then $r_1 = 0$, $r_2 = 1$, and

$$\mathrm{PSL}_2\left(\mathbb{Q}\left(\sqrt{d}\right) \otimes_{\mathbb{Q}} \mathbb{R}\right) = \mathrm{PSL}_2\,\mathbb{C}\ . \tag{1.13}$$

In this case the maximal compact is $\mathrm{SO}_3\,\mathbb{R}$, and the quotient:

$$\mathbb{H}^3 = \mathrm{PSL}_2\,\mathbb{C}/\,\mathrm{SO}_3\,\mathbb{R} \tag{1.14}$$

is hyperbolic 3-space. Now we need to mod out (on the left) by a lattice, and the result is some hyperbolic manifold which is a 3-dimensional version of the picture in fig. 1.1.

Remark 3. The point is that the real group we get after varying the number field is not that interesting, just some copies of PSL_2. But the lattice we are modding out by depends more strongly on the number field, so this is the interesting part.

Conductor/ramification data

Fixing the number field $F = \mathbb{Q}$, we can vary the "conductor" or "ramification data". The idea is as follows. The locally symmetric space $\Gamma\backslash\mathbb{H}$ has a bunch of covering spaces of the form $\Gamma'\backslash\mathbb{H}$, where Γ' is some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. So we can replace Γ by Γ'.

We won't define congruence subgroups in general, but there are basically two types. For $N \in \mathbb{Z}$, we fix subgroups:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \mathrm{id} \bmod N \right\} \tag{1.15}$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\}\ . \tag{1.16}$$

The idea is that we start with the conductor N and the lattice Γ, and then we modify Γ at the divisors of N. Note that even in this setting we have the choice of $\Gamma(N)$ or $\Gamma_0(N)$. Really the collection of variants has a lot more structure. The local data at p has to do with the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$.

Action of Hecke algebra

We have seen that our Hilbert space depends on the group, the number field, and some ramification data. A very important aspect of this theory is that this vector space (of functions) carries a lot more structure. There is a huge "degeneracy" here in the sense that the eigenspaces of the Laplacian are much bigger than one might have guessed (not one-dimensional).

This degeneracy is given by the theory of *Hecke operators*. This says that the Laplacian Δ is actually a part of a huge commuting family of operators. In particular, these all act on the eigenspaces of the Laplacian. For p a prime (p unramified, i.e. $p \nmid N$) we have the Hecke operator at p, T_p. Then

$$\bigoplus_p \mathbb{C}[T_p] \circlearrowright L^2(\Gamma\backslash G/K)\ . \tag{1.17}$$

This is some kind of "quantum integrable system" because having so many operators commute with the Hamiltonian tells us that a lot of quantities are conserved.[3]

[3] This example is often included in the literature as an example of quantum chaos (the opposite of

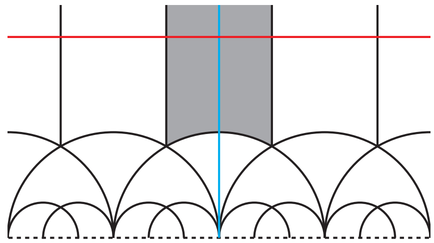

Figure 1.2: Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

Periods/states

There is a special collection of measurements we can take of modular forms, called periods. A basic example is given by integrating a modular form on the line $i\mathbb{R}_+ \subset \mathbb{H}$ as in fig. 1.2. This is how Hecke defined the L-function.

The takeaway is that we have a collection of measurements/states with very good properties, and then we can study modular forms by measuring them with these periods.

Langlands functoriality

There is a collection of somewhat mysterious operators whose action corresponds to varying the group G.

1.2 The Langlands program and TFT

1.2.1 Overview

We have seen that for a choice of reductive algebraic group G and number field $F/\mathbb{Q}$, we get a locally symmetric space

$$\mathcal{M} = \mathcal{M}_{G,F} = \text{“arithmetic lattice”}\backslash\text{real group/maximal compact} . \tag{1.18}$$

This can be thought of as some space of G-bundles

$$\mathcal{M}_{G,F} = \text{“}\mathrm{Bun}_G\left(\mathrm{Spec}\,\mathcal{O}_F\right)'' . \tag{1.19}$$

Then we linearize this space by taking either L^2 or H^*.

Starting with this theory of automorphic forms, we spectrally decompose under the action of the Hecke algebra. Then the Langlands program says that the pieces of this decomposition correspond to Galois representations. We can think of the theory of automorphic forms as being fed into a prism, and the colors coming out on the other side are Galois representations as in fig. 1.3. More specifically, the "colors" are representations:

$$\mathrm{Gal}\left(\overline{F}/F\right) \to G^\vee_{\mathbb{C}} . \tag{1.20}$$

integrability). The chaotic aspect has nothing to do with the discrete subgroup Γ. Specifically this fits into the study of "arithmetic quantum chaos" which more closely resembles the study of integral systems.

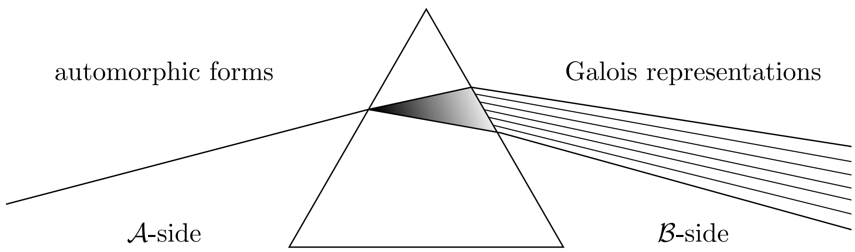

Figure 1.3: Just as light is decomposed by a prism, this spectral decomposition breaks automorphic forms ($\mathcal{A}$-side) up into Galois representations of number fields ($\mathcal{B}$-side).

Example 4. If $G = \mathrm{GL}_2 \mathbb{R}$, then $G^\vee = \mathrm{GL}_2 \mathbb{C}$. Let E be an elliptic curve. Then

$$H^1 \left(E/\mathbb{Q}\right) \tag{1.21}$$

is a 2-dimensional representation of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. This is the kind of representation you get in this setting.

Example 5. The representations in example 4 are very specific to GL_2. If we started with $\mathrm{GL}_3(\mathbb{R})$ instead, the associated locally symmetric space $\mathrm{O}_3 \setminus \mathrm{GL}_3 \mathbb{R} / \mathrm{GL}_3 \mathbb{Z}$ is not a complex manifold.

The goal is to match all of this structure in section 1.1.2 with a problem in physics, but ordinary quantum mechanics will be too simple. On the physics side we will instead consider quantum field theory.

Slogan: the Langlands program is part of the study of 4-dimensional (arithmetic, topological) quantum field theory.

The idea is that the Langlands program is an equivalence of 4-dimensional arithmetic topological field theories (TFTs):

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \text{automorphic} & & \text{spectral} \\ \text{magnetic} & & \text{electric} \end{array} \tag{1.22}$$

called the $\mathcal{A}$ and $\mathcal{B}$-side theories.

Remark 4. This is what one might call "four-dimensional mirror symmetry". The $\mathcal{A}$ and $\mathcal{B}$ are in the same sense as usual mirror symmetry.

An n-dimensional TFT is a beast which assign a quantum mechanics problem (or just a vector space, chain complex, etc.) to every $(n-1)$-manifold. So a 4-dimensional TFT sends a 3-manifold to some kind of vector space. It assigns more complicated data to lower-dimensional manifolds and less complicated data to higher-dimensional manifolds as in table 1.1.

Table 1.1: Output of a four-dimensional topological field theory. Numbers are the easiest to understand, but are usually the trickiest to produce (often requires analysis). Vector spaces are also pretty simple, but three-manifolds are hard. So the sweet spot is kind of in 2-dimensions, since we understand surfaces and categories aren't that complicated.

Dimension	Output
4	$z \in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$-category
0	$(\infty, 3)$-category? (rarely understood)

The *topological* means we are throwing out the dynamics and only looking at the ground states. This is the analogue of only looking at the harmonic forms rather than the whole spectrum of the Laplacian. The *arithmetic* means that we're following the paradigm of *arithmetic topology*. The idea is that we will eventually make an analogy between number fields and three-manifolds. Then we can plug a number field into the TFT (instead of an honest manifold) to get a vector space which turns out to be $L^2(\mathcal{M}_{G,F})$ (or $H^*(\mathcal{M}_{G,F})$).

1.2.2 Arithmetic topology

Weil's Rosetta Stone

In a letter to Simone Weil [Kri05], André Weil explained a beautiful analogy, now known as *Weil's Rosetta Stone*. This establishes a three-way analogy between number fields, function fields, and Riemann surfaces.

The general idea is as follows. $\operatorname{Spec}\mathbb{Z}$ is some version of a curve, with points $\operatorname{Spec}\mathbb{F}_p$ associated to different primes. $\operatorname{Spec}\mathbb{Z}_p$ is a version of a disk around the point, and $\operatorname{Spec}\mathbb{Q}_p$ is a version of a punctured disk around that point. This is analogous to the usual picture of an algebraic curve.

Curve	$\operatorname{Spec}\mathbb{F}_q[t]$	$\operatorname{Spec}\mathbb{Z}$
Point	$\operatorname{Spec}\mathbb{F}_p$	$\operatorname{Spec}\mathbb{F}_p$
Disk	$\operatorname{Spec}\mathbb{F}_t[[t]]$	$\operatorname{Spec}\mathbb{Z}_p$
Punctured disk	$\operatorname{Spec}\mathbb{F}_q((t))$	$\operatorname{Spec}\mathbb{Q}_p$

In general, let $F/\mathbb{Q}$ be a number field. Then we can consider $\mathcal{O}_F$, and $\operatorname{Spec}\mathcal{O}_F$ has points corresponding to primes in $\mathcal{O}_F$. The analogy between number fields and function fields is as follows. Start with a smooth projective curve $C/\mathbb{F}_q$ over a finite field. Then the analogue to F is the field of rational functions, $\mathbb{F}_q(C)$. The analogue to $\mathcal{O}_F$ is the ring of regular functions, $\mathbb{F}_q[C]$. Finally points of $\operatorname{Spec}\mathcal{O}_F$ correspond to points of C.

Now we might want to replace C with a Riemann surface. So let $\Sigma/\mathbb{C}$ be a compact Riemann surface. Then primes in $\mathcal{O}_F$ (and so points of C) correspond to points of Σ. The field of meromorphic rational functions on Σ, $\mathbb{C}(\Sigma)$, is the analogue of F. To get an analogue of $\mathcal{O}_F$ we have to remove some points of Σ (we wouldn't get any functions on the compact curve). The point is that number fields have some points at ∞, so the analogue isn't really a compact Riemann surface, but with some marked points. So the analogue of $\mathcal{O}_F$ consists of functions on Σ which are regular away from these points.

This is summarized in table 1.2.

Table 1.2: Weil's Rosetta stone, as it was initially developed, establishes an analogy between these three columns. We will eventually refine this dictionary. Let $F/\mathbb{Q}$ be a number field, $C/\mathbb{F}_q$ be a smooth projective curve over a finite field, and let $\Sigma/\mathbb{C}$ be a compact Riemann surface. $\mathbb{F}_q(C)$ denotes the field of rational functions, $\mathbb{F}_q[C]$ denotes the ring of regular functions, and $\mathbb{C}(\Sigma)$ denotes the meromorphic rational functions on Σ.

Number fields	Function fields	Riemann surfaces
$F/\mathbb{Q}$	$\mathbb{F}_q(C)$	$\mathbb{C}(\Sigma)$
$\mathcal{O}_F$	$\mathbb{F}_q[C]$	f'ns regular away from marked points of Σ
$\operatorname{Spec}\mathcal{O}_F$	points of C	$x \in \Sigma$

Missing chip

Now we want to take the point of view that there was a chip missing from this Rosetta stone, and we were supposed to consider 3-manifolds rather than Riemann surfaces. The idea is that $\Sigma/\mathbb{C}$ really corresponds to $C/\overline{\mathbb{F}_q}$. This is manifested in the following way. To study points, we study maps:

$$\operatorname{Spec}\mathbb{F}_q \hookrightarrow C\ . \tag{1.23}$$

But from the point of view of étale topology, $\operatorname{Spec}\mathbb{F}_q$ is not really a point. It is more like a circle in the sense that

$$\operatorname{Gal}\left(\overline{\mathbb{F}_q}/\mathbb{F}_q\right) = \widehat{\mathbb{Z}} = \pi_1^{\text{étale}}(\operatorname{Spec}\mathbb{F}_q) \tag{1.24}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion. So it's better to imagine this as a modified circle, where this $\widehat{\mathbb{Z}}$ is generated by the Frobenius. There is always a map

$$\operatorname{Spec}\overline{\mathbb{F}_q} \to \operatorname{Spec}\mathbb{F}_q \tag{1.25}$$

and we can lift our curve to $\overline{\mathbb{F}}_q$. This corresponds to unwrapping these circle, i.e. replacing them by their universal cover. So their is some factor of $\mathbb{R}$ which doesn't play into the topology/cohomology. So we have realized that curves over $\mathbb{F}_q$ have too much internal structure to match with a Riemann surface.

Remark 5. The map $\operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is analogous to the usual n-fold cover of the circle.

To fix the Rosetta Stone, we replace a Riemann surface Σ by certain a Σ-bundle over S^1. Explicitly, if we have Σ and a diffeomorphism φ, we can form the mapping torus:

$$\Sigma \times I/\left((x,0) \sim (\varphi(x), 1)\right)\ . \tag{1.26}$$

The idea is that if we start with a curve over a finite field, the diffeomorphism φ is like the Frobenius.

Similarly $\operatorname{Spec}\mathcal{O}_F$ looks like a curve where each "point" carries a circle. So this is again some kind of 3-manifold.

Remark 6. These circles don't talk to one another because they all have to do with a Frobenius at a different prime. So they're less like a product or a fibration, and more like a 3-manifold with a foliation.

This fits with the existing theory of arithmetic topology, sometimes known as the "knots and primes" analogy. The theory was started in a letter from Mumford to Mazur, but can be attributed to many people such as Mazur [Maz73], Manin, Morishita [Mor10], Kapranov [Kap95], and Reznikov. The recent work [Kim15, CKK+19] of Minhyong Kim plays a central role.

Remark 7. Lots of aspects of this dictionary are spelled out, but one should be wary of using it too directly. Rather we should think of this as telling us that there are several classes of '3-manifolds': ordinary 3-manifolds, function fields over finite fields, and number fields.

Lecture 2; January 21, 2021

Updated Rosetta Stone

The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. In particular, we're thinking of $\operatorname{Spec}\mathcal{O}_F$ (e.g. $\operatorname{Spec}\mathbb{Z}$) as a 3-manifold, so for any prime p we have the loop $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathcal{O}_F$, which we can interpret as a knot in the 3-manifold. Let F_v be the completion of the local field F at the place v (e.g. $\mathbb{Q}_p$). Then $\operatorname{Spec} F_v$ turns out to be the boundary of a tubular neighborhood of the knot. The point is that if F_v is a non-Archimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) then the "fundamental group" is the Galois group, and it has a quotient:

$$\operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \; . \tag{1.27}$$

This group is called a *Baumslag-Solitar* group. Explicitly it is:

$$\operatorname{BS}(1,p) = \left\{\sigma, u \,\middle|\, \sigma u \sigma^{-1} = u^p\right\} \tag{1.28}$$

where we think of σ as the Frobenius, so corresponding to $\widehat{\mathbb{Z}}$, and u as the generator of $\mathbb{Z}_\ell$. The kernel is

$$p\text{-group} \times \prod_{\ell^r \neq \ell, p} \mathbb{Z}_{\ell^r} \hookrightarrow \operatorname{Gal}\left(\overline{F_v}/F_v\right) \twoheadrightarrow \mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}} \; . \tag{1.29}$$

This tells us that there is For $p = 1$, this group is $\mathbb{Z} \times \mathbb{Z} = \pi_1\left(T^2\right)$. For $p = -1$, this is the fundamental group of the Klein bottle. This is evidence that the étale fundamental group of $\operatorname{Spec} F_v$ looks like some kind of p-dependent version of the fundamental group of the torus. So we can think of $\operatorname{Spec} F_v$ as a 2-manifold (fibered over S^1).

After this discussion we can identify an updated (multi-dimensional) Rosetta Stone. In three-dimensions we have: $\operatorname{Spec}\mathcal{O}_F$, $C/\mathbb{F}_q$, and a mapping torus $T_\varphi(\Sigma)$. The first two comprise the *global arithmetic setting*. In two-dimensions we first have local fields, which come in two types. One is finite extensions $F_v/\mathbb{Q}_p$ and the other is $\mathbb{F}_q((t))$. Spec of either of these is "two-dimensional" and the latter is some kind of punctured disk D^*. These two comprise the *local arithmetic setting*. A curve $\overline{C/\mathbb{F}_q}$ over an algebraically closed field (of positive characteristic) and a Riemann surface (projective curve Σ over $\mathbb{C}$) are also both "two-dimensional". These comprise the *global geometric setting*. The only 4-manifolds we will consider are of the form $M^3 \times I$ or $M^3 \times S^1$ where M^3 is a three-dimensional object of arithmetic or geometric origin. This discussion is summarized in table 1.3.

Table 1.3: The columns correspond to the three aspects of Weil's Rosetta Stone, and the rows correspond to dimension. The four-dimensional objects we consider are just products of three-dimensional objects with S^1 or I. M^3 is a three-dimensional object of arithmetic or geometric origin. The three-dimensional objects are number fields, function fields and mapping tori of Riemann surfaces. F is a number field, φ is some diffeomorphism of Σ, and T_φ denotes the corresponding mapping torus construction. The two-dimensional objects are local fields and curves. F_v is a finite extension of $\mathbb{Q}_p$. The 1-dimensional objects are both versions of circles, and the 0-dimensional objects are points.

<table>
<tr><th>Dimension</th><th>Number fields</th><th>Function fields</th><th>Geometry</th></tr>
<tr><td>4</td><td colspan="3">$M^3 \times S^1$, $M^3 \times I$</td></tr>
<tr><td rowspan="2">3</td><td colspan="2">Global arithmetic</td><td>-</td></tr>
<tr><td>$\operatorname{Spec} \mathcal{O}_F$</td><td>$C/\mathbb{F}_q$</td><td>$T_\varphi(\Sigma)$</td></tr>
<tr><td rowspan="2">2</td><td colspan="2">Local arithmetic</td><td>Global geometric</td></tr>
<tr><td>$\operatorname{Spec} F_v$</td><td>$\operatorname{Spec} \mathbb{F}_q((t)) = D^*$</td><td>$\overline{C}/\overline{\mathbb{F}}_q, \quad \Sigma/\mathbb{C}$</td></tr>
<tr><td rowspan="2">1</td><td colspan="2">-</td><td>Local geometric</td></tr>
<tr><td colspan="2">$\operatorname{Spec} \mathbb{F}_q$</td><td>$D^*_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}((t))$,
$D^*_{\overline{\mathbb{F}}_q} = \operatorname{Spec} \overline{\mathbb{F}}_q((t))$</td></tr>
<tr><td>0</td><td colspan="2">$\operatorname{Spec} \overline{\mathbb{F}_q}$</td><td>$\operatorname{Spec} \mathbb{C}$</td></tr>
</table>

1.2.3 $\mathcal{A}$-side

The $\mathcal{A}$-side (or automorphic/magnetic side) TFT $\mathcal{A}_G$ is a huge machine which does many things, as in table 1.1. So far, the only recognizable thing is that it sends a 3-manifold M to some vector space $\mathcal{A}_G(M)$. We're thinking of $\operatorname{Spec}\mathcal{O}_F$ as a 3-manifold, and the assignment is the vector space we've been discussing:

$$\mathcal{A}_G(\operatorname{Spec}\mathcal{O}_F) = L^2(\mathcal{M}_{G,F}) \text{ or } H^*(\mathcal{M}_{G,F}) \ . \tag{1.30}$$

Remark 8. As suggested in eq. (1.19), note that $\Gamma\backslash G/K$ is a moduli space of something over the 3-manifold in question, not the 3-manifold itself.

1.2.4 Structure (reprise)

As it turns out, all the bells and whistles from the theory of automorphic forms in section 1.1.2 line up perfectly with the bells and whistles of TFT.

Number field

The assignment in eq. (1.30) formalizes the idea that we got a vector space $L^2(\mathcal{M}_{G,F})$ labelled by a group and a number field.

Conductor/ramification data

Recall the ramification data was a series of primes. This is manifested as a link (collection of knots) in the 3-manifold, where we allow singularities. These appear as *defects* (of codimension 2) in the physics. So the structure we saw before is manifested as defects of the theory.

Hecke algebra

The Hecke operators correspond to line defects (codimension 3) in the field theory. Physically this is "creating magnetic monopoles" alone some loop in spacetime.

Periods/states

These correspond to boundary conditions, i.e. codimension 1 defects.

Langlands functoriality

Passing from G to H can be interpreted as crossing a domain walls (also a codimension 1 defect).

1.2.5 $\mathcal{B}$-side

The $\mathcal{B}$-side (or spectral side) is the hard part from the point of view of number theory because Galois groups of number fields (and their representations) are very hard. I.e. the $\mathcal{B}$-side is the question, and the $\mathcal{A}$-side is the answer. But from the point of view of geometry, it is the other way around because fundamental groups of Riemann surfaces are really easy.

The $\mathcal{B}$-wide is about studying the algebraic geometry of spaces of Galois representations.

Recall that given a three-manifold (or maybe a number field F) the $\mathcal{A}$-side is concerned with the topology of the arithmetic locally symmetric space $\mathcal{M}_{G,F}$. $\mathcal{M}_{G,F}$ has to do with the geometry of F, so the $\mathcal{A}$-side is concerned with the topology of the geometry of F.

The $\mathcal{B}$-side concerns itself with the algebra of the topology of F. This means the following. For a manifold M (of any dimension), we can construct $\pi_1(M)$. Then the collection of rank n local systems on M is:

$$\operatorname{Loc}_n M = \{\pi_1(M) \to \mathrm{GL}_n \mathbb{C}\} \ . \tag{1.31}$$

A local system looks like a locally constant sheaf of rank n (or vector bundles with flat connection). These are sometimes called *character varieties*. Then we can study $\mathbb{C}[\operatorname{Loc}_n M]$. We can also replace GL_n with our favorite complex Lie group G to get:

$$\operatorname{Loc}_{G^\vee} M = \{\pi_1(M) \to G^\vee\} \ . \tag{1.32}$$

This depends only on the topology of M.

If we're thinking of a number field as a three-manifold, then π_1 is a stand-in for the Galois group so this is a space of representations of Galois groups. The TFT sends any three-dimensional M^3 to functions on $\operatorname{Loc}_{G^\vee}$:

$$\mathcal{B}_{G^\vee}\left(M^3\right) = \mathbb{C}[\operatorname{Loc}_{G^\vee} M] \ . \tag{1.33}$$

Remark 9. This side was a lot easier to write down than the $\mathcal{A}$-side, but if M is a number field, the Galois group is potentially very hard to understand. All the other bells and whistles are also easy to define here.

1.2.6 All together

In all of the setting in table 1.3, we can either make and automorphic measurement (attach $\mathcal{M}_{G,F}$ and study its topology) or we could take the Galois group (or π_1), construct a variety out of it, and study algebraic functions on it. The idea we will explain is that the Langlands program is an equivalence of these giant packages, but for "Langlands dual groups" G and $G^\vee$:

$$\mathcal{A}_G \simeq \mathcal{B}_{G^\vee} \ . \tag{1.34}$$

Remark 10. More is proven in the geometric setting than the arithmetic, but even geometric Langlands for a Riemann surface is still an open question.

This is really a conjectural way of organizing a collection of conjectures.

Chapter 2

Spectral decomposition

2.1 What is a spectrum?

Lecture 3; January 26, 2021

The basic idea is that we start in the world of geometry, meaning we have a notion of a "space" (e.g. algebraic geometry, topology, ...), and given one of these spaces X we attach some kind of collection of functions $\mathcal{O}(X)$. These functions can have many different flavors, but they always form some kind of commutative algebra, possibly with even more structure. We will access the geometry of spaces using these functions. The operation $\mathcal{O}$ turns out to be a functor, i.e. if we have a morphism $\pi\colon X \to Y$ of spaces, we get a pullback morphism $\pi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.

A fundamental question about this setup is to what degree we can reverse this operation. So starting with any commutative algebra, we would like to understand the extent to which we can we get geometry out of it. Category theory tells us that the right sort of thing to consider is a right adjoint to $\mathcal{O}$, which we call Spec. The fact that they form an adjunction means:

$$\mathrm{Map}_{\mathbf{Spaces}}(X, \operatorname{Spec} R) = \mathrm{Hom}_{\mathbf{Ring}^{\mathrm{op}}}(\mathcal{O}(X), R) = \mathrm{Hom}_{\mathbf{Ring}}(R, \mathcal{O}(X)) \ . \tag{2.1}$$

In the language of analysis, we might regard $\operatorname{Spec} R$ as a "weak solution" in the sense that it's a formal solution to the problem of finding a space associated to R. It is a functor assigning a set to any test space, but there is no guarantee that there is an honest space out there which would agree with it.

Remark 11. We might have to adjust the categories we're considering so that

$$\mathcal{O}(\operatorname{Spec} R) = R \ , \tag{2.2}$$

since this doesn't just fall out of the adjunction.

The point is that for some nice class of spaces, one might hope that we can recover a space from functions on that space: $X = \operatorname{Spec} \mathcal{O}(X)$.

Remark 12. The word spectrum is used in many places in mathematics. They are basically all the same, except spectra from homotopy theory.

2.1.1 Finite set

Let X be a finite set and k a field. Then the k-valued functions $\mathcal{O}(X)$ can be expressed as

$$\mathcal{O}(X) = \prod_{x \in X} k \tag{2.3}$$

which can be thought of as diagonal $X \times X$ matrices.

2.1.2 Compactly supported continuous functions

Gelfand developed the following version of this philosophy. For our category of spaces, consider the category of locally compact Hausdorff spaces X with continuous maps as the morphisms. For our space of functions we take $C_v(X)$, which is the space of continuous $\mathbb{C}$-valued functions which vanish at ∞. This is in the category of commutative C^*-algebras. These are Banach $\mathbb{C}$-algebras with a $*$ operation (to be thought of as conjugation) which is $\mathbb{C}$-antilinear and compatible with the norm. Given any commutative C^*-algebra A, the associated spectrum m-Spec A is called the *Gelfand spectral*, and as a set it consists of the maximal ideals in A. We can write this as:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) \tag{2.4}$$

i.e. the unitary 1-dimensional representations of A. The adjunction is then saying that:

$$\text{m-Spec}\, A = \mathrm{Hom}_{C^*}(A, \mathbb{C}) = \mathrm{Map}(\mathrm{pt}, \text{m-Spec}\, A) \ . \tag{2.5}$$

Theorem 1 (Gelfand-Naimark)**.** *C_v and* m-Spec *give an equivalence of categories.*

See e.g. [aHRW10] for more details on this theorem.

2.1.3 Measure space

There is a coarser version where we start with a measure space X, and take attach the bounded functions $L^\infty(X)$. This forms a commutative von Neumann algebra. Again this is an equivalence of categories.

2.1.4 Algebraic geometry

We will focus on the setting of algebraic geometry. The category on the commutative algebra side will just be the category **cRing** of commutative rings. The geometric side will be the category of locally ringed spaces. This just means that there is a notion of evaluation at each point.

The functor:

$$(X, \mathcal{O}_X) \mapsto \mathcal{O}(X) \tag{2.6}$$

has an adjoint:

$$\mathrm{Spec}\, R \leftarrow\!\shortmid R \ . \tag{2.7}$$

Affine schemes comprise the image of Spec:

$$\begin{tikzcd} \text{locally-ringed spaces} \arrow[r] & \mathbf{cRing} \arrow[dl, "\mathrm{Spec}"] \\ \text{affine schemes} \arrow[u, hook] & \end{tikzcd} \ . \tag{2.8}$$

Then it is essentially built into the construction that affine schemes are equivalent to commutative rings.

This doesn't really capture a lot of what we want to study in algebraic geometry, because one is usually interested in general schemes, which are locally ringed spaces that locally look affine. One way to deal with this is to really think of our geometric objects as:

$$\mathrm{Fun}\left(\mathbf{Ring}^{\mathrm{op}}, \mathbf{Set}\right) = \mathrm{Fun}\left(\mathbf{Aff}, \mathbf{Set}\right) \subset \text{Geometry} \; . \tag{2.9}$$

2.1.5 Topology

In homotopy theory we might start with the homotopy category of spaces and pass to some notion of functions, e.g. cohomology $H^*(X;\mathbb{Z})$. This sits in the category of graded commutative rings. This doesn't directly lead to a nice spectral theory, but it does if we remember a bit more structure. Instead, start with the category of spaces with continuous maps. Then we can take rational chains, $C^*_{\mathbb{Q}}$, and get a commutative differential graded $\mathbb{Q}$-algebra. This is the Quillen-Sullivan rational homotopy theory.[1] As it turns out, the category of simply connected spaces up to rational homotopy equivalence is equivalent to commutative differential graded algebras which are $\mathbb{Q}$ in degree 0 and 0 in degree 1.

2.2 Spectral decomposition

Let R be a commutative ring over some field k. Commutative rings usually arise via the study of modules over it. So let $V \in R$-**mod**, i.e. a map

$$R \to \mathrm{End}\left(V\right) \; . \tag{2.10}$$

Now we want to decompose V into some sheaf $\underline{V}$ over $\mathrm{Spec}\, R$. We use that R-**mod** is symmetric monoidal, so we have a tensor product $\otimes_R$ and we can define

$$\underline{V}\left(U\right) = V \otimes_R \mathcal{O}\left(U\right) \tag{2.11}$$

for open $U \subset \mathrm{Spec}\, R$. Then we can talk about the *support* of $v \in V$:

$$\mathrm{Supp}\left(v\right) \subset X \; . \tag{2.12}$$

Example 6. If X is finite, so $R = \prod_{x \in X} k$, then by asking for vectors supported at a single point, we get a decomposition $V = \oplus_{x\in X} V_x$. But to every point x, we get an evaluation morphism:

$$\lambda\colon R \xrightarrow{\mathrm{ev}_x} k \tag{2.13}$$

associated to the point x. Just like before we can think of this as a one-dimensional R-module, since $k = \mathrm{End}\left(k\right)$. Therefore, changing notation, we can write the decomposition as

$$V = \oplus_{\lambda \in X} V_\lambda w \; . \tag{2.14}$$

In this language the spaces in the decomposition are just the λ-eigenspaces:

$$V_\lambda = \mathrm{Hom}_{R\text{-}\mathbf{mod}}\left(k_\lambda, V\right) = \left\{v \in V \mid r \cdot v = \lambda\left(r\right) v\right\} \tag{2.15}$$

[1] This should also be attributed to Mandell. See also Yuan's paper [Yua19].

where k_λ is the one-dimensional module over R, where R acts via the map λ.

This description is via evaluation at a point, but points are both open and closed so we can also describe this as restriction:

$$V_\lambda = V \otimes_{\mathcal{O}(X)=R} k_\lambda \ , \tag{2.16}$$

so this vector spaces is realized as both a Hom and a tensor because these points happened to be both open and closed.

Define the category of quasi-coherent sheaves to be:

$$\begin{array}{ccc} \mathbf{QCoh}\,(X = \operatorname{Spec} R) & \coloneqq & R\text{-}\mathbf{mod} \\ \big\uparrow & & \\ \mathbf{Coh}\,(X) & \coloneqq & R\text{-}\mathbf{mod}_{\text{f.g.}} \end{array} \ . \tag{2.17}$$

Note that in general X is only locally of the form $\operatorname{Spec} R$, so $\mathbf{QCoh}\,(X)$ is only locally of the form R-**mod**.

2.3 The spectral theorem

2.3.1 Algebraic geometry version

Let V be a vector space and $M \in \operatorname{End} V$. We can think of $M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{End} V)$, but $\operatorname{End} V$ is not just a set. It is an associative algebra over k, so really

$$M \in \operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) \ . \tag{2.18}$$

This sets us up for an adjunction with the free k-algebra construction. The free algebra in one generator is just $k[x]$ so the adjunction says that:

$$\operatorname{Hom}_{\mathbf{Set}}(\mathrm{pt}, \operatorname{Forget} \operatorname{End} V) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(k[x], \operatorname{End} V) \ . \tag{2.19}$$

What we have seen here is that equipping V with some $M \in \operatorname{End} V$ is equivalent to making V a module over $k[x]$. I.e. equipping V with $M \in \operatorname{End} V$ is equivalent to V being global sections of some quasi-coherent sheaf on $\operatorname{Spec} k[x] = \mathbb{A}^1$. I.e. V spreads out over $\mathbb{A}^1$.

To make this precise, assume V is finitely generated, i.e. the sheaf $\underline{V}$ is coherent. Then V has a sort of decomposition as a quotient and a subspace:

$$V_{\text{torsion}} \hookrightarrow V \twoheadrightarrow V_{\text{tor. free}} \tag{2.20}$$

where

$$V_{\text{torsion}} = \bigcup_{\lambda \in \mathbb{A}^1} \{ v \in V \mid \operatorname{Supp} v = \lambda \} \ . \tag{2.21}$$

For general modules over a PID ($k[x]$ is a PID) we have:

$$V \simeq \underbrace{V_{\text{tor}}}_{\text{discrete spectrum}} \oplus \underbrace{V_{\text{free}}}_{\text{continuous spectrum}} \tag{2.22}$$

where

$$V_{\text{free}} = k[x]^{\oplus r} \tag{2.23}$$

and

$$V_{\mathrm{tor}} = \bigoplus_{\lambda \in \mathrm{Spec}} V_{\hat{\lambda}} \tag{2.24}$$

where $V_{\hat{\lambda}}$ is the subspace supported at λ. As it turns out

$$V_{\hat{\lambda}} = \bigoplus_i k\left[x\right] / \left(x-\lambda\right)^{\ell_i} \ , \tag{2.25}$$

i.e. for any element of this some power of $(x-\lambda)$ annihilates it, so these are generalized eigenspaces. This decomposition is precisely the Jordan normal form.

An eigenvector is some element $v \in V$ such that $\lambda v = xv$ (or by definition Mv). But this is the same as an element of:

$$\mathrm{Hom}_{k[x]}\left(k_\lambda, V\right) \ . \tag{2.26}$$

So this is what one might call a section supported "scheme-theoretically" at $\lambda \in \mathbb{A}^1$.

On the other hand, the fibers:

$$V \otimes_{k[x]} k_\lambda \tag{2.27}$$

are naturally quotients of V (rather than a sub), and so they're some kind of coeigenvectors.

In the continuous spectrum there are no eigenvectors: as a free module, $k\left[x\right]$ doesn't contain any eigenvectors.

Example 7. Consider the free case. It is sufficient to consider $V = k\left[x\right]$, since otherwise it is just a direct sum of copies of this. Then $\underline{X} = \mathcal{O}_{\mathbb{A}^1}$. There are no generalized eigenvectors (because $\left(x-\lambda\right)^N = 0$ for some $N \gg 0$ and some $\lambda \in \mathbb{A}^1$). There are lots of coeigenvectors, though. For any $\lambda \in \mathbb{A}^1$ we have a map

$$V \twoheadrightarrow k_\lambda \ . \tag{2.28}$$

This is a distribution, i.e. an element of

$$\mathrm{Hom}\left(V, k\right) \ . \tag{2.29}$$

So for every $\lambda \in \mathbb{A}^1$, we get

$$\mathrm{Hom}\left(V, k\right) \ni \delta_\lambda \colon V \twoheadrightarrow k_\lambda \ . \tag{2.30}$$

The basic example is $V = L^2\left(\mathbb{R}\right)$ and $M = x$. Then V_λ consists of functions supported at x, but there are none. We would like to say δ_λ, but this is not L^2. The dual operator $M^\vee = d/dx$ has eigenvectors which are roughly $e^{i\lambda x}$, but these are not L^2 either.

This is what continuous spectra look like. When you decompose functions on $\mathbb{A}^1$ under the action of x or d/dx, there is a sense in which it is a direct integral, which is different from a direct sum. The things you're integrating aren't actually subsets. So we can think of functions on $\mathbb{A}^1$ as being some kind of continuous direct sum of functions on a point, but those functions don't live as subspaces. In this case they lived as quotients, but not subspaces. This is simpler in the torsion-free case, but is a general feature of continuous spectra. This is not a weird/special fact about analysis, because we see it even at the level of algebra (polynomials).

2.3.2 Measurable version

Instead of the matrix M, we consider a self-adjoint operator A on a Hilbert space $V = \mathcal{H}$. Then von Neumann's spectral theorem tells us that there is a "sheaf" (projection valued measure) π_A on $\mathbb{R}$ and

$$A = \int_{\mathbb{R}} x d\pi_A \ . \tag{2.31}$$

A projective valued measure can be thought of as a sheaf $\underline{\pi_A}$ as follows. For $U \subset \mathbb{R}$ measurable, we attach the image under the projection:

$$\underline{\pi_A}(U) = \pi_A(U) \ . \tag{2.32}$$

So this is some kind of sheaf of Hilbert spaces. There is no topology to be compatible with, but it does satisfy the additivity property that the rule:

$$U \mapsto \langle w, \pi_A(U) v\rangle \tag{2.33}$$

defines a $\mathbb{C}$-valued measure on $\mathbb{R}$. So this is the version of a sheaf in the measurable world.

So now eq. (2.31) is saying that the Hilbert space $\mathcal{H}$ sheafifies over $\mathbb{R}$ in such a way that A acts by the coordinate function x, just like in the algebro-geometric setting above. Then the spectrum is a measurable subset

$$\operatorname{Spec}(A) \coloneqq \operatorname{Supp} \pi_A \subset \mathbb{R} \ . \tag{2.34}$$

Example 8. If $\mathcal{H}$ is finite-dimensional then the spectrum is a discrete set of points, and the decomposition is just into eigenspaces.

2.3.3 Homotopical version

We saw that we had a quasi-coherent sheaf in algebraic geometry, a projection-valued measure in measure theory, and now in algebraic topology we have the following. If $R = C^*(X)$, then

$$R\textbf{-mod} \hookrightarrow \operatorname{Loc}(X) \tag{2.35}$$

where $\operatorname{Loc}(X)$ consists of locally-constant complexes on X.

The basic idea is that if we have a model for cochains on X:

$$\to R^{\oplus j} \to R^{\oplus i} \to M \tag{2.36}$$

then we get a presentation of $\underline{M}$ by constant sheaves:

$$\to \underline{k_X}^{\oplus j} \to \underline{k_X}^{\oplus i} \to \underline{M} \tag{2.37}$$

where the key point is that

$$C^*(X) = \operatorname{End}^*\left(\underline{k_X}, \underline{k_X}\right) \ . \tag{2.38}$$

Or more directly we can define the sheaf to be:

$$\underline{M}(U) = M \otimes_{C^*(X)} C^*(X) \ . \tag{2.39}$$

Lecture 4; January 28, 2021

2.3.4 Physics interpretation: observables and states

We have seen that, starting with some flavor of commutative algebra A, we can construct a geometric object $\operatorname{Spec} A$. Then a module M over A gets spread out into a sheaf over $\operatorname{Spec} A$. The algebra A can be thought of as the *algebra of observables* of some physical system. Then the *space of states* forms a module over A, and so fits into this framework.

Let's back up. The general idea is that we're trying to get a grip on the geometry of this space via the functions on it, i.e. making observations. Recall that the defining property of $\operatorname{Spec} A$ is that whenever we have a space X and a map $A \to \mathcal{O}(X)$, where A is commutative, then we get a map

$$X \to \operatorname{Spec} A\ . \tag{2.40}$$

We can think of the map $A \to \mathcal{O}(X)$ as picking out some functions (observables) on the space which satisfy the relations of A. E.g. if we just have one function, this is a map from the space down to the line, and then the space will decompose over this. In general the space will decompose over a higher-dimensional base. So this is a way of measuring the space with functions.

Spectral decomposition of modules is a linearized version of this. We replace X by a linearized version of it, e.g. $\mathcal{O}(X)$, and this becomes a module over A. I.e. a module M over A is a linearized version of a map $X \to \operatorname{Spec} A$. If we just have a single function, then this corresponds to a map $X \to \operatorname{Spec} k[x] = \mathbb{A}^1$. Likewise, a single matrix (endomorphism of a vector space) gives rise to a sheaf over $\mathbb{A}^1$.

In quantum mechanics, we don't have a phase space. We only have a vector space $\mathcal{H}$ (some linearized version of the phase space), called the *Hilbert space of states*. Observables are operators on $\mathcal{H}$. In physics we're interested in reality, so we might insist on the condition that observables are self-adjoint:

$$\mathcal{O} = \mathcal{O}^*\ . \tag{2.41}$$

Typically we won't impose this condition. For an observable $\mathcal{O} \circlearrowright \mathcal{H}$, spectral decomposition tells us that $\mathcal{H}$ sheafifies (as a projection-valued measure) over $\mathbb{R}$. The base is $\mathbb{R}$ because this is Spec of the algebra generated by a single operator. This is the analogue of starting with a classical phase space M and a single observable $M \to \mathbb{R}$, and then decomposing M over $\mathbb{R}$.

A *state* is an element $|\varphi\rangle \in \mathcal{H}$. Given a state and an operator $\mathcal{O}$, this state becomes a section of the sheaf $\mathcal{H}$, i.e. we get an eigenspace decomposition of this vector. Given a section, the first thing we can ask for is the support. This is just where the measurement we made "lives".

We can do something more precise by using the norm. As it turns out, $\|\varphi\|^2$ is a probability measure on $\mathbb{R}$ which tells us where to expect the state to be located. For example, we can take the expectation value of the observable $\mathcal{O}$ in the state φ:

$$\langle \mathcal{O} \rangle_\varphi = \frac{\langle \varphi | \mathcal{O} | \varphi \rangle}{\langle \varphi | \varphi \rangle}\ . \tag{2.42}$$

This is a continuous version of

$$\frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda \in \operatorname{Spec} \mathcal{O}} \lambda \left\| \operatorname{Proj}_{\mathcal{H}_\lambda} |\varphi\rangle \right\|^2 = \frac{1}{\langle \varphi | \varphi \rangle} \sum_{\lambda, \psi_i} \lambda \langle \psi_i | \varphi \rangle |\psi_i\rangle \tag{2.43}$$

where ψ_i is a basis of eigenvectors.

Remark 13. To give a quantum-mechanical system, we also need to specify the *Hamiltonian* H. This is a specific observable (self-adjoint operator on $\mathcal{H}$) which plays the role of the energy functional. The eigenstates for H are the steady states of the system. This lets us spread $\mathcal{H}$ out over $\mathbb{R}$ to get the energy eigenstates. We will be working in the *topological* setting where $H = 0$, i.e. we're just looking at the 0 eigenspace. So this decomposition is kind of orthogonal to our interests.

Chapter 3

Fourier theory/abelian duality

We have seen that whenever we have a "spectral dictionary", we get a notion of spectral decomposition: modules become sheaves, where the notion of a sheaf depends on the context. This gives us a way of spreading out the algebra of modules over the geometry or topology of our space.

For this to be useful, we need interesting sources of commutative algebras. A natural source for commuting operators is when we have an abelian group G acting on a vector space V: given a morphism

$$\rho\colon G \to \operatorname{Aut}(V) \tag{3.1}$$

we get a family of operators $\{\rho(g)\}_{g\in G}$ and we can spectrally decompose V using these operators. This is what Fourier theory is about. So we're thinking of Fourier theory as some kind of special case of spectral decomposition.

3.1 Characters

Let V be a representation of an abelian group G, i.e. we have a map

$$\begin{aligned} G &\longrightarrow \operatorname{Aut}(X) \ \subset \ \operatorname{End}(V) \\ g &\longmapsto T_g \end{aligned} \tag{3.2}$$

such that $T_g T_h = T_{gh}$.

Example 9. If G acts on a space X, and V is functions on X, then we get an action of G on V.

Example 10. G always acts on itself, so therefore it acts on functions on G itself. This is the *regular representation*.

Now we want to spectrally decompose. First we need to know what the spectrum is, so we ask the following question.

Question 1. What are the possible eigenvalues?

Let $v \in V$ be an eigenvector:

$$g \cdot v = \chi(g)\, v \tag{3.3}$$

where

$$\chi \colon G \to \operatorname{Aut} \mathbb{C}V = \mathbb{C}^\times \subset \mathbb{C} \tag{3.4}$$

is a group homomorphism, i.e. a *character* of G. So the possible eigenvalues are the characters:

$$G^\vee = \{\text{characters}\} = \operatorname{Hom}_{\mathbf{Grp}}\left(G, \mathbb{C}^\times\right) \ . \tag{3.5}$$

This is the spectrum, i.e. we will be performing spectral decomposition over $G^\vee$.

3.2 Finite Fourier transform

Now let G be a finite group.[1] We will eventually assume G is abelian, but we don't need this yet. We want to see $G^\vee$ appear at the spectrum. For a complex representation V we have a group map $G \to \operatorname{Aut} V$, but the composition with the inclusion

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend right, "\text{monoid map}"'] & \operatorname{Aut} V \subset & \operatorname{End}(V) \end{tikzcd} \tag{3.6}$$

is a monoid map. In other words V gives rise to an element of

$$\operatorname{Hom}_{\mathbf{Monoid}}(G, \operatorname{Forget}(\operatorname{End} V)) \ . \tag{3.7}$$

Just like before, we have an adjunction:

$$\operatorname{Hom}_{\mathbf{Monoid}}(G, \operatorname{Forget}(\operatorname{End} V)) = \operatorname{Hom}_{\mathbb{C}\text{-}\mathbf{Alg}}(?, \operatorname{End} V) \ , \tag{3.8}$$

where the missing entry should be some kind of free construction. As it turns out, the answer is the *group algebra*:

$$? = \mathbb{C}G \ . \tag{3.9}$$

This is the algebra freely generated by scalar multiplication and sums of elements of G. Since the group is finite this is just:

$$\mathbb{C}G = \left\{ \sum_{g \in G} f(g) \cdot g \right\} \tag{3.10}$$

where $f \colon G \to \mathbb{C}$ is any function. Really we should think of f as a measure rather than a function. There is no difference when G is finite, but for any $g \in G$ would would like a canonical element

$$\delta_g = 1 \cdot g \in \mathbb{C}G \ , \tag{3.11}$$

which means the coefficients come from some f which is 1 at g and 0 elsewhere, which is not a function in general.

[1]This is a simplifying assumption so we don't need to worry about what "kind" of functions we're considering.

We can think of $\mathbb{C}G$ as being generated by the elements δ_g for $g \in G$. The algebra structure comes from convolution:

$$\delta_f * \delta_g = \delta_{fg} \ . \tag{3.12}$$

In general

$$f_1 * f_2 = \sum_g f_1(g)\, g * \sum_h f_2(h)\, h \tag{3.13}$$

$$= \sum_k \left(\sum_{gh=k} f_1(g)\, f_2(h) \right) k \tag{3.14}$$

$$= \sum_k \sum_g f_1(g)\, f_2\left(kg^{-1}\right) \cdot k \ . \tag{3.15}$$

We can express the convolution in terms of the multiplication map $\mu \colon G \times G \to G$ as follows. We have he two projections π_1 and π_2:

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu"] & & G \\ G & & G & \end{tikzcd} \ . \tag{3.16}$$

We can pull f_1 back along π_1 and f_2 back along π_2 to get a function on $G \times G$:

$$f_1 \boxtimes f_2 \coloneqq \pi_1^* f_1 \pi_2^* f_2 \ . \tag{3.17}$$

Then we can push this along μ, and the result is the convolution:

$$f_1 * f_2 = \mu_* \left(f_1 \boxtimes f_2\right) = \int_\mu f_1 \boxtimes f_2 \ . \tag{3.18}$$

The upshot is that we can define the group algebra in this way whenever we have things which can be pulled and pushed like this.

For $G \circlearrowright V$, we have extended this to an action of $\mathbb{C}G \circlearrowright V$. Then G is **abelian** iff $(\mathbb{C}G, *)$ is a **commutative** algebra. So now the fundamental object over which representation theory of G will sheafify is:

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) \ , \tag{3.19}$$

and as it turns out

$$\operatorname{Spec}\left(\mathbb{C}G, *\right) = G^\vee \ , \tag{3.20}$$

i.e.

$$\left(\mathbb{C}G, *\right) \simeq \left(\mathcal{O}\left(G^\vee\right), \cdot\right) \ . \tag{3.21}$$

This is a first version of the Fourier transform. The idea is that a map $\operatorname{Spec} k \to \operatorname{Spec} A$ is the same as a morphism $A \to k$, which is exactly a 1-dimensional representation of G, i.e. a character. Under this correspondence, the characters $\chi_t \in \mathbb{C}G$ for $t \in G^\vee$ correspond to points $\delta_t \in \mathcal{O}\left(G^\vee\right)$ for $t \in G^\vee$. Moreover, translation by g corresponds to multiplication by $g^\vee$, i.e. the character

$$t \mapsto \chi_t(g) \ . \tag{3.22}$$

We can rephrase this equivalence slightly to make it more evident that this is some version of the Fourier transform. If f is a function on G then we can write

$$f = \sum_{t \in G^\vee} \widehat{f}(t) \cdot \chi_t \ . \tag{3.23}$$

This is just expressing f in terms of the basis of characters. Then we can recover $\widehat{f}(t)$ as the coefficient of f in this orthonormal basis. We also have that

$$f * (-) = \widehat{f} \cdot (-) \ . \tag{3.24}$$

3.2.1 Secret symmetry

There is a secret symmetry here. $G^\vee$ is an abelian group itself under the operation of pointwise multiplication. I.e.

$$\chi_{t \cdot s} \coloneqq \chi_t \cdot \chi_s \ . \tag{3.25}$$

Call the corresponding abelian group the *dual group to G*.

To see that this is a good duality, note that there is a tautological map

$$\begin{aligned} G &\longrightarrow G^{\vee\vee} \\ g &\overset{\sim}{\longmapsto} \{\chi \mapsto \chi(g)\} \ , \end{aligned} \tag{3.26}$$

which turns out to be an isomorphism.

We could have set this up in a more symmetric way. We have two projections:

$$\begin{tikzcd} & G \times G^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G^\vee \end{tikzcd} \tag{3.27}$$

and there is a tautological object, called the universal character, living over $G \times G^\vee$:

$$\begin{array}{c} \chi(-,-) \\ \downarrow \\ G \times G^\vee \end{array} \ , \tag{3.28}$$

i.e. a function on $G \times G^\vee$ given by:

$$\chi(g,t) = \chi_t(g) = \chi_g(t) \ . \tag{3.29}$$

Then the Fourier transform of $f \in \mathrm{Fun}(G)$ is given by pulling up to $G \times G^\vee$, multiplying by χ, and then summing up by pushing forward by π_2:

$$f \mapsto \pi_{2*}(\pi_1^* f \cdot \chi) \ . \tag{3.30}$$

Explicitly the Fourier transform is:

$$\widehat{f}(t) = \sum_g f(g)\, \overline{\chi}(g,t) \tag{3.31}$$

$$f(g) = \sum_t \widehat{f}(t)\, \chi(g,t) \ . \tag{3.32}$$

We have simultaneously diagonalized the action of all $g \in G$ on $\mathrm{Fun}(G)$.

For any V with a G action, we get a $(\mathbb{C}G, *)$ action on V so V spectrally decomposes over $G^\vee$, i.e.

$$V = \bigoplus_{t \in G^\vee} V_{\chi_t} \ , \tag{3.33}$$

where G acts by the eigenvalue specified by χ_t on the subspace V_{χ_t}.

This gives a complete picture of the complex representation theory of finite abelian group. The exact same formalism works in any setting with abelian groups. We will focus on the setting of topological groups and algebraic groups. Everything will mostly look the same, with the difference being what kind of functions we consider.

3.3 Pontrjagin duality

Let G be a locally compact abelian (LCA) group.

Example 11. $\mathbb{Z}$, $\mathrm{U}(1)$, $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathbb{Q}_p^*$ are all (non-finite) examples.

Define the dual to be the collection of *unitary* characters

$$G^\vee = \mathrm{Hom}_{\mathbf{TopGrp}}(G, \mathrm{U}(1)) \ . \tag{3.34}$$

Remark 14. We shouldn't be too shocked by replacing $\mathbb{C}^\times$ by $\mathrm{U}(1) \subset \mathbb{C}^\times$. If G is finite, all of the character theory was captured by $\mathrm{U}(1)$ anyway.

3.3.1 Group algebra

The spectrum will again be Spec of the group algebra, but we need to determine the appropriate definition of the group algebra in this context.

Endow G with a Haar measure. Before we had the counting measure, and could translate freely between functions and measures (so, in particular, they could both push and pull). Now $L^1(G)$ has a convolution structure in exactly the same way as in eq. (3.18):

$$f_1 * f_2 = \int_{g \in G} f_1(h) f_2\left(gh^{-1}\right) dg \ . \tag{3.35}$$

Just as before, this convolution comes from an adjunction. I.e. it satisfies a universal property in the world of C^*-algebras. If we have a representation:

$$G \to \mathrm{End}(V) \tag{3.36}$$

then this will correspond to a morphism

$$\left(L^1(G), *\right) \to \mathrm{End}(V) \tag{3.37}$$

of C^*-algebra.

The spectrum is the Gelfand spectrum:

$$\text{m-Spec}\left(L^1(G), *\right) = G^\vee \ . \tag{3.38}$$

This is a version of the Fourier transform which says that:

$$\left(L^1(G), *\right) \simeq \left(C_v\left(G^\vee\right), \cdot\right) \ , \tag{3.39}$$

where C_v denotes functions vanishing at ∞. Another version says that there is a tautological map:

$$G \to G^{\vee\vee} \tag{3.40}$$

which is an isomorphism.

Again, this can be written in a symmetric way:

$$f \mapsto \pi_{2*}\left(\pi_1^* f \cdot \chi\right) \tag{3.41}$$

where

$$\begin{tikzcd} & \chi \arrow[d] & \\ & G \times G^{\vee} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G^{\vee} \end{tikzcd} \quad . \tag{3.42}$$

For any notion of functions or distributions on G, we can perform this Fourier transform operation. The question is, given the type of functions we feed in, what type of functions do we get in the other side? For L^2 functions we simply get:

$$L^2(G) \xrightarrow{\sim} L^2(G) \ . \tag{3.43}$$

Any of these notions of a Fourier transform have the same general features. Some of which are as follows.

(i) Translation by a group element becomes pointwise multiplication.

(ii) Convolution also becomes pointwise multiplication.

(iii) Characters correspond to points.

3.3.2 Fourier series

Take $G = \mathrm{U}(1)$. Then

$$G^{\vee} = \mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathrm{U}(1), \mathrm{U}(1)\right) = \mathbb{Z} \tag{3.44}$$

where $n \in \mathbb{Z}$ corresponds to

$$\left\{x \mapsto e^{2\pi i n x}\right\} \ . \tag{3.45}$$

Then the Fourier transform established an equivalence

$$L^2(\mathrm{U}(1)) \xrightarrow{\sim} L^2(\mathbb{Z}) = \ell^2 \ . \tag{3.46}$$

As before, this is symmetric. There is a universal character:

$$\chi\colon\ (x, n) \mapsto e^{2\pi i n x} \tag{3.47}$$

living over $\mathrm{U}(1) \times \mathbb{Z}$, and we have the usual projections:

$$\begin{tikzcd} & \mathrm{U}(1) \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \mathrm{U}(1) & & \mathbb{Z} \ . \end{tikzcd} \tag{3.48}$$

Then we can read it backwards. A character

$$\mathbb{Z} \to \mathrm{U}(1) \tag{3.49}$$

is determined by the image of $1 \in \mathbb{Z}$, so characters of $\mathbb{Z}$ are labelled by points of $\mathrm{U}(1)$.

In general Pontrjagin duality, G is compact iff $G^\vee$ is discrete. Concretely, for $n \in \mathbb{Z}$ we have

$$e^{2\pi inx} \in L^2 \tag{3.50}$$

because G is compact. Similarly, because $\mathbb{Z}$ is discrete,

$$\delta_n \in \ell^2 \ . \tag{3.51}$$

Lecture 5; February 2, 2021

3.4 Fourier transform

Recall that the Pontrjagin dual of $\mathrm{U}(1)$ is $\mathbb{Z}$. The characters $\mathrm{U}(1) \to \mathrm{U}(1)$ are given by $z \mapsto z^n$ for any $n \in \mathbb{Z}$. Similarly, the dual of $\mathbb{Z}$ is $\mathrm{U}(1)$.

We can replace $\mathrm{U}(1)$ by a torus T, which is defined as:

$$T = \mathbb{R}^d/\Lambda = \Lambda \otimes_{\mathbb{Z}} \mathrm{U}(1) \tag{3.52}$$

where $\Lambda \subset \mathbb{Z}^d$ is a full-rank lattice. The dual of T is the dual lattice:

$$T^\vee = \Lambda^\vee \ . \tag{3.53}$$

Similarly, the dual of the lattice Λ is the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathrm{U}(1) \ . \tag{3.54}$$

The classical Fourier transform takes G to be a real vector space. For $G = \mathbb{R}_x$ the dual is another copy of $\mathbb{R}$: $G^\vee = \mathbb{R}_t$. Performing the same operations as before, with universal character

$$\chi(x,t) = e^{2\pi ixt} \ , \tag{3.55}$$

we get the usual Fourier transform. For a general real vector space the dual is the dual vector space, and the character is

$$\chi(x,t) = e^{2\pi i\langle x,t\rangle} \ , \tag{3.56}$$

where the pairing is the usual one between the vector space and its dual.

Recall that in the context of the duality between $\mathrm{U}(1)$ and $\mathbb{Z}$, characters corresponded to points. This situation differs in the sense that characters are not L^2 anymore (since $\mathbb{R}$ is not compact, like $\mathrm{U}(1)$ is) and the points are not isolated (since $\mathbb{R}$ is not discrete like $\mathbb{Z}$). But we still have:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(t)\, e^{2\pi ixt} dt \ . \tag{3.57}$$

The operation of differentiation d/dx corresponds with multiplication by t on the other side. We should think of d/dx as an infinitesimal version of group translation, which went to multiplication before. So this is part of the same framework where group theory on one side goes to geometry on the other.

In general, let G be a Lie group with abelian Lie algebra $\mathfrak{g}$. We have a map from $\mathfrak{g}$ to vector fields on G:

$$\mathfrak{g} \to \mathrm{Vect}\, G \subset \mathrm{Diff}\,(G) \tag{3.58}$$

and there is an adjunction between

$$\text{Forget}\colon \mathbf{Alg}_{\text{Assoc.}} \to \mathbf{Lie\text{-}Alg} \tag{3.59}$$

and the functor

$$U\colon \mathbf{Lie\text{-}Alg} \to \mathbf{Alg}_{\text{Assoc.}} \tag{3.60}$$

which sends a Lie algebra to the *universal enveloping algebra*. $\mathfrak{g}$ is abelian so the universal enveloping algebra is just the symmetric algebra

$$U\,(\mathfrak{g}) = \mathrm{Sym}^*\, \mathfrak{g}\ . \tag{3.61}$$

$U\mathfrak{g}$ is now a commutative algebra acting on $C^\infty\,(G)$. Therefore we can spectrally decompose/sheafify over

$$\mathrm{Spec}\, U\mathfrak{g} = \mathfrak{g}^*\ . \tag{3.62}$$

E.g. for d/dx acting on $\mathbb{R}_x$,

$$\mathfrak{g}^* = \mathbb{R}_t = \mathrm{Spec}\, RR\,[d/dx = t]\ . \tag{3.63}$$

Example 12. The dual of $\mathrm{U}\,(1)$ is $\mathbb{Z} \subset \mathbb{R}_t$.

3.5 In quantum mechanics

Before we see where the Fourier transform comes up in quantum mechanics, we consider classical mechanics. If we want to a model a particle moving around in a manifold M, then the phase space is the cotangent bundle T^*M with positive coordinates q on M, and momenta coordinates p in the fiber direction. The observables form:

$$\text{functions on } M \otimes \mathrm{Sym}\, TM\ . \tag{3.64}$$

In quantum mechanics the space of states is replaced by $\mathcal{H} = L^2\,(M)$. The observables contain $\mathrm{Diff}\,(M)$, the differential operators on M. Inside of this we have two pieces:

$$\begin{tikzcd}[column sep=small] \text{f'ns on } M \arrow[dr, hook] & \\ & \mathrm{Diff}\,(M) \\ TM \arrow[ur, hook] & \end{tikzcd}\ , \tag{3.65}$$

with

$$p_j = i\frac{d}{dq_j}\ . \tag{3.66}$$

In classical mechanics the analogous pieces commute. But here they commute up to a term: a tangent vector $\xi \in TM$ and a function $f \in \mathrm{Fun}\,(M)$ must satisfy

$$\xi f = f\xi + \hbar f'\ . \tag{3.67}$$

To summarize, states look like "$\sqrt{\text{observables}}$". The position operators:

$$q_j \cdot (-) \tag{3.68}$$

are diagonalized, on the other hand the momentum operators act as derivates.

We can also pass to the momentum picture where we diagonalize the p_i's (derivatives) instead. For $M = \mathbb{R}^n$ we have a natural basis of invariant vector fields (this is the advantage of having a group). Now we can simultaneously diagonalize

$$p_j = i\frac{d}{dx_j} \tag{3.69}$$

to identify

$$L^2\left(\mathbb{R}^n_q\right) \simeq L^2\left(\mathbb{R}^n_p\right) \ , \tag{3.70}$$

which is the Fourier transform. One might say that this is identifying quantum mechanics for G with quantum mechanics for the Pontrjagin dual $G^\vee$. This is the one-dimensional case of abelian duality, or "one-dimensional mirror symmetry".

3.6 Cartier duality

This is the same Fourier theory we've been doing, but in the context of algebraic geometry, i.e. instead of continuous, etc. functions we're considering algebraic functions. We will eventually see that this duality shows up in physics (electric-magnetic duality), as well as number theory (class field theory).

To say what a group is in the world of algebraic geometry, we need to review the notion of the functor of points. To a variety X we can associate a functor $\mathbf{cRing} \to \mathbf{Set}$ by sending a ring R to

$$X(R) = \mathrm{Hom}\left(\operatorname{Spec} R, X\right) \ . \tag{3.71}$$

As it turns out, specifying this functor is equivalent to specifying X itself.

A variety G is an *algebraic group* if the associated functor of points $\mathbf{cRing} \to \mathbf{Set}$ lifts to a functor landing in groups:

$$\begin{tikzcd} \mathbf{cRing} \arrow[r] \arrow[dr, dashed] & \mathbf{Set} \\ & \mathbf{Grp} \arrow[u] \end{tikzcd} \tag{3.72}$$

i.e. that

$$G(R) = \mathrm{Hom}\left(\operatorname{Spec} R, G\right) \tag{3.73}$$

is a group.

Remark 15. Sometimes other things are assumed in the definition of an algebraic group, which we do not assume here.

Example 13. Consider $\mathbb{A}^1 = \mathbb{G}_a$. As a functor, this sends

$$R \mapsto (R, +) \ . \tag{3.74}$$

This is saying that $\mathrm{Map}\left(X, \mathbb{A}^1\right) = \mathcal{O}(X)$.

Example 14. Consider $\mathbb{A}^1 \setminus 0 = \mathbb{G}_m = \operatorname{Spec} k\left[t, t^{-1}\right]$. As a functor this sends

$$R \mapsto \left(R^\times, \cdot\right) \ . \tag{3.75}$$

Example 15. The integers form an algebraic group with functor of points given by:

$$\mathbb{Z} \colon R \mapsto (\mathbb{Z}, +) \ . \tag{3.76}$$

Let G be an abelian (algebraic) group. The *Cartier dual* of G is

$$G^\vee = \operatorname{Hom}_{\mathbf{Grp}_{\mathrm{Alg}}}\left(G, \mathbb{G}_m\right) \ . \tag{3.77}$$

where the dualizing object $\mathbb{G}_m$ comes from $\operatorname{Aut}_k = \mathbb{G}_m$.

Example 16. Let $G = \mathbb{Z}/n$. The Cartier dual is

$$\mathbb{Z}/n^\vee = \operatorname{Hom}\left(\mathbb{Z}/n, \mathbb{G}_m\right) \simeq \mu_n \tag{3.78}$$

where μ_n denotes the nth roots of unity. As a functor, this sends:

$$R \mapsto n\text{th roots of unity in } R \ . \tag{3.79}$$

Example 17. The dual of the integers is $\mathbb{Z}^\vee \simeq \mathbb{G}_m$, since $\mathbb{Z} \to \mathbb{G}_m$ is determined by the image of 1. Similarly:

$$\mathbb{G}_m^\vee = \operatorname{Hom}_{\mathbf{TopGrp}}\left(\mathbb{G}_m, \mathbb{G}_m\right) \simeq \mathbb{Z} = \left\{z \mapsto z^n\right\} \ . \tag{3.80}$$

Example 18. More generally, the dual to a lattice Λ will be the dual torus:

$$T^\vee = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m \ , \tag{3.81}$$

so (over $\mathbb{C}$) this looks roughly like $\left(\mathbb{C}^\times\right)^{\operatorname{rank}\Lambda}$. Similarly a torus T gets exchanged with the dual lattice $\Lambda^\vee$.

To avoid technicalities, assume G is finite.[2] Consider the collection of functions on G, $\mathcal{O}(G)$. Dualizing and taking Spec gives the dual group:

$$G^\vee = \operatorname{Spec} \mathcal{O}(G)^* \ . \tag{3.82}$$

Rather than talking about a group algebra, functions on G has the structure of a group coalgebra as follows. The multiplication map on G induces a coproduct on $\mathcal{O}(G)$:

$$\begin{gathered} G \times G \xrightarrow{\quad \mu \quad} G \\ \\ \mathcal{O}(G) \xrightarrow{\Delta := \mu^*} \mathcal{O}(G) \otimes \mathcal{O}(G) \ . \end{gathered} \tag{3.83}$$

This makes $\mathcal{O}(G)$ into a *Hopf algebra*, i.e. $\mathcal{O}(G)$ has a multiplication, and a comultiplication Δ. In fact, this is a finite-dimensional commutative and cocommutative Hopf algebras. The study of these Hopf algebras turns out to be equivalent to the study of finite abelian group schemes.

[2]Formally we're assuming that G is a finite abelian group scheme.

Example 19 ("Fourier transform")**.** Let $\operatorname{char} k = 0$. The Cartier dual of $\mathbb{G}_a$ is the formal completion of itself:

$$\mathbb{G}_a^\vee = \widehat{\mathbb{G}_a} \ , \tag{3.84}$$

where the formal completion of $\mathbb{G}_a$ is given by:

$$\widehat{\mathbb{G}_a} = \bigcup \operatorname{Spec} k\left[t\right]/\left(t^n\right) \ . \tag{3.85}$$

The Cartier dual of an n-dimensional vector space V is the completion of the dual:

$$V^\vee = \widehat{V^*} \ . \tag{3.86}$$

The character of

$$\mathbb{G}_a = \operatorname{Spec} k\left[x\right] \tag{3.87}$$

should be

$$e^{xt} = \sum \frac{(xt)^n}{n!} \ , \tag{3.88}$$

but we need this to be a finite sum. This makes sense if t is nilpotent, i.e. there is $N \gg 0$ such that $t^N = 0$.

So when this is true, i.e. t is "close to 0", the function $e^{\langle x,t \rangle}$ is well-defined for $x \in V$ and $t \in V^*$, and is the character for V.

Similarly the dual of the completion is $\mathbb{G}_a$:

$$\mathbb{G}_a = \widehat{\mathbb{G}_a}^\vee \ . \tag{3.89}$$

Recall that Fourier duality exchanges

$$\left(\operatorname{Fun}\left(G\right), *\right) \simeq \left(\operatorname{Fun}\left(G^\vee\right), \cdot\right) \ . \tag{3.90}$$

Then $\mathbf{Rep}\left(G\right)$ became spectrally decomposed over $G^\vee$. In algebraic geometry, a representation V is a *comodule* for $\mathcal{O}\left(G\right)$. I.e. we have a map $G \times V \to V$, and passing to functions gives us a map

$$V \to \mathcal{O}\left(G\right) \otimes V \ . \tag{3.91}$$

Furthermore,

$$\mathbf{Rep}\left(G\right) \simeq \mathcal{O}\left(G\right)\textbf{-comod} \simeq \mathcal{O}\left(G^\vee\right)\textbf{-mod} \simeq \mathbf{QC}\left(G^\vee\right) \ . \tag{3.92}$$

3.6.1 Fourier series examples

Example 20. For $G = \mathbb{Z}$, the category of representations is given by

$$\mathbf{Rep}\left(\mathbb{Z}\right) = k\left[z, z^{-1}\right]\textbf{-mod} \tag{3.93}$$

where z is the action of $1 \in \mathbb{Z}$. This action must be invertible, which is why z^{-1} is included. Then the duality tells us that:

$$\mathbf{Rep}\left(\mathbb{Z}\right) = k\left[z, z^{-1}\right]\textbf{-mod} = \mathcal{O}\left(\mathbb{G}_m\right)\textbf{-mod} = \mathbf{QC}\left(\mathbb{G}_m\right) \ . \tag{3.94}$$

Example 21. A vector space and an endomorphism (matrix) gives us

$$\mathbf{QC}\left(\mathbb{A}^1 = \operatorname{Spec} k\left[z\right]\right) \tag{3.95}$$

but if we have an automorphism (invertible matrix) then we get

$$\mathbf{Rep}\left(\mathbb{Z}\right) \leftrightarrow \mathbf{QC}\left(\mathbb{A}^1 \setminus \{0\}\right) \ . \tag{3.96}$$

Example 22. In algebraic geometry

$$\mathbf{Rep}\left(\mathbb{G}_m\right) = \mathbb{Z}\text{-graded vector space} = \mathbf{QC}\left(\mathbb{Z}\right) \tag{3.97}$$

where

$$V \simeq \bigoplus_{n\in\mathbb{Z}} V_n \tag{3.98}$$

and $z \in \mathbb{G}_m$ acts on V_n by z^n.

Example 23 (Topological example)**.** The following is an example of Fourier series from topology. Let M^3 be a compact oriented three-manifold.[3] Let

$$G = \operatorname{Pic} M^3 \tag{3.99}$$

consist of complex line bundles (or $\mathrm{U}(1)$-bundles) on M^3 up to isomorphism. This forms an abelian group under tensor product. We can think of this as:

$$G = \operatorname{Map}\left(M^3, B\,\mathrm{U}\left(1\right)\right) \tag{3.100}$$

where $B\,\mathrm{U}(1)$ denotes the classifying space of $\mathrm{U}(1)$. Up to homotopy we can think of $B\,\mathrm{U}(1)$ as:

$$B\,\mathrm{U}\left(1\right) \simeq \mathbb{CP}^\infty \simeq K\left(\mathbb{Z}, 2\right) \ . \tag{3.101}$$

For $\mathcal{L} \in G$, we can attach the first Chern class:

$$c_1\left(\mathcal{L}\right) \in H^2\left(M, \mathbb{Z}\right) \ , \tag{3.102}$$

which is a complete invariant of the line bundle. So we can take

$$G = \Lambda = H^2\left(M, \mathbb{Z}\right) \ . \tag{3.103}$$

The Cartier dual is

$$G^\vee \simeq \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m = \operatorname{Hom}\left(H_1\left(M, \mathbb{Z}\right), \mathbb{G}_m\right) = \operatorname{Hom}\left(\pi_1\left(M\right), \mathbb{G}_m\right) \tag{3.104}$$

where the last line follows from the fact that $H_1 = \pi_1^{\mathrm{ab}}$, and Hom from the abelianization is the same as Hom from the whole thing. But this is just flat $\mathbb{C}^\times$-bundles on M, i.e.

$$G^\vee = \operatorname{Loc}_{\mathbb{C}^\times}\left(M\right) \ . \tag{3.105}$$

This shouldn't be that surprising since G looks something like $\mathbb{Z}^n$, and the Cartier dual $G^\vee$ looks like $\left(\mathbb{C}^\times\right)^n$.

We can also replace line bundles by torus bundles, i.e. we can pass from $\mathrm{U}(1)$ to $\mathrm{U}(1)^n \simeq T$. Then

$$\operatorname{Bun}_T\left(M\right) \leftrightarrow \operatorname{Loc}_{T^\vee} M \ . \tag{3.106}$$

where $\operatorname{Loc}_{T^\vee} M$ consists of isomorphism classes of flat $T^\vee$-bundles over M. The LHS still looks like a lattice Λ, and the RHS still looks like a torus $\left(\mathbb{C}^\times\right)^n$.

transition between lectures 2021

Example 24 (Fourier series)**.** Take $\mathbb{Z}$ to be my abelian group G. Then we have some notions of a dual. One is

$$\mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathbb{Z}, \mathrm{U}\left(1\right)\right) = \mathrm{U}\left(1\right) \tag{3.107}$$

another is

$$\mathrm{Hom}_{\mathbf{TopGrp}}\left(\mathbb{Z}, \mathbb{G}_m\right) = \mathbb{G}_m \ , \tag{3.108}$$

i.e. $\mathbb{C}^\times$ if we're over $\mathbb{C}$.

The difference is the kind of function theory we're consider. In the first case we have an equivalence

$$\ell^2 = L^2\left(\mathbb{Z}\right) \simeq L^2\left(\mathrm{U}\left(1\right)\right) \tag{3.109}$$

and in the second case we have

$$\mathbb{C}\mathbb{Z} \simeq \mathcal{O}\left(\mathbb{C}^\times\right) \ . \tag{3.110}$$

Notice that

$$\mathcal{O}\left(\mathbb{C}^\times\right) = \mathbb{C}\left[z, z^{-1}\right] \subset L^2\left(\mathrm{U}\left(1\right)\right) \ , \tag{3.111}$$

and

$$\mathbb{C}\mathbb{Z} \subset \ell^2 \ . \tag{3.112}$$

So the algebraic version kind of its inside of the analytic story.

Digression 1*.* For any group G, there are many different version of representation theory, which we can think of as coming from different versions of the group algebra. On the dual side, this corresponds to a different structure on $G^\vee$.

For example, if we take the group algebra in the sense of a von Neumann algebra, then we get $G^\vee$ as a measure space. If we start with the group algebra as a C^*-algebra, then we get $G^\vee$ as a (locally compqct) topological space. If we start with a discrete/algebraic group algebra, then we get $G^\vee$ as an algebraic variety.

3.7 Pontrjagin-Poincaré duality

Now we want to give a series of examples of Pontrjagin/Fourier duality which are basically the same example, but they will be more interesting because we will introduce some topology. A summary of all of these duality statements is in table 3.1.

Let M^n be a compact oriented manifold. The cohomology group of M will be the abelian groups we study the duality theory of. This duality says that for G an abelian group

$$H^i\left(M, G\right)^\vee \simeq H^{n-i}\left(M, G^\vee\right) \ . \tag{3.113}$$

To convince ourselves that this is on the correct footing, we can check that the following is a nondegenerate pairing:

$$H^i\left(M, G\right) \otimes H^{n-i}\left(M, G^\vee\right) \to H^n\left(M, G \otimes G^\vee\right) \to H^n\left(M, \mathrm{U}\left(1\right)\right) \to \mathrm{U}\left(1\right) \tag{3.114}$$

where the last map is integration.

The "dimension" of the following examples will eventually correspond to the dimension of the appropriate field theory, i.e. the geometric objects we consider are actually a dimension less than the listed dimension so as to get a vector space. The side with $\mathbb{Z}$ coefficients will be called the A-side, and the side with $\mathbb{C}^\times$ coefficients will be called the B-side.

[3]This will be the three-manifold on which we do electromagnetism. This Cartier duality will give us electric-magnetic duality. If we think of a number field as a three-manifold, then this duality fits into the framework of class field theory.

3.7.1 One dimension

In this case we just have that the A-side $\mathrm{Map}\,(\mathrm{pt}, \mathbb{Z}) = \mathbb{Z}$ gets exchanged with the B-side $\mathrm{Map}\,(\mathrm{pt}, \mathbb{C}^\times) = \mathbb{C}^\times$. We can think of this as saying that quantum mechanics on $\mathbb{Z}$ is QM of $\mathbb{C}^\times$ in the sense that some kind of spaces of functions on these are identified by Fourier theory.

3.7.2 Two dimensions

One duality in two dimensions is between:

$$H^0\left(S^1, \mathbb{Z}\right) \qquad \text{and} \qquad H^1\left(S^1, \mathbb{C}^\times\right) \ , \tag{3.115}$$

in the sense that functions on these are identified. Note that we can write the A-side as:

$$H^0\left(S^1, \mathbb{Z}\right) = \pi_0\left(\mathrm{Map}\left(S^1, \mathbb{Z}\right)\right) = \left[S^1, \mathbb{Z}\right] \ . \tag{3.116}$$

The B-side is

$$H^1\left(S^1, \mathbb{C}^\times\right) = \left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right| \ , \tag{3.117}$$

where $\left|\mathrm{Loc}_{\mathbb{C}^\times} S^1\right|$ denotes the underlying space of $\mathrm{Loc}_{\mathbb{C}^\times} S^1$, which is a stack.

The A-side just looks like $\mathbb{Z}$, and the B-side just looks like $\mathbb{C}^\times$. This is the sense in which we're doing the "same" example as before.

The A-side is the theory of maps to $\mathbb{Z}$, and the B-side is some kind of $\mathrm{U}\,(1)$ Gauge theory. By Gauge theory we just mean that we're dealing with local systems or principal bundles.

Digression 2 (Local systems). Let X be a topological space. Let G be a group (with the discrete topology). Then $\mathrm{Loc}_G X$ consists of principal G-bundles on X. These are sheaves which are locally isomorphic to the constant sheaf $\underline{G}$. We can also think of this as:

$$\mathrm{Loc}_G X = \{\pi_1(X) \to G\} \tag{3.118}$$

where, given a principal G-bundle, the associated representation of π_1 is the monodromy representation.

If we only care about isomorphism classes, then this is

$$H^1(X, G) = |\mathrm{Loc}_G X| \ . \tag{3.119}$$

3.7.3 Two dimensions

Another version of this duality in two-dimensions has

$$H^1\left(S^1, \mathbb{Z}\right) \tag{3.120}$$

on the A-side. On the B-side we have

$$H^0\left(S^1, \mathbb{C}^\times\right) \ , \tag{3.121}$$

which is just the maps to $\mathbb{C}^\times$ with the discrete topology.

On the A-side:

$$H^1\left(S^1, \mathbb{Z}\right) = \left[S^1, B\mathbb{Z}\right] \ , \tag{3.122}$$

and

$$B\mathbb{Z} = K(\mathbb{Z}, 1) = S^1 \ . \tag{3.123}$$

So the A-side is maps to S^1, and the B-side is also maps to $\mathbb{C}^\times$ (some version of S^1). These two spaces consist of what are sometimes called *scalar fields*.

This duality theory is known as T-duality or mirror symmetry. It is also called $R \leftrightarrow 1/R$ duality.

Again, as a group, the A-side is just $\mathbb{Z}$ and the B-side is just some version of S^1. One thing that is useful, in all of these examples, is to replace $\mathbb{Z}$ by a lattice Λ. The duality turns out to be between

$$H^1\left(S^1, \Lambda\right) \qquad \text{and} \qquad H^0\left(S^1, T^\vee_{\mathbb{C}}\right) \ , \tag{3.124}$$

where we can think of

$$H^1\left(S^1, \Lambda\right) = \left[S^1, T\right] \tag{3.125}$$

where T is the compact lattice

$$\Lambda \otimes_{\mathbb{Z}} S^1 \ . \tag{3.126}$$

The upshot is that we are exchanging maps into the torus with maps into the dual torus.

3.7.4 Three dimensions

Let Σ be a compact oriented two-manifold. Then we have a duality between

$$H^1(\Sigma, \mathbb{Z}) \qquad \text{and} \qquad H^1\left(\Sigma, \mathbb{C}^\times\right) \ . \tag{3.127}$$

The B side is $|\mathrm{Loc}_{\mathbb{C}^\times} \Sigma|$, so the B-side is three-dimensional gauge theory (i.e. principal bundles are involved). The A-side is

$$H^1(\Sigma, \mathbb{Z}) = \left[\Sigma, B\mathbb{Z} = S^1\right] \ , \tag{3.128}$$

so this is a scalar field. So maps into a circle get exchanged with principal bundles.

3.7.5 Three dimensions

Alternatively, in three dimensions, we get a duality between

$$H^2(\Sigma, \mathbb{Z}) \qquad \text{and} \qquad H^0\left(\Sigma, \mathbb{C}^\times\right) \ . \tag{3.129}$$

We can think of

$$H^2(\Sigma, \mathbb{Z}) = H^1(\Sigma, \mathrm{U}(1)) = [\Sigma, B\,\mathrm{U}(1) = K(\mathbb{Z}, 2)] \tag{3.130}$$

as consisting of $\mathrm{U}(1)$-principal bundles (i.e. line bundles) so this is some kind of gauge theory. On the other hand,

$$H^0\left(\Sigma, \mathbb{C}^\times\right) = \left[\Sigma, \mathbb{C}^\times\right] \tag{3.131}$$

consists of scalar fields. So principal bundles got exchanged with maps into a circle. Again, this can be upgraded to tori and lattices.

Digression 3 (Dictionary between vector bundles and principal bundles)*.* Let $\mathcal{P} \to X$ be a $\mathrm{U}(1)$-bundle. Then I can form a line bundle by taking the product $\mathcal{P} \times_{\mathrm{U}(1)} \mathbb{C}$. In general if I have a principal G-bundle I have a representation of the group, so I get a vector bundle.

Conversely, if $\mathcal{L} \to X$ is a line bundle we can take Hom with the trivial bundle: $\mathrm{Hom}(\mathcal{L}, X \times \mathbb{C})$ and this is a principal $G\,\mathrm{U}(1)$-bundle.

Table 3.1: Summary of examples of Pontrjagin-Poincaré duality in various dimensions.

dimension	A-side	B-side
1	$\mathrm{Map}\,(\mathrm{pt},\mathbb{Z})=\mathbb{Z}$	$\mathrm{Map}\,(\mathrm{pt},\mathbb{C}^\times)=\mathbb{C}^\times$
2	$H^0\left(S^1,\mathbb{Z}\right)$	$H^1\left(S^1,\mathbb{C}^\times\right)=\left\lvert\mathrm{Loc}_{\mathbb{C}^\times}S^1\right\rvert$
2	$H^1\left(S^1,\mathbb{Z}\right)=\left[S^1,S^1\right]$	$H^0\left(S^1,\mathbb{C}^\times\right)$
3	$H^1\left(\Sigma,\mathbb{Z}\right)=\left[\Sigma,S^1\right]$	$H^1\left(\Sigma,\mathbb{C}^\times\right)=\lvert\mathrm{Loc}_{\mathbb{C}^\times}\Sigma\rvert$
3	$H^2\left(\Sigma,\mathbb{Z}\right)=\left[\Sigma,B\,\mathrm{U}\,(1)\right]$	$H^0\left(\Sigma,\mathbb{C}^\times\right)$

3.7.6 Four dimensions

The case of four dimensions is important because both sides will be gauge theories. Assume there is no torsion in our cohomology. The duality is between:

$$H^2\left(M^3,\mathbb{Z}\right) \qquad \text{and} \qquad H^1\left(M^3,\mathbb{C}^\times\right)\ . \tag{3.132}$$

The B-side can be written as

$$H^1\left(M^3,\mathbb{C}^\times\right)=\lvert\mathrm{Loc}_{\mathbb{C}^\times}M\rvert\ , \tag{3.133}$$

i.e. some kind of gauge theory since we're dealing with $\mathbb{C}^\times$-bundles. The A-side is

$$[M,K\,(\mathbb{Z},2)] \tag{3.134}$$

and as a homotopy type:

$$K\,(\mathbb{Z},2)=B\,\mathrm{U}\,(1)=\mathbb{CP}^\infty\ . \tag{3.135}$$

So the A-side is

$$\begin{aligned}[M,K\,(\mathbb{Z},2)] &= \pi_0\,\mathrm{Map}\,(M,\mathbb{CP}^\infty) && (3.136)\\ &= \pi_0\,(\text{line bundles on } M) && (3.137)\\ &= \text{line bundles}/\sim\ , && (3.138)\end{aligned}$$

i.e. line bundles up to isomorphism, so also some kind of gauge theory.

Then the equivalent vector spaces are functions on the two sides. The formal statement is that finitely supported functions (or locally constant if we're not thinking of π_0 of the space) on the A-side are equivalent to algebraic functions on the B-side.

A summary of all of these duality statements is in table 3.1.

Chapter 4

Electric-magnetic duality

The idea is that these duality statements we have seen are a shadow of electric-magnetic duality. We will follow Witten [DEF+99], Freed [Fre00], and Freed-Moore-Segal [FMS07].

4.1 Classical field theory

4.1.1 dimension 2

We will first study classical field theory in dimension 2. Let Σ be a Riemannian 2-manifold. Consider a periodic scalar field, i.e. a smooth map:

$$\varphi \in \mathrm{Map}_{C^\infty}\left(\Sigma, S^1\right) \ . \tag{4.1}$$

Now there is a natural classical field equation for this to satisfy: the harmonic map equation. It is convenient to introduce the 1-form $u = d\varphi$ on Σ, so we can write down the set of equations:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{4.2}$$

The first one is automatically satisfied, since u is already exact. The second equation is equivalent to $\star d \star u = 0$, which is equivalent to φ being harmonic.

We write eq. (4.2) like this because they are symmetric under sending $u \rightsquigarrow \star u$. In other words, the theory of a scalar φ (where $d\varphi = u$) is the same as the theory associated to some $\varphi^\vee$ (where $d\varphi^\vee = \star u$). If we were keeping track of metrics, then we would be studying harmonic maps into a circle of some radius R, and the dual theory is studying harmonic maps into a circle of radius $1/R$. This is $R \leftrightarrow 1/R$ duality.

Remark 16. For differential forms, Poincaré duality is realized by the Hodge star $\star$, so we are not so far from where we were before.

4.1.2 Three-dimensions

In dimension three, we can do the same trick. Let $\varphi \in \mathrm{Map}_{C^\infty}\left(\Sigma, S^1\right)$ be a scalar field, and let $u = d\varphi$. Again we write the harmonic map equation in this nice form:

$$\begin{cases} du = 0 \\ d \star u = 0 \end{cases} . \tag{4.3}$$

Now when we pass from u to $\star u$, we're passing from a 1-form to a 2-form so it's not even the same kind of beast anymore. Write $F = \star u$. Locally, we can write

$$F = dA \tag{4.4}$$

for A some 1-form, and we get the equations:

$$\begin{cases} dF = 0 \\ d * F = 0 \end{cases} , \tag{4.5}$$

which are the three-dimensional Maxwell equations. So the theory with a scalar field φ is dual to a theory with a field F satisfying Maxwell's equations.

4.2 Four-dimensions/Maxwell 101

The *electric field* is a 1-form $E \in \Omega^1\left(\mathbb{R}^3\right)$, and the magnetic field is a 2-form $B \in \Omega^2\left(\mathbb{R}^3\right)$. Relativistically, it is better to think of a composite beast, called the field strength:

$$F = B - dt \wedge E \in \Omega^2\left(\mathbb{M}^4\right) . \tag{4.6}$$

where $\mathbb{M}^4$ is Minkowski space. I.e. we need to turn E into a 2-form, so we wedge it with time so we can subtract it from B.

Maxwell's equations in a vacuum are:

$$\begin{cases} dF = 0 \\ d \star F = 0 \end{cases} . \tag{4.7}$$

Observe that this is symmetric under exchanging F and $\star F$. Some algebra reveals that:

$$\star F = B^\vee - dt \wedge E^\vee , \tag{4.8}$$

where $B^\vee$ is the three-dimensional hodge star of E:

$$B^\vee = - \star_3 E \tag{4.9}$$

and similarly

$$E^\vee = \star_3 B . \tag{4.10}$$

The upshot is that Maxwell's equations are symmetric under exchanging E with B.

If we're not in a vacuum, i.e. we have some currents, then Maxwell's equations become:

$$\begin{cases} dF = j_B \\ d \star F = j_E \end{cases} \tag{4.11}$$

where j_B is the magnetic current and j_E is the electric current. These are three-forms $j_B, j_E \in \Omega^3\left(\mathbb{M}^4\right)$.

The meaning of these currents is that they're related to the total charge:

$$\int_{M^3} j_B = Q_B \qquad \int_{M^3} j_E = Q_E \tag{4.12}$$

where we're taking $\mathbb{M}^4 = M^3 \times \mathbb{R}$. If we use Stokes' theorem, we get Gauss' law. This says that for a closed surface $\Sigma \subset M^3$, the magnetic flux can be expressed as:

$$b_\sigma = \int_\Sigma F = \# Q_{B,\Sigma} \tag{4.13}$$

for some scalar $\#$, where $Q_{B,\Sigma}$ is the total magnetic charge in Σ. Similarly, the electric flux is:

$$e_\Sigma = \int_\Sigma \star F = \# Q_{E,\Sigma} \ . \tag{4.14}$$

If you've ever learned about electromagnetism, you probably remember that there is a 0 on the RHS of (4.13). This is because, in reality, there don't seem to be any magnetic monopoles, i.e. $Q_B = 0$ and Gauss' law says that $\int_\Sigma F = 0$. Because of this property, we can introduce A, the electromagnetic potential by writing:

$$F = dA \ . \tag{4.15}$$

Note that this is breaking the symmetry between electricity and magnetism since $dF = 0$ is automatically satisfied. In particular this means that $j_B = 0$. Recall we should consider

$$\nabla = d + A \ , \tag{4.16}$$

regarded as a connection on a $\mathrm{U}(1)$-bundle on M.

The field strength F was something meaningful, but the potential A has some kind of ambiguity, since it is a sort of anti-derivative of F. This ambiguity is captured by the gauge transformations

$$\nabla \rightsquigarrow g^{-1} \nabla g \tag{4.17}$$

where $g \colon M \to \mathrm{U}(1)$. Similarly

$$A \mapsto A + g^{-1} dg \ , \tag{4.18}$$

i.e. it shifts A by some derivative, which **does not affect** F.

In fact, this is something which can be experimentally measured. This is called the *Bohm-Aharonov effect*. Even when $F = 0$ (flat connection, i.e. $E = 0$, $B = 0$), the monodromy of this connection is observable, i.e. a charged particle acquires a phase when it travels along loops.

Now we can think of this as a connection on any oriented Riemannian four-manifold. When we pass from F to $\nabla = d + A$, this implements Dirac charge quantization, which roughly says that the charges of elementary particles have to be integer valued (up to some renormalization). Let $M^4 = M^3 \times \mathbb{R}$. So now ∇ is a connection on some arbitrary line bundle $\mathcal{L}$ and F is a 2-form just like before. Even without charged particles, we still have fluxes:

$$b_\Sigma = \frac{1}{2\pi i} \int_\sigma F = \langle c_1(\mathcal{L}), [Sigma] \rangle \tag{4.19}$$

where this is the pairing between H^2 and H_2 since

$$c_1(\mathcal{L}) \in H^2(M, \mathbb{Z}) \qquad\qquad [\Sigma](\mathcal{L}) \in H_2(M, \mathbb{Z}) \tag{4.20}$$

for $\Sigma \subset M^3$ a closed surface. Similarly

$$e_\Sigma = \int_\Sigma \star F \ . \tag{4.21}$$

4.3 Quantum field theory

We will work in the Hamiltonian formalism. Roughly speaking, quantum field theory (QFT) on $M^{d-1} \times \mathbb{R}$ (thought of as space crossed with time) will be quantum mechanics on some space of fields on M^3. So we will study a Hilbert space attached to a fixed time slide, which is roughly

$$L^2\left(\text{fields on } M^{d-1}\right) \ . \tag{4.22}$$

In electromagnetism we started with the field strength F, and replaced it with this connection A. Following the above heuristic, this means that in Maxwell theory, the Hilbert space is roughly:

$$\mathcal{H} = \text{“}L^{2\prime\prime}\left(\mathcal{C}\left(M^3\right)\right) \tag{4.23}$$

where $\mathcal{C}$ takes the isomorphism classes of line bundles $\mathcal{L}$ and $\mathrm{U}(1)$ connection ∇. I.e. this consists of isomorphism classes of connections $|\mathrm{conn}(M)|$. Connections always have automorphisms (circle rotation at least), but we'll just look at the set of isomorphism classes.

It is useful to note hat $\mathcal{C}(M)$ is an (∞-dimensional) abelian Lie group with group operation given by tensor product. Abelian Lie groups all look like:

$$\Lambda \times T \times V \tag{4.24}$$

for Λ some finite lattice, T some finite-dimensional torus, and V some infinite-dimensional vector space. We will kind of ignore V. Basically the idea is that we have both the lattice and torus present on both sides of the duality.

Now we follow [FMS07]. We can take the first Chern class (isomorphism class of underlying line bundle)

$$\mathcal{C}(M) \xrightarrow{c_1} H^2(M,\mathbb{Z}) \ . \tag{4.25}$$

This is where the lattice comes from. This is the *total* magnetic flux in the sense that when we pair with a surface, we get the flux through the surface. We can also take the curvature

$$\mathcal{C}(M) \xrightarrow{F} \Omega^2_{\mathbb{Z}}(M) \ . \tag{4.26}$$

Integral differential forms "talk" to integral H^2, and we have a short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) & \longrightarrow & \mathcal{C}(M) & \xrightarrow{c_1} & H^2(M,\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{C}_{\text{flat}}(M) & \longrightarrow & \mathcal{C}(M) & \xrightarrow{F} & \Omega^2_{\mathbb{Z}}(M) & \longrightarrow & 0 \end{array} \tag{4.27}$$

where

$$\mathcal{C}_{\text{flat}}(M) = \left|\mathrm{Loc}_{\mathrm{U}(1)} M\right| = H^1(M,\mathbb{R}/\mathbb{Z}) = \mathrm{U}(1) \otimes_{\mathbb{Z}} H^1(M,\mathbb{Z}) \tag{4.28}$$

is a torus of dimension $b_1(M)$. There is a map

$$\mathcal{C}(M) \to \Lambda = H^2(M,\mathbb{Z}) \tag{4.29}$$

to the magnetic flux, and inside of here there is a torus:

$$T = \left|\mathrm{Loc}_{\mathrm{U}(1)} M\right| \to \mathcal{C}(M) \ . \tag{4.30}$$

Now we will see that electric-magnetic duality corresponds exactly to Pontrjagin duality on $\mathcal{C}(M)$. We will be able to write the same Hilbert space in two ways, and there is a sense in which magnetic measurements on one side are dual to electric measurements on the other side.

4.4 Quantum field theory

4.4.1 Lagrangian formalism

transition between lectures

Lecture 7: February 9, 2021

We will be doing euclidean quantum field theory in dimension d. First we will give a schematic overview of the Lagrangian formalism. The idea is that, to a d-manifold M, we will attach a space of *fields* $\mathcal{F}(M)$. These are some local quantities on our space, for example functions or sections of a bundle.

To any field $\varphi \in \mathcal{F}(M)$, we can attach the *action* $S(\varphi) \in \mathbb{C}$. This is a way of prescribing the classical equations of motion. Instead of finding solutions to some equations of motion one studies critical points of this function S.

In quantum field theory, we do some kind of "probability theory" on $\mathcal{F}(M)$ with "measure" given by

$$e^{-S(\varphi)/\hbar} D\varphi \ . \tag{4.31}$$

We can think of this thing as being a "vanilla" measure on the space of states that is then weighted by the action S. The idea is that, as $\hbar \to 0$, this concentrates on solutions to the equations of motion. We won't try to make mathematical sense of this, but this is the schematic.

For M^d closed, we can attach the partition function, which is the volume (total measure) of this space of fields:

$$Z(M) = \int_{\mathcal{F}(M)} e^{-S(\varphi)/\hbar} D\varphi \ . \tag{4.32}$$

This isn't a very interesting quantity, and we often normalize so that this is 1. The more interesting thing to calculate are the expectation values of operators.

Example 25. One example of an operator, specifically a local operator $\mathcal{O}_x$ at $x \in M$, is the functional on $\mathcal{F}(M)$ given by evaluation (making a measurement) at $x \in M$.

Now we can take the expectation value of this measurement:

$$\langle \mathcal{O}_x \rangle = \int_{\mathcal{F}(M)} \frac{\mathcal{O}_x(\varphi)\, e^{-S(\varphi)/\hbar} D\varphi}{Z(M)} \ , \tag{4.33}$$

where we're dividing by $Z(M)$ to normalize the measure. Then we might calculate correlation functions, where we're taking several different measurements at different points.

There are also "disorder" operators, where inserting the operator means we look at fields with a prescribed singularity at x. I.e. we're looking at all fields $\mathcal{F}(M \setminus \{x\})$, and the operator is something like the delta function on some prescribed singularity space, i.e. it's picking out fields with some prescribed singularity at x.

4.4.2 Time evolution

Let M be a Riemannian manifold with boundary:

$$\partial M = \partial M_{\text{in}} \sqcup \partial M_{\text{out}} \ . \tag{4.34}$$

This is a Riemannian *bordism* from ∂M_{in} to ∂M_{out}. See fig. 4.1 for a picture.

Example 26. Let N be an $(n-1)$-manifold. Then $M = N \times I$ is a d-dimensional bordism from N to itself.

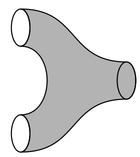

Figure 4.1: A bordism from the disjoint union of two copies of S^1 to a single copy of S^1.

This gives us a correspondence of fields:

$$\begin{tikzcd}
 & \mathcal{F}(M) \arrow[dl] \arrow[dr] & \\
\mathcal{F}(\partial M_{\mathrm{in}}) & & \mathcal{F}(\partial M_{\mathrm{out}})
\end{tikzcd} \quad . \tag{4.35}$$

The functional $e^{-S(\varphi)/\hbar}D\varphi$ lives over $\mathcal{F}(M)$, and so we get an integral transform given by pulling, multiplying by this functional, and integrating (pushing forward).

This integral transform gives us an operator $Z(M)$ between

$$\mathcal{H}_{\mathrm{in}} = \text{functionals on } \mathcal{F}(\partial_{\mathrm{in}}) \tag{4.36}$$

and

$$\mathcal{H}_{\mathrm{out}} = \text{functionals on } \mathcal{F}(\partial_{\mathrm{out}}) \ . \tag{4.37}$$

Then

$$\begin{aligned}
(Z(M)(f))(\varphi_{\mathrm{out}}) &= \int_{\varphi|_{\partial_{\mathrm{out}}} = \varphi_{\mathrm{out}}} f(\varphi|_{\partial_{\mathrm{in}}})\, e^{-S(\varphi)/\hbar} D\varphi \\
&= \pi_{\mathrm{out}*}\left(\pi_{\mathrm{in}}^{*}(f)\, e^{-S(\varphi)/\hbar} D\varphi\right) \ .
\end{aligned}$$

This operator is the time evolution operator.

Remark 17 (Hamiltonian versus Lagrangian). In the Hamiltonian formulation we start with a Hilbert space $\mathcal{H}$, which is associated to a time slice. Then we're also supposed to give a Hamiltonian operator H, and then time evolution is given by the operator $e^{iTH/\hbar}$. Then we have an algebra of observables and make various measurements.

On the other hand in the Lagrangian formalism we start with the space of fields and the action. The Lagrangian formulation is very flexible in the sense that it allows us to define these push-pull operators, which is how we recover a Hilbert space and time evolution operator (Hamiltonian formalism) from the Lagrangian formalism.

Example 27. One-dimensional QFT is quantum mechanics. Let X be a Riemannian target. Then the space of fields might be $\mathcal{F}(\mathbb{R}) = \mathrm{Maps}(\mathbb{R}, X)$. The critical points of S will be geodesics in X. The Hilbert space is

$$\mathcal{H} = L^2(\mathrm{Maps}(\mathrm{pt}, X)) = L^2(X) \ , \tag{4.38}$$

and the Hamiltonian is the Laplace operator Δ.

To formalize all of this would require a great deal of work. We also don't want to perform any perturbative techniques since we don't necessarily want $\hbar$ to be small. We will use this as a schematic guide, and formally pass to the topological setting.

4.5 Quantum Maxwell theory

We will consider a four-dimensional quantum field theory. Our space of fields on a 4-manifold M^4 consists of $\mathrm{U}(1)$ bundles with connection $d+A$ (A is the electromagnetic potential) up to gauge equivalence, i.e. we mod out by the action of the gauge transformations.

The classical equations of motion are Maxwell's equations:

$$\begin{cases} dF = 0 \\ d \star F = 0 \ , \end{cases} \tag{4.39}$$

where F is the curvature. Then we get an action

$$S = \frac{g}{2\pi i} \int_M F \wedge \star F + \theta \int_M F \wedge F \ . \tag{4.40}$$

The integral in the second term is just calculating $c_1^2 = p_1$ of the line bundle. This term is constants?
called the topological term.

The idea is that the Hilbert space is attached to a specific time slice. So assume we can write $M^4 = M^3 \times \mathbb{R}$, and then the Hilbert space is

$$\mathcal{H} = L^2\left(\mathcal{C}\left(M^3\right)\right) \tag{4.41}$$

where $\mathcal{C}\left(M^3\right)$ denotes the collection of line bundles with connections modulo gauge transformations on M^3. $\mathcal{C}\left(M^3\right)$ is an abelian Lie group, and has a map to the lattice of possible line bundles up to isomorphism:

$$\mathcal{C}\left(M^3\right) \xrightarrow{c_1} \Lambda = H^2(M, \mathbb{Z}) \tag{4.42}$$

and also a sub given by the torus of flat connections:

$$T = \text{flat connections} \simeq H^1(M, \mathbb{R}/\mathbb{Z} = \mathrm{U}(1)) \simeq BH^1(M, \mathbb{Z}) \to \mathcal{C}\left(M^3\right) \tag{4.43}$$

where $\mathrm{U}(1)$ is taken to have the discrete topology. Then the final factor of $\mathcal{H}$ is given by an infinite-dimensional vector space V. I.e. there is a (noncanonical) splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$.

4.5.1 Operators

Now we want to identify some operators on $\mathcal{H}$ (i.e. observables) that appear naturally from the setup. The first thing to say is that $\mathcal{H}$ has an obvious grading by $\Lambda = H^2(M, \mathbb{Z})$. This already picks out some operators, e.g. by telling you which component you're on. This is the magnetic flux, so it is said that this is a grading by magnetic fluxes.

Dirac/'t Hooft operators

Λ also acts on $\mathcal{H}$ to yield 't Hooft operators.[1] If we've already chosen a splitting $\mathcal{C}(M) \simeq \Lambda \times T \times V$, then the action is just by translation. I.e. these operators shift the magnetic

[1] Technically 't Hooft operators are the nonabelian generalization of these. These operators really have to do with Dirac monopoles, but this would be confusing since the term *Dirac operator* already has another meaning.

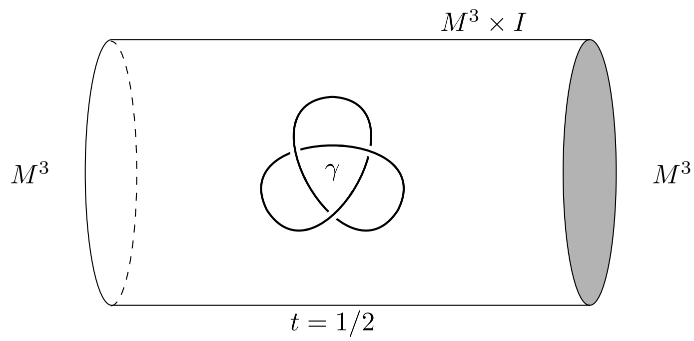

Figure 4.2: The loop γ lives in some time slice in $M^3 \times I$, say $t = 1/2$. We can excise this, and fields on the resulting 4-manifold will possibly have singularities along γ. Asking for the integral around the boundary of a neighborhood of γ to be 1 gives us a space of fields with controlled singularity inside the neighborhood of γ. Physically this introduces a magnetic monopole along γ.

flux. Recall the flux roughly records the number of enclosed monopoles. So physically these operators create magnetic monopoles.

More precisely, let γ be a simple closed curve in M. Note that this defines a class

$$[\gamma] \in H_1(M, \mathbb{Z}) \simeq H^2(M, \mathbb{Z}) \simeq \Lambda \ . \tag{4.44}$$

The associated operator introduces a monopole along γ as follows. The idea is that γ lives at a particular time slice in $M^3 \times I$, and we're studying electromagnetism on $M \times I$ with a monopole along γ as in fig. 4.2.

In other words, we're considering fields $\mathcal{C}(M \times I \setminus \gamma)$, i.e. the connections are possibly singular along γ. So we have excised the knot, and this has introduced a new boundary component of our 4-manifold: the link of the knot (boundary of tubular neighborhood of the knot) which looks like $S^2 \times S^1$. Then we can look at connections which have a specific integral over this. In particular, we can ask for

$$\frac{1}{2\pi i} \int_{S^2} F = 1 \ . \tag{4.45}$$

Write the resulting space of fields as:

$$\mathcal{C}(M \times I \setminus \gamma)_{c_1 = 1} \ . \tag{4.46}$$

Note that, physically, this is saying that this surface $S^2 \times S^1 \subset M^3 \times I$ contains magnetic charge 1. Mathematically we're seeing that we can't extend the connection over this curve. Physically we're seeing a monopole along this curve.

The actual operator is defined as follows. The 4-manifold $M \times I \setminus \gamma$ is a bordism from M^3 to itself, and therefore defines a correspondence and integral transform (Feynman path integral) as usual:

$$\begin{tikzcd}[column sep=small] & \mathcal{C}\left(M \times I \setminus \gamma\right)_{c_1=1} \arrow[dl, "\pi_{\text{in}}"'] \arrow[dr, "\pi_{\text{out}}"] & \\ \mathcal{C}\left(M^3_{\text{in}}\right) & & \mathcal{C}\left(M^3_{\text{out}}\right) \end{tikzcd} \tag{4.47}$$

$$\mathcal{H} \xrightarrow{\pi_{\text{out}*}\left(e^{-S(\varphi)/\hbar} D\varphi \pi^*_{\text{in}}(-)\right)} \mathcal{H}\ .$$

Physically this operator takes the space of states and evolves it through time, during which a monopole is introduced along γ and then removed. Mathematically it is shifting the Chern class by $[\gamma]$, i.e. it is just tensoring with a flat line bundle with prescribed Chern class

$$c_1 = [\gamma] \in H^2(M, \mathbb{Z})\ . \tag{4.48}$$

Wilson loop operators

On the other hand, there are much simpler observables called the Wilson loop operators. Again let $\gamma \subset M$ be a simple closed curve. This determines a function W_γ on $\mathcal{C}(M)$ given by:

$$W_\gamma\colon\ (\mathcal{L}, \nabla) \mapsto \text{holonomy along } \gamma \in \mathrm{U}(1) \subset \mathbb{C}\ . \tag{4.49}$$

We can draw the same picture as fig. 4.2, but we're doing something very different with it. We are still thinking of γ as living on some time slice inside $M \times I$. Now, as time evolves, we make the measurement of the holonomy along γ. So this is a measurement just like the example of a local operator in example 25 only now the measurement is along a loop, so this is called a line/loop operator. This still gives us a correspondence and an integral transform, only now the kernel of the transform is given by the function W_γ:

$$\begin{tikzcd}[column sep=small] & \mathcal{C}\left(M \times I\right) \arrow[dl, "\pi_{\text{in}}"'] \arrow[dr, "\pi_{\text{out}}"] & \\ \mathcal{C}\left(M^3_{\text{in}}\right) & & \mathcal{C}\left(M^3_{\text{out}}\right) \end{tikzcd} \tag{4.50}$$

$$\mathcal{H} \xrightarrow{\pi_{\text{out}*}\left(W_\gamma \pi^*_{\text{in}}(-)\right)} \mathcal{H}\ .$$

The point is that we can multiply functions on this space of connections by this function W_γ. So these Wilson operators are already diagonalized given the way we've presented the Hilbert space.

The Wilson loops W_γ are eigenfunctions for the action of the space of flat connections:

$$T = \mathcal{C}_\flat(M) \simeq H^1(M, \mathrm{U}(1))\ . \tag{4.51}$$

This acts on $\mathcal{C}(M)$ (by tensoring), so therefore it also acts on $L^2(\mathcal{C}(M))$. The eigenvalues are given by multiplying by the monodromy along γ: given a flat connection (element of $H^1(M, \mathrm{U}(1))$) we canonically get an element of $\mathrm{U}(1)$ by taking the monodromy along γ, and this is exactly doing Fourier series. These W_γ are the characters of this torus T.

4.5.2 Electric-magnetic duality

Notice that when we were studying 't Hooft operators we were paying attention to the action of the lattice, and we had a grading by magnetic fluxes. Now we're focusing on the torus action, and we can diagonalize. This gives a decomposition by characters of T, which comprise the lattice $H^2(M, \mathbb{Z})$:

$$\mathcal{H} = \bigoplus_{e \in H^2(M,\mathbb{Z})} \mathcal{H}_e \ , \tag{4.52}$$

and this can be interpreted as a grading by electric fluxes.

Passing from the magnetic grading to the electric grading is exactly performing a Fourier transform with respect to the torus part of our space of fields. In general, electric-magnetic duality can be thought of as doing Fourier series on our Lie group of fields. It specifically identifies:

$$L^2\left(\mathcal{C}_{\mathrm{U}(1)}(M)\right) \simeq L^2\left(\mathcal{C}_{\mathrm{U}(1)^\vee}(M)\right) \tag{4.53}$$

where the left side is the F side, and the right side is the $\star F$ side. The group always splits as a lattice, a torus, and a vector space (which we are ignoring). The F and $\star F$ sides are respectively associated with the splittings:

$$\Lambda_B \times T_E \times \mathbf{Vect} \qquad\qquad \Lambda_E \times T_B \times \mathbf{Vect} \ . \tag{4.54}$$

Λ_B is the lattice of magnetic fluxes, T_B is its dual torus, Λ_E is the lattice of electric fluxes, and T_E is its dual torus. On the F (left) side, Wilson loop operators are diagonalized, and on the $\star F$ (right) side, the 't Hooft operators are diagonalized. Similarly, the magnetic b grading goes to the electric e grading.

4.6 Abelian duality in two-dimensions

Recall this is called T-duality, or mirror symmetry. The space of fields is $\mathrm{Map}\left(M, S^1\right)$, or more generally into a torus T. Our Hilbert space $\mathcal{H}$ on S^1 was then

$$L^2\left(\mathrm{Map}\left(S^1, S^1_R\right)\right) \ . \tag{4.55}$$

Note that again $\mathrm{Map}\left(S^1, S^1\right)$ is an abelian Lie group under the operation of pointwise multiplication, i.e. we're using the group structure of the target not the source. There is also a map to a lattice: this is graded by $H^1\left(S^1, \mathbb{Z}\right)$ which is given by taking the winding number.

Dually we can study:

$$L^2\left(\mathrm{Maps}\left(S^1, \left(S^1_R\right)^\vee = S^1_{1/R}\right)\right) \ . \tag{4.56}$$

This is graded by $H^0\left(S^1, \mathrm{U}(1)\right)$, which is dual to the grading above. We have an operator given by shifting the winding number (analogues of the 't Hooft operators) and operators given by evaluation at a point (analogues of the Wilson operators) and the duality exchanges them.

4.7 Topological quantum mechanics/quantum field theory

The idea is that quantum mechanics is hard because it involves analysis. In ordinary quantum mechanics, we started with a point and assigned a Hilbert space $\mathcal{H}$. Then to an interval of length T, we assigned unitary time evolution operator

$$e^{iTH/\hbar} \ . \tag{4.57}$$

In *topological* quantum mechanics (TQM) we want to kill time, i.e. we want $H = 0$. The naive way to interpret this is to restrict to ground states. But a closed manifold does not admit any nonconstant bounded harmonic functions. So we killed the entire theory.

Witten [Wit82] introduced the following technique to kill time in a derived sense via super-symmetry (SUSY). Let X be a Riemannian manifold. Instead of $L^2(X)$, we expand our Hilbert space to be L^2 differential forms, i.e. we add new fields, and now we get a bigger symmetry group given by a *super Lie group.* So now we consider: what acts on differential forms? The first thing is the Laplace operator $H = \Delta$. We also have the de Rham differential $Q = d$ and its adjoint and $Q^* = d^*$. We also have a $\mathrm{U}(1)$ action which gives a $\mathbb{Z}$-grading. This package of operators is called the $\mathcal{N} = 1$ SUSY algebra. So we have operators where H has degree 0, Q has degree 1, and Q^* has degree -1. Most of these operators commute:

$$[Q^*, Q^*] = [Q, Q] = 0 \tag{4.58}$$
$$[Q, H] = [Q^*, H] = 0 \tag{4.59}$$

except

$$[Q, Q^*] = H \ . \tag{4.60}$$

$[Q, Q] = 0$ can be interpreted as $d^2 = 0$. The other relation is that

$$\Delta = dd^* + d^*d \ , \tag{4.61}$$

which says that Q^* gives a homotopy from H to 0. In particular, this implies that, on Q-cohomology, H acts by 0.

So we added new fields, and we got a larger symmetry algebra. One of the operators in this algebra is an odd operator squaring to 0, and we call it the differential. On the associated cohomology, H is zero. So we've killed time in a derived sense. So topological quantum mechanics assigns, to X, the complex

$$(\Omega^\bullet(X), d) \ , \tag{4.62}$$

or $H^*(X)$. In the first case H is homotopic to 0, and in the latter $H = 0$. This is the way de Rham cohomology gets recovered from quantum mechanics.

We will never be dealing with honest quantum field theories. Instead, we will be doing this topological twist. Note that in this setting, the theory only depends topologically on X. The upshot is that we're killing the dependence on Riemannian metrics, which will make life much easier.

Lecture 8; February 11, 2021

4.8 Topological quantum mechanics

The way to set this up formally, is that we have the $d = 1$, $\mathcal{N} = 1$ SUSY algebra acting on $\mathcal{H}$. The $d = 1$ means we're doing quantum mechanics and $\mathcal{N} = 1$ just means the smallest amount of supersymmetry.

transition between lectures

There are two natural realizations of TQM. The first is the A-type TQM. Take X to be a Riemannian manifold, and then the Hilbert space consists of differential forms:

$$\mathcal{H} = \Omega^{\bullet}(X) \ , \tag{4.63}$$

the Hamiltonian $H = \Delta$ is the Laplace operator, $Q = d$ is the de Rham differential, and $Q^* = d^*$ is its adjoint with respect to the metric. The grading is the usual one on forms, and the cohomology is

$$H^{\bullet}(\mathcal{H}) = H^{\bullet}_{\mathrm{dR}}(X) \ . \tag{4.64}$$

There is another realization, called B-type TQM. Now take X to be a complex manifold. The Hilbert space is

$$\mathcal{H} = \Omega^{0,\bullet}(X) \tag{4.65}$$

i.e. forms with $\overline{\partial}$ in them. The SUSY operators are given by $Q = \overline{\partial}$, $Q = \overline{\partial}^*$, and $H = \Delta_{\overline{\partial}}$. The cohomology is Dolbeault cohomology:

$$H^0(\mathcal{H}) = H^{0,*}(X) = R\Gamma(\mathcal{O}_X) \ . \tag{4.66}$$

These were both realizations of the 1-dimensional $\mathcal{N} = 1$ SUSY algebra. There is also $\mathcal{N} = 2$ TQM. One such example comes from studying a Kähler manifold X. The Hilbert space is still differential forms:

$$\Omega^{\bullet,\bullet}(X) \ . \tag{4.67}$$

This is bigraded, rather than having a single grading like before, and this has a bunch of operators, e.g. ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$. The Kähler identities tell us that the associated Laplacians agree: $\Delta_{\overline{\partial}} = \Delta_d$.

For a mathematician, there is the following deep theorem. We have an action of $\mathrm{SU}(2)$ on $\Omega^{\bullet,\bullet}(X)$, which can be thought of as coming from its complexification $\mathrm{SU}(2) \subset \mathrm{SL}_2\mathbb{C}$. These are the Lefschetz operators. The diagonal part is giving a cohomological grading, and then there is a raising operator (intersecting with Kähler form) and a lowering operator. Mathematically this is a very deep statement about global cohomology of a Kähler manifold (the hard Lefschetz theorem). Physically this is a calculation of which SUSY algebra acts on our Hilbert space, and then this $\mathrm{SU}(2)$ actions is the R-symmetry. The point is that we found a big SUSY algebra.

The $\mathcal{N} = 1$ SUSY algebra encoded Hodge theory of Riemannian manifolds. This $\mathcal{N} = 2$ SUSY algebra encodes complex Hodge theory. There is also $\mathcal{N} = 4$ TQM. In this case we're studying differential forms on a compact hyperkähler manifold X: $\Omega^{\bullet}(X)$. The analogue of Lefschetz SL_2 action for hyperkähler is an action of $\mathrm{Spin}\,5$. The idea is that if we assemble all of the Lefschetz SL_2s together, you get a $\mathrm{Spin}\,5$. This is part of a super-Lie group that we won't write down. From the point of view of the physics, this is just a natural calculation about what operators are sitting around when we're studying the theory of particles in this manifold.

Remark 18. There is also $\mathcal{N} = 8$ SUSY. These different SUSY algebras are really distinguished by the number of odd operators (a.k.a. surpercharges Q) are present. E.g. $\mathcal{N} = 1$ has two, $\mathcal{N} = 2$ has 4, $\mathcal{N} = 4$ has 8, and $\mathcal{N} = 8$ had 16.

The general philosophy of TQM and eventually TQFT is to add fields so that we have an action of a big super-Lie group (SUSY algebra). We're looking for two things:

1. that the Hamiltonian is exact: $H = [Q, -]$ (i.e. we're killing time[2]), and
2. $T = [Q, -]$, i.e. the metric dependence is exact (i.e. we're killing geometry).

The stress energy tensor T is an object in QFT which measures the dependence on the metric. Mathematically T is a derived version of invariance under isometry. This is explained nicely in Costello-Gwilliam [CG17].

4.8.1 Topological Maxwell theory

We want to take the quantum theory of light (ordinary Maxwell theory), add some fields, and find the SUSY algebra. Then we will pick some Q and pass to cohomology. This will be an $\mathcal{N} = 4$ GL-twisted theory. In particular, this $\mathcal{N} = 4$ means that there is a lot of super-symmetry: there will be sixteen supercharges.

Before, our space of fields consisted of a line bundle and a choice of connection $\nabla = d + A$. Now we will add:

- a 1-form σ on the manifold (the Higgs field),
- a complex scalar u, and
- four fermions (odd fields) (we will not pay much attention to these).

The Hilbert space used to be $\mathcal{H} = L^2(\mathcal{C}(M))$. Now, in the A-twist, the 3-manifold M^3 gets attached to the cohomology of this space: $H^\bullet(\mathcal{C}(M))$. We have a much bigger space of fields now, but it doesn't actually make a difference at the level of the cohomology, since introducing these new fields didn't change the topology of the space. But as it turns out, we don't want ordinary cohomology, we want cohomology which is equivariant with respect to the automorphisms of the connections, i.e. we want:

$$\mathcal{H} = H^\bullet(\text{connections/gauge equivalence}) \tag{4.68}$$
$$= H^\bullet_{\mathrm{U}(1)}(\mathcal{C}(M)) \ . \tag{4.69}$$

This is doing topological quantum mechanics in the *de Rham* sense on the space of connections.

There is another version called the B-twist. Now we modify the fields by thinking of:

$$\nabla + i\sigma = d + (A + i\sigma) \tag{4.70}$$

as a connection on a $\mathbb{C}^\times$-bundle rather than a $\mathrm{U}(1)$-bundle. The vector space attached to M^3 is Dolbeault TQM on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$. This just means the vector space is (some derived version of) holomorphic functions on $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.

Now we reformulate the A-side to connect with what we've seen. Write $\mathrm{Pic}(M)$ for the underlying topological space of the space of connections $\mathcal{C}(M)$, i.e. it doesn't detect the Riemannian geometry of M. $H^0(\mathrm{Pic}(M))$ consists of locally constant functions on the space of connections $\mathcal{C}(M)$, i.e.

$$H^0(\mathrm{Pic}(M)) = \mathbb{C}[\pi_0(\mathcal{C}(M))] \tag{4.71}$$
$$= \mathbb{C}\left[H^2(M, \mathbb{Z})\right] \ , \tag{4.72}$$

[2] I.e. time evolution is made trivial.

which is the same vector space we were previously attaching to M in the discussion summarized in table 3.1. The full cohomology of this space is:

$$H^\bullet(\mathrm{Pic}(M)) = H^\bullet(\Lambda \times T \times V \times B\,\mathrm{U}(1)) \ . \tag{4.73}$$

Before we only saw $\Lambda = H^2(M,\mathbb{Z})$, but now we will see

- the cohomology of $T = H^1(M, \mathrm{U}(1))$ (an exterior algebra),
- the vector space doesn't contribute to cohomology, but we also see
- the cohomology of $B\,\mathrm{U}(1)$, which will look like $\mathbb{C}[u]$.

So this is the shape of our Hilbert space after making a topological twist, and on the B-side there will also be factors corresponding to these extra exterior and symmetric pieces.

On the B-side we're looking at the space $\mathrm{Loc}_{\mathbb{C}^\times} M$. Up to some derived factors, this looks like a complex torus $\mathrm{Loc}_{\mathbb{C}^\times} M \simeq T^\vee_{\mathbb{C}}$ of maps $\pi_1(M) \to \mathbb{C}^\times$ which factor as:

$$\begin{tikzcd} \pi_1(M) \arrow[rr] \arrow[dr] & & \mathbb{C}^\times \\ & H_1(M) \arrow[ur] & \end{tikzcd} \ . \tag{4.74}$$

$H_1(M)$ looks like Λ (by Poincaré duality), so this part is the dual torus to the lattice Λ. Recall the Fourier series identifies:

$$\mathbb{C}\left[H^2(M,\mathbb{Z})\right] \qquad \text{and} \qquad \mathbb{C}\left[T^\vee_{\mathbb{C}}\right] \ , \tag{4.75}$$

and now we have some extra (derived) factors, coming from $H^2(M,\mathbb{C})$ and $H^3(M,\mathbb{C})$. The $H^2(M,\mathbb{C})$ factor corresponds to the exterior algebra factor on the A-side, and the $H^3(M,\mathbb{C})$ factor corresponds to the symmetric algebra factor on the A-side.

So again, we had this notion of a Fourier transform, which exchanges:

$$\Lambda = H^2(M,\mathbb{Z}) \qquad \text{and} \qquad T^\vee_{\mathbb{C}} = |\mathrm{Loc}_{\mathbb{C}^\times}(M)| \ , \tag{4.76}$$

and this turns out to just be the degree 0 part of the duality between these A and B-twists, where we have this extra exterior algebra, and extra symmetric algebra. **Mottos:**

- The A-twisted super Maxwell theory with gauge group T studies the topology of $\mathrm{Pic}(M)$, and
- the B-twisted version of super Maxwell theory with gauge group $T^\vee$ studies the algebraic geometry of $\mathrm{Loc}_{\mathbb{C}^\times}(M)$.
- E-M duality switches these two.

See table 4.1 for a summary.

Remark 19. We looked at two different twists, i.e. two different realizations of the SUSY algebra, i.e. two different charges. We could also take any linear combination of these two and get a new topological theory. In other words, this is a $\mathbb{P}^1$ family of possible topological theories which all arise from quantum Maxwell theory. The A and B-twists are then two extreme points of $\mathbb{P}^1$. On one end only magnetic phenomena are left, and on the other only electric phenomena are left.

Table 4.1: Summary of the topological A and B twists of super-Maxwell theory.

A-side	B-side
topology of Pic	AG of Loc
$\Lambda = H^2(M,\mathbb{Z})$	$T^\vee_{\mathbb{C}} = \lvert\mathrm{Loc}_{\mathbb{C}^\times}(M)\rvert$
't Hooft operators (shifting lattice (magnetic flux) by $\gamma^\vee \in H^2(M,\mathbb{Z})$)	Wilson operators (multiply by monodromy along γ)
Create magnetic monopole	Create electric particle.
Magnetic side	Electric side

4.8.2 Defects

There are two other "physics operations" done in electromagnetism called defects, which we will describe at the topological level. These amount to changing the fields we're considering, e.g. introducing singularities.

Time-like line defects

The first type of defect we will consider is a "timelike" 't Hooft loop/line. Consider a point $x \in M^3$. Then we can consider electromagnetism in the presence of a monopole at x. Mathematically this means we look at the space of connections:

$$\mathrm{Pic}(M \setminus x) = \lvert\mathcal{C}(M \setminus x)\rvert = \coprod_n \mathrm{Pic}(M, nx) \tag{4.77}$$

where

$$\mathrm{Pic}(M, nx) = \{A \in \mathrm{Pic}(M \setminus x) \mid c_1 = n \text{ on sphere at } x\} \ . \tag{4.78}$$

E.g. $\mathcal{C}(M, x)$ is the charge 1 component. So now we get a new Hilbert space by linearizing $\mathcal{C}(M \setminus x)$ (via taking L^2 or H^*).

Dually (on the B-side) we have a "timelike" Wilson loop/line. For $x \in M$ we take our fields to be:

$$\mathrm{Loc}_{\mathbb{C}^\times}(M, x) \tag{4.79}$$

which are flat $\mathbb{C}^\times$ connections equipped with a trivialization of the fiber at x. Physically this is interpreted as adding a heavy[3] charged particle, but it breaks the gauge symmetry at this point. This new space of fields $\mathrm{Loc}_{\mathbb{C}^\times}(M, x)$ is a $\mathbb{C}^\times$-bundle over $\mathrm{Loc}_{\mathbb{C}^\times}(M)$:

$$\begin{tikzcd} \mathrm{Loc}_{\mathbb{C}^\times}(M, x) \arrow[d, "\mathbb{C}^\times"] \\ \mathrm{Loc}_{\mathbb{C}^\times}(M) \end{tikzcd} \ . \tag{4.80}$$

On the B-side we've introduced an extra factor of $\mathbb{C}^\times$, and on the A-side we've introduced an extra factor of $\mathbb{Z}$, by allowing different charges. Fourier series will identify these factors:

$$H^*(\mathrm{Pic}(M \setminus x)) \simeq \mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(M, x)] \ . \tag{4.81}$$

This is another instance of electric-magnetic duality: creating a magnetic monopole corresponds to creating an electrically charged particle.

[3] Heavy just means we're not adding a new field and doing field theory with it.

Surface defects

Another type of defect is a *surface defect*, also known as ramification, or introducing a solenoid. The idea is that we have a long tube, with a wire coiled around the outside, and then we send a current through it and this creates a magnetic field inside of the tube. So this is 1-dimensional subspace of a 3-manifold M^3, and inside of this crossed with time, $M \times I$, this defines a surface.

On the B-side, this introduces a singularity of the electric field, i.e. our fields are:

$$\mathrm{Loc}_{\mathbb{C}^\times}(M \setminus \beta) \ . \tag{4.82}$$

The holonomy around β introduces an extra factor of $\mathbb{Z}$ in H_1.

The magnetic (A-side) version of this has fields given by the same space of connections $\mathcal{C}(M)$, but equipped with a trivialization along β. How does this help you? A trivialization (or the difference between two trivializations) along a loop is a map from $\beta = S^1$ to $\mathrm{U}(1)$. This has a winding number, so we get $\mathbb{Z}$-many components to these fields. This is dual to the extra $\mathbb{C}^\times$ from the holonomy around β on the B-side. So this is another version of Fourier series. Explicitly this identifies:

$$\mathbb{C}\left[\mathrm{Loc}_{\mathbb{C}^\times}(M \setminus \beta)\right] \simeq H^*\left(\mathbb{C}\left(M \text{ trivialized along } \beta\right)\right) \ . \tag{4.83}$$

Chapter 5

Class field theory

Topology and physics gave us one source of interesting duality statements. Number theory, specifically class field theory (CFT), is another source. The topology (e.g. the locally constant functions) of Pic will still be exchanged with the algebraic geometry (e.g. the algebraic functions) of Loc. But we need to interpret Pic and Loc in this context.

Let $F/\mathbb{Q}$ be a number field. Write $\mathcal{C}\ell\,(F)$ for the ideal class group. We can think of this as:

$$\mathcal{C}\ell\,(F) = \mathrm{Pic}\,(\mathrm{Spec}\,\mathcal{O}_F) \ , \tag{5.1}$$

which consists of line bundles on $\mathrm{Spec}\,\mathcal{O}_F$, i.e. rank 1 projective $\mathcal{O}_F$-modules modulo isomorphism. Concretely, the *class group* is

$$\mathcal{C}\ell\,(F) = \text{fractional ideals of } \mathcal{O}_F/\text{principal ideals} \ . \tag{5.2}$$

Note that this is an abelian group. As it turns out $\mathcal{C}\ell\,(F)$ is also a finite group, and its order is an important invariant of F called the *class number*. The general philosophy of CFT is to relate the class group to the Galois group.

5.1 Unramified class field theory

Unramified CFT identifies:

$$\mathbb{C}\,[\mathcal{C}\ell\,(f)] \simeq \mathbb{C}\,[\mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F)] \ . \tag{5.3}$$

But we need to specify what Loc is in this context. We can always think of Loc as consisting of representations of π_1, and we can also write:

$$\pi_1\,(M) = \mathrm{Aut}\left(\widetilde{M}\right) \ , \tag{5.4}$$

where $\widetilde{M}$ is the universal cover of M. In this context, we have:

$$\mathrm{Gal}\,(E/F) = \mathrm{Aut}_F\,(E) \ , \tag{5.5}$$

and the analogue of the universal cover is the *maximal unramified extension* of F, written F^{ur}. So the analogue of π_1 is

$$\pi_1^{\mathrm{et}}\,(\mathrm{Spec}\,\mathcal{O}_F) = \mathrm{Gal}\,(F^{\mathrm{ur}}/F) \ . \tag{5.6}$$

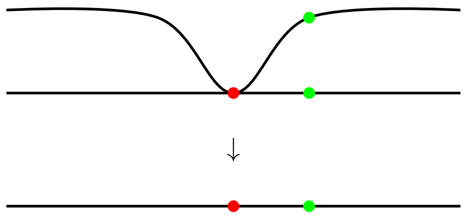

Figure 5.1: An extension which is ramified at the red point (on the left), and unramified at the green point (on the right).

We take F^{ur} because we want to consider coverings of Spec $\mathcal{O}_F$, not of F itself. We say E/F is *unramified at a prime p* if

$$\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F/p\,(=\mathcal{O}_E/p) \tag{5.7}$$

is a product of fields, i.e. it has no nilpotents. The idea is that we have Spec $\mathcal{O}_E$ living over Spec $\mathcal{O}_F$, and we don't want there to be any branching as in fig. 5.1. I.e. if we look at the preimage of a points in the base, we want a product of fields. If we have some nilpotence, then this tells us there is some interesting geometry at that point, which we want to avoid for now. Arbitrary extensions will stay play the role of covering spaces away from the ramification point. E/F is unramified at ∞ if there are no $\mathbb{C}/\mathbb{R}$ extensions between them. E.g. if F is totally imaginary. The idea is that tensing with $\mathbb{R}$ corresponds to restricting to the neighborhood of infinity. But this is just a bunch of copies of $\mathbb{R}$ and $\mathbb{C}$:

$$\mathcal{O}_F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \tag{5.8}$$

and then we insist that going from F to E never involves extending from $\mathbb{R}$ to $\mathbb{C}$.

Now that we have established the basics, we can state that unramified CFT tells us that

$$\mathcal{C}\ell \simeq \mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right) = \mathrm{Aut}\,(\text{Hilbert class field}) \ , \tag{5.9}$$

where the Hilbert class field is the maximal abelian extension of F which is unramified everywhere.

To make contact with what we have seen before, we should study characters of this Galois group. I.e. on one side we have an analogue to Pic: $\mathcal{C}\ell$, and on the other side our analogue to Loc is:

$$\mathrm{Hom}\left(\mathrm{Gal}\,(F^{\mathrm{ur}}/F)\,, \mathbb{C}^\times\right) \ . \tag{5.10}$$

The point is that homomorphisms into an abelian group factor into the abelian quotient:

$$\mathrm{Hom}\left(\mathrm{Gal}\,(F^{\mathrm{ur}}/F)\,, \mathbb{C}^\times\right) = \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) \ . \tag{5.11}$$

Remark 20. $\mathrm{Gal}\,(F^{\mathrm{ur}}/F)$ is the analogue of $\pi_1\,(\mathrm{Spec}\,\mathcal{O}_F)$, and $\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right)$ is the analogue of $H_1\,(\mathrm{Spec}\,\mathcal{O}_F)$. So the equality between these Hom spaces is analogous to when morphisms from $\pi_1\,(\mathrm{Spec}\,\mathcal{O}_F)$ factor through its abelianization $H_1\,(\mathrm{Spec}\,\mathcal{O}_F)$.

Therefore, from (5.9), this is the Pontrjagin dual of the class group:

$$\mathrm{Loc}_{\mathbb{G}_m}(\mathcal{O}_F) = \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur}}/F\right), \mathbb{C}^\times\right) \tag{5.12}$$
$$= \mathrm{Hom}\left(\mathrm{Gal}\left(F^{\mathrm{ur,ab}}/F\right), \mathbb{C}^\times\right) \tag{5.13}$$
$$= \mathrm{Hom}\left(\mathcal{C}\ell(F), \mathbb{C}^\times\right) \tag{5.14}$$
$$= \mathcal{C}\ell(F)^\vee\ . \tag{5.15}$$

Remark 21. Artin-Verdier duality is an analogue of Poincaré duality for $\operatorname{Spec}\mathcal{O}_F$. This was our basis for the arithmetic topology dictionary which says that a number field (or really $\operatorname{Spec}\mathcal{O}_F$) is analogous to a 3-manifold. Recall that the duality between Pic and Loc for a 3-manifold was Poincaré duality, and this reformulation of CFT is a statement of Artin-Verdier duality. But the development of Artin-Verdier duality depends on CFT, so this isn't the direction one takes to prove CFT.

Remark 22. The arithmetic topological dictionary told us that $\operatorname{Spec}\mathcal{O}_F$ corresponds to a 3-manifold. As it turns out, it is an *unoriented* one. That is, the dualizing object is not the constants. This lack of orientation is analogous to the difference between:

- $\mathbb{Z}/n$ and roots of unity μ_n,
- $\mathbb{Q}/\mathbb{Z}$ and μ_∞,
- $\mathbb{C}^\times$ and $\mathbb{G}_m$, etc.

As it turns out, rather than $\mathbb{Z}$ as the dualizing object, we have the Tate-twist $\mathbb{Z}(1)$ (μ_n with respect to the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$). In Gauss' law, or when calculating the Chern classes of line bundles, we encountered some factors of $2\pi i$. These are accounted for by this phenomenon.

The upshot of the identification of characters of $\mathrm{Gal}^{\mathrm{ur}}$ with $\mathcal{C}\ell_F^\vee$ is that we get a Fourier transform: Lecture 9; February 23, 2021

$$\mathbb{C}[\mathrm{Cl}_F] \simeq \mathbb{C}[\mathrm{Loc}_1(\operatorname{Spec}\mathcal{O}_F)] \tag{5.16}$$

where Loc_1 is defined as the characters of $\mathrm{Gal}^{\mathrm{ur}}$.

Example 28. If $F = \mathbb{Q}$ then both sides are trivial.

5.2 Function fields

Let C be a smooth projective curve over some field k. To this we can attach the function field $F = k(C)$, which is the field of rational functions on C. We can also consider the Picard group, $\mathrm{Pic}(C)$, which has k-points given by:

$$\mathrm{Pic}(C)(k) = \{\text{line bundles on } C\}/\sim \tag{5.17}$$
$$= \{\text{loc. free rank 1 } \mathcal{O}_C - \text{modules}\} \tag{5.18}$$
$$= \{\text{divisors}\}/\{\text{pricipal divisors}\} \tag{5.19}$$
$$= H^1\left(C, \mathcal{O}^\times\right) \tag{5.20}$$

where in the last line we're describing line bundles by their transition functions, i.e. Čech cocycles. Each line bundle has a degree, which is an integer. If we think of a line bundle as being given by a sum $\sum_{x\in C} n_x x$, then this is explicitly given by the map:

$$\begin{array}{ccc} \mathrm{Pic} & \xrightarrow{\deg} & \mathbb{Z} \\ \sum_{x\in C} n_x x & \longmapsto & \sum n_x \end{array} , \tag{5.21}$$

which is surjective with kernel given by:

$$0 \to \mathrm{Jac} \to \mathrm{Pic} \to \mathbb{Z} \to 0 , \tag{5.22}$$

where Jac is the Jacobian, i.e. line bundles of degree 0. For any choice of $x \in C$, we get a copy $\mathbb{Z}x \hookrightarrow \mathrm{Pic}$, i.e. this extension is split, with a section for every point of the curve..

If $k = \mathbb{F}_q$ is finite, this is similar to something we can see from the (étale) fundamental group of C. Instead of having $\mathbb{Z}$ as a quotient, this has the profinite completion[1] $\widehat{\mathbb{Z}}$ as a quotient:

$$\pi_1^{\text{ét}}(C) \twoheadrightarrow \widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(\mathrm{Spec}\, k) . \tag{5.23}$$

The idea is that the curve C lives over $\mathrm{Spec}\, k$, so for a covering of $\mathrm{Spec}\, k$, we can pull this back to a covering $\widetilde{C}$ of C. This corresponds to the map from the fundamental group from C to the fundamental group of $\mathrm{Spec}\, k$. This looks very similar to the above situation, where Pic has $\mathbb{Z}$ as a quotient, with two main differences:

1. $\pi_1^{\text{ét}}$ is not abelian, and
2. $\widehat{\mathbb{Z}}$ is not $\mathbb{Z}$.

We can address this in two different ways. The first thing we can do is replace $\pi_1^{\text{ét}} = \mathrm{Gal}^{\mathrm{ur}}(F)$ by the *unramified Weil group of C*, written W_C^{ur}. The idea is that we want to make $\mathrm{Spec}\, k$ look more like S^1, so we replace $\widehat{\mathbb{Z}}$ by $\mathbb{Z}$, and W_C^{ur} is the preimage of $\mathbb{Z}$ in $\pi_1^{\text{ét}}$:

$$\begin{tikzcd} \pi_1^{\text{ét}} \arrow[r] & \widehat{\mathbb{Z}} \\ W_C^{\mathrm{ur}} \arrow[u, hook] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd} . \tag{5.24}$$

The other thing we can do is pass to the abelianization: $W_C^{\mathrm{ur,ab}}$, which has $\mathbb{Z}$ as a quotient:

$$W_C^{\mathrm{ur,ab}} \twoheadrightarrow \mathbb{Z} . \tag{5.25}$$

There is also a section of this for every point $x \in C(k)$, just like in the Pic case above. A point is a map $x\colon \mathrm{Spec}\, k \to C$, and geometrically this gives us a map

$$\widehat{\mathbb{Z}} = \pi_1^{\text{ét}}(k) \to \pi_1^{\text{ét}}(C) , \tag{5.26}$$

which is the corresponding section. The picture is that $\mathrm{Spec}\, k$ is an analogue of the circle, and C is an analogue of a 3-manifold fibered over it. Then a k-point of C is a section, i.e. a loop in C. To summarize, the claim is that Pic and $W_C^{\mathrm{ur,ab}}$ both

[1] Recall the profinite completion is the inverse limit of all finite quotients.

- surject onto $\mathbb{Z}$,
- are abelian, and
- have a section for every $x \in C(k)$.

Instead of altering the finite field picture, we could have altered Pic. Recall that Pic was an extension of $\mathbb{Z}$ by the Jacobian, which is a finite group over a finite field. Therefore all of the "infinity" of Pic is coming from $\mathbb{Z}$. So we could replace Pic by its profinite completion $\widehat{\mathrm{Pic}}$. So now we get something with a surjection to $\widehat{\mathbb{Z}}$ and a section for every point $x \in C$. In any case, the theorem is that they match.

Theorem 2 (Unramified CFT)**.** *There is a map* $\mathrm{Pic} \to \pi_1^{\acute{e}t,ab}$ *which is an isomorphism on profinite completions. Equivalently we have an isomorphism*

$$\mathrm{Pic} \simeq W_C^{ur,ab} \tag{5.27}$$

respecting the operation of taking the degree of a line bundle and respecting the sections associated to $x \in C$.

Example 29. If $C = \mathbb{P}^1$ then both sides are just $\mathbb{Z}$. This is more interesting for higher-genus curves.

Now we want to reformulate this in a way which is more reminiscent of the Pontrjagin duality that we have seen. Theorem 2 implies that the characters of Pic are the same as $W_C^{\mathrm{ur}\ ,\ \mathrm{ab}}$, but once we take characters, we can't detect the abelianization, so

$$\begin{aligned}
\text{characters of } \mathrm{Pic} &\simeq \text{characters of } W_C^{\mathrm{ur,ab}} &(5.28)\\
&\simeq \text{characters of } W_C^{\mathrm{ur}} &(5.29)\\
&\simeq \text{characters of } \pi_1^{\text{ét}}(C) &(5.30)\\
&\simeq \text{rank 1 local systems on } C\ . &(5.31)
\end{aligned}$$

I.e. the Pontrjagin dual to $\mathrm{Pic}(C)$ is:

$$(\mathrm{Pic}(C))^\vee \simeq \mathrm{Loc}_1(C)\ . \tag{5.32}$$

One kind of statement we can make from this, is that there is a Fourier transform identifying functions on Pic with functions on Loc.

Inside of function on Pic, we have the characters. The character condition, which asks that $\chi(gh) = \chi(g)\chi(h)$, is equivalent to asking for

$$\mu^*\chi = \pi_1^*\chi \boxtimes \pi_2^*\chi \tag{5.33}$$

where

$$\begin{tikzcd}
& \mathrm{Pic}\times\mathrm{Pic} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[rr, "\mu"] & & \mathrm{Pic} \\
\mathrm{Pic} & & \mathrm{Pic} &
\end{tikzcd} . \tag{5.34}$$

Equivalently these are eigenfunctions for the translation action of Pic on itself. But Pic is generated by $\mathbb{Z}_x$ for $x \in C$, so a character is the same as a function f on Pic which is an eigenfunction for the action of $\mathbb{Z}_x$ for all $x \in C$. Explicitly, for $x \in C$, this action sends

$\mathcal{L} \in \mathrm{Pic}$ to the line bundle $\mathcal{L}(x)$ which is $\mathcal{L}$ with an extra $1 \cdot x$ added to the divisor. Being an eigenfunction means that:

$$f(\mathcal{L}(x)) = \gamma_x \cdot f(\mathcal{L}) \ , \tag{5.35}$$

where γ_x is a number.

These eigenfunctions should go to something like delta functions on Loc under this Fourier transform. An element $\rho \in \mathrm{Loc}$ is a representation:

$$\rho \colon \pi_1^{\text{ét}} \to e^{\times} \tag{5.36}$$

for e some field of coefficients (e.g. $\mathbb{C}$). Then f is an eigenfunction if

$$f(\mathcal{L}(x)) = \rho(\mathrm{Fr}_x) \cdot f(\mathcal{L}) \ . \tag{5.37}$$

The idea is that a local system is giving you a collection of eigenvalues for each point of the curve $\gamma_x = \rho(\mathrm{Fr}_x)$, i.e. $\rho \in \mathrm{Loc}$ determines the eigenvalues for $\mathbb{Z}x$ (for $x \in C$).

The operator $\mathcal{L} \mapsto \mathcal{L}(x)$ is a *Hecke operator*. These are playing the role of the 't Hooft/Dirac monopole operators from section 4.5.1 which acted on $H^2(M) \simeq \Lambda$ by translation, and were labelled by a loop $\gamma \in H_1(M)$. Recall that we were thinking of this as $H^2(M) = \pi_0(\mathrm{Pic})$.

On the other hand, we had the Wilson operators from section 4.5.1. For M a 3-manifold we sent:

$$\mathrm{Loc}_1(M) \ni \rho \mapsto W_\gamma(\rho) = \text{monodromy of } \rho \text{ around } \gamma \ . \tag{5.38}$$

Now we're sending:

$$\mathrm{Loc}_1 \ni \rho \mapsto \rho(\mathrm{Fr}_x) \ . \tag{5.39}$$

Again, in all of these cases, we're studying the algebraic geometry of Pic as a realization of the topology of Loc and vice versa.

5.2.1 Loc

Now we explain a bit about how to think about Loc. In algebraic geometry, we can't fully "access" π_1 because we only have finite covers.

Example 30. Consider the punctured affine line $\mathbb{A}^1 \setminus \{0\}$. We cannot access the universal cover because it is the exponential map $\exp\colon \mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$, which is not an algebraic function. We can however access finite covers of this since $t^{1/n}$ is algebraic.

So the first think we might study is something like a map:

$$\pi_1 \to \mu_n \subset \mathbb{Q}/\mathbb{Z} \ . \tag{5.40}$$

Moreover, for $\operatorname{char} k = p$, only the theory of prime-to-p-order covers works "as expected". So we need to pick a prime $\ell \neq p = \operatorname{char} k$, and then we can study maps:

$$\pi_1 \to \mathbb{Z}/\ell^n \ . \tag{5.41}$$

But then taking the inverse limit over ℓ gives us $\mathbb{Z}_\ell$, so we can make sense of representations into $\mathbb{Z}_\ell$, but we can also tensor up, i.e. we can lift the representation along:

$$\mathbb{Z}/\ell^n \leftarrow \mathbb{Z}_\ell \subset \mathbb{Q}_\ell \subset \overline{\mathbb{Q}_\ell} \ . \tag{5.42}$$

The upshot is that this leads to a good theory of ℓ-adic representations:

$$\pi_1 \to \mathrm{GL}_n\left(\overline{\mathbb{Q}_\ell}\right) \ , \tag{5.43}$$

i.e. a theory of ℓ-adic local systems in characteristic p. One nice thing is that $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ as fields, but this does not respect the topology.

So when we say rank 1 local systems we really mean continuous morphisms:

$$\mathrm{Loc}_1 = \mathrm{Hom}_{\mathrm{cts}}\left(\mathrm{Gal}^{\mathrm{ur}}(C), \overline{\mathbb{Q}_\ell}^{\times}\right) \ . \tag{5.44}$$

More generally, whenever we're discussing $\mathbb{C}$-functions we should really be taking $\overline{\mathbb{Q}_\ell}$-valued functions. The resulting theory is independent of ℓ as long as $\ell \neq p$.

5.2.2 Pic

We want to describe $\mathrm{Pic}(C)$ for C/k a smooth projective curve as something along the lines of:

$$\begin{aligned} \mathrm{Pic}(C) &= \text{Divisors/principal divisors} && (5.45)\\ &= \bigoplus_{x\in C} \mathbb{Z}/\,\mathrm{unit}\, k(C)\ . && (5.46) \end{aligned}$$

Let $\mathcal{L}$ be a line bundle. We want to describe it via its transition functions. We can trivialize $\mathcal{L}$ generically, i.e. there exists a meromorphic section s of $\mathcal{L}$ or equivalently there is an isomorphism of functions away from finitely many points $\{x_i\}_i$:

$$s\colon\ \mathcal{O}|_{C\setminus\{x_i\}_i} \xrightarrow{\sim} \mathcal{L}|_{C\setminus\{x_i\}_i} \ . \tag{5.47}$$

On the other hand, we can (more democratically) trivialize $\mathcal{L}$ very close to any $x \in C$. Formally, the disk around x is:

$$D_x = \mathrm{Spec}(\mathcal{O}_x) = \mathrm{Spec}(k[[t]]) \ , \tag{5.48}$$

where $\mathcal{O}_x$ is the completed local ring at x and we have chosen a coordinate t on C. And when we pull $\mathcal{L}$ back to D_x, it is automatically trivialized. Therefore the section s has a *nonzero* Laurent expansion around these finitely many points, i.e. we have

$$[s] \in K_x \simeq k((t)) \tag{5.49}$$

where K_x is the field of fractions of $\mathcal{O}_x$.

Now we can measure the degree of the section s at x:

$$\deg_x s \in K_x^\times/\mathcal{O}_x^\times \tag{5.50}$$

where we are quotienting out by changes of the trivialization of $\mathcal{L}$ on D_x. Expressed in the coordinate t, this is:

$$\begin{aligned} K_x^\times/\mathcal{O}_x^\times &= k((t))^\times/k[[t]]^\times && (5.51)\\ &= \left\{a_{-N}t^{-N} + \dots\right\}/\left\{b_0 + b_1 t + \dots \,|\, b_0 \neq 0\right\}\ . && (5.52) \end{aligned}$$

Now it's an exercise in algebra to check that we can force $a_{-N} = 1$, and all other $a_i = 0$. Therefore this is identified with the integers:

$$K_x^\times / \mathcal{O}_x^\times \simeq \mathbb{Z} \,. \tag{5.53}$$

This description is not very "efficient". For varying line bundles, we have to vary the open set $U = C \setminus \{x_i\}$ where we are able to trivialize $\mathcal{L}$. The way we deal with this is by "removing all points". So consider the space of line bundles $\mathcal{L}$ equipped with a rational section[2] and a trivialization of $\mathcal{L}|_{D_x}$ for all $x \in C$. Now we want to describe the space of such data. We know we get a nonzero Laurent series for every point $x \in C$, so our first guess might be a product of $K_x^\times$, but this is actually a *restricted product* because for any particular line bundle, there are only finitely many points where there was a problem:

$$\begin{aligned} \prod_{x\in C}{}' K_x^\times &= \left\{(\gamma_x) \in K_x^\times \,\middle|\, \gamma_x \in \mathcal{O}_x^\times \text{ for all but finitely many } x\right\} && (5.54)\\ &\subset \prod_{x\in C} K_x^\times \,. && (5.55)\end{aligned}$$

So this was kind of "overkill", and we got a huge amount of data, and now we will kind of "strip it away". So consider Line bundles equipped only with just a rational section (not a trivialization everywhere). We can access these by quotienting out by changes of trivialization, i.e. line bundles equipped with a rational section comprise:

$$\begin{aligned} \prod{}' K_x^\times / \prod \mathcal{O}_x^\times &= \prod{}' \left(K_x^\times / \operatorname{unit} \mathcal{O}_x\right) && (5.56)\\ &= \prod{}' \mathbb{Z} && (5.57)\\ \left\{\text{finite } \sum_{x\in C} a_x x\right\} & && (5.58)\\ &= \text{Divisors} \,. && (5.59)\end{aligned}$$

Now we need to get rid of our choice of rational section by modding out on the left:

$$\begin{aligned} \operatorname{Pic} &= \{\text{line bundles}\} && (5.60)\\ &= F^\times \backslash \prod{}' K_x^\times / \prod \mathcal{O}_x^\times && (5.61)\\ &= F^\times \backslash \prod{}' \mathbb{Z} \,, && (5.62)\end{aligned}$$

which is exactly divisors moduli principal ones. The left quotient is by changes of the rational section, and the right quotient is by changes of the trivialization on D_x.

transition

This is the adélic description of $\operatorname{Pic}(C)$. We can rewrite this as: February 25, 2021

$$\operatorname{Pic}(C) = \operatorname{GL}_1(F) \backslash \operatorname{GL}_1(\mathbb{A}_F) / \operatorname{GL}_1(\mathcal{O}_{\mathbb{A}_F}) \tag{5.63}$$

where $\mathbb{A}_F$ is the adéles for F:

$$\mathbb{A}_F \coloneqq \prod_{x\in C}{}' K_x \tag{5.64}$$

[2]Note we don't bother saying where the poles are. So we just trivialize $\mathcal{L}$ over the function field.

where K_x is the completed local field. Inside of this is the ring of integers:

$$\mathcal{O}_{\mathbb{A}} = \prod_{x \in C} \mathcal{O}_x \tag{5.65}$$

where $\mathcal{O}_x$ is the completed local ring.

Remark 23. This description might seem like overkill, in the sense that we're writing it has a quotient of something huge. This is the same sense in which defining Pic via divisors is overkill: the collection of all divisors is huge before we mod out by principal ones.

The *(unramified) idéle class group (idéles)* is:

$$\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^{\times} . \tag{5.66}$$

Arithmetic version

Now we will relate this discussion to the analogous ones in the arithmetic setting. For F a number field, we can construct the idéles of F as:

$$\mathbb{A}_F = \prod_{v \text{ places}}{}' F_v , \tag{5.67}$$

where F_v is a completed local field (e.g. $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$, $\ldots$) and similarly

$$\mathcal{O}_{\mathbb{A}} = \prod_{\text{primes}} \mathcal{O}_v . \tag{5.68}$$

Then, in an attempt to get our hands on an analogue of Pic, we can follow our nose from the function field setting and guess that the unramified idéle class group is:

$$\mathrm{GL}_1(F) \backslash \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) = F^{\times} \backslash \prod{}' F_v^{\times} / \prod_v \mathcal{O}_v^{\times} . \tag{5.69}$$

This is missing a factor coming from the place at infinity, which we can add as follows. At a finite place, the inclusion $F^{\times} \supset \mathcal{O}^{\times}$ looks something like $\mathbb{Q}_p^{\times} \supset \mathbb{Z}_p^{\times}$. One way of describing $\mathbb{Z}_p^{\times}$, is that is is a maximal compact subgroup with respect to the p-adic topology. So now we can use this to determine what to do at infinity. In particular, we let K_∞ be the maximal compact subgroup of $F_v^{\times}$, and we additionally quotient out by this on the right:

$$\mathrm{GL}_1(F) \backslash \mathrm{GL}_1(\mathbb{A}_F) / \mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) \cdot K_\infty = F^{\times} \backslash \prod{}' F_v^{\times} / \prod_v \mathcal{O}_v^{\times} \cdot K_\infty . \tag{5.70}$$

Note that, at infinity, $F_v^* \simeq \mathbb{R}^{\times}$ or $\mathbb{C}^{\times}$ so the maximal compacts are $\mathbb{Z}/2$ and $\mathrm{SO}(2)$ respectively. So we're only removing part of these extra copies of $\mathbb{R}^{\times}$ and $\mathbb{C}^{\times}$. Explicitly we still have some factors of $\mathbb{R}_+$.

The idea is that this is very close to $\mathcal{C}\ell$:

$$\mathcal{C}\ell_F = \pi_0 \left(F^{\times} \backslash \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty \right) . \tag{5.71}$$

In other words: there is a quotient from the unramified idéle class group to $\mathcal{C}\ell_F$ given by taking π_0, i.e. by quotienting out by the leftover copies of $\mathbb{R}_+$ from the places at infinity. Now recall $\mathcal{C}\ell$ is isomorphic to $\mathrm{Gal}^{\mathrm{ur,ab}}$:

$$F^{\times} \backslash \mathrm{GL}_1(\mathbb{A}) / \mathrm{GL}_1(\mathcal{O}) \cdot K_\infty \twoheadrightarrow \mathcal{C}\ell_F \simeq \mathrm{Gal}^{\mathrm{ur,ab}} . \tag{5.72}$$

In particular if we look at characters that don't detect copies of $\mathbb{R}$ and $\mathbb{C}$ (e.g. finite order or continuous ones) then the characters on these groups will agree.

This is close to what we saw in topology. Recall we were consider the space of line bundles and connections, $\operatorname{Pic}\left(M^3\right) = \operatorname{Map}\left(M^3, B\,\mathrm{U}(1)\right)$, which had π_0 given by $H^2(M,\mathbb{Z})$. The identity component of the idéle class group can be described as follows. Our first guess is: Pic notation

$$\mathbb{R}_+^r \times \mathbb{R}_+^s \tag{5.73}$$

where r is the number of real embeddings of F, and s is the number of pairs of complex embeddings. This is because the infinite part of our field is:

$$F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \ , \tag{5.74}$$

and then when we quotiented out by the maximal compacts, we just got copies of $\mathbb{R}_+$ for both $\mathbb{R}$ and $\mathbb{C}$. Then we quotient out by a boring diagonal copy on the right, and by the units in F on the left:

$$\mathcal{O}_F^\times \backslash \mathbb{R}_+^r \times \mathbb{R}_+^s / \mathbb{R}_+ \ . \tag{5.75}$$

Theorem 3 (Dirichlet unit theorem)**.** *$\mathcal{O}_F^\times \simeq \mu_\infty(\mathcal{O}_F) \times \mathbb{Z}^{r+s-1}$, i.e. the units are just the obvious ones (all roots of unity in F) and a lattice.*

Therefore the connected component of the idéle class group is:

$$\pi_0\left(F^\times \backslash \operatorname{GL}_1(\mathbb{A}) / \operatorname{GL}_1(\mathcal{O}) \cdot K_\infty\right) = \simeq \mathrm{U}(1)^{r+s-1} \ . \tag{5.76}$$

So if we study the cohomology of the idéle class group, we get an exterior algebra of rank $r+s-1$.

This is the complete story for the unramified case. But this is a very restrictive condition to ask for, especially in number theory.

5.3 Ramification

5.3.1 Physics

Recall in the physics context (section 4.8.1) we had $\operatorname{Pic}\left(M^3\right)$, which consisted of line-bundles equipped with a connection. By line-bundles and connections we just mean $\operatorname{Map}(M, B\,\mathrm{U}(1))$. Pic notation
In particular this has

$$\pi_0 = H^2(M,\mathbb{Z}) \qquad\qquad \pi_1 = H^1(M,\mathbb{Z}) \ . \tag{5.77}$$

We matched this with

$$\operatorname{Loc}_1\left(M^3\right) = \operatorname{Hom}\left(\pi_1(M), \mathbb{C}^\times\right) \tag{5.78}$$

in the sense that functions on them were identified:

$$H^0\left(\operatorname{Pic}\left(M^3\right)\right) \leftrightarrow \mathbb{C}\left[\operatorname{Loc}_1 M^3\right] \ . \tag{5.79}$$

Then, in section 4.8.2, we had surface defects/solenoids where we looked at $\operatorname{Loc}_1\left(M^3 \setminus \gamma\right)$ for some closed 1-manifold $\gamma \hookrightarrow M^3$, e.g. a knot or link. I.e. we're allowing singularities along γ. Loc_1 now has an extra copy of $\mathbb{C}^\times$ for every component of the link, so we need to change Pic too.

To alter Pic, instead of introducing singularities along γ, we work relative to it, so we consider: $\operatorname{Pic}\left(M^3, \gamma\right)$, which consists of line bundles equipped with a trivialization on γ. This introduces an extra factor of $\mathbb{Z}$ in Pic per component of γ.

Remark 24. If we're thinking topologically, before we introduced ramification, we had the Poincaré duality (with coefficients) between: $\pi_0 \operatorname{Pic} = H^2(M, \mathbb{Z})$ and $H^1(M, \mathrm{U}(1))$. Introducing ramification corresponds to passing from Poincaré duality to Lefschetz-duality, which is a version for relative cohomology:

$$H^2((M,N), \mathbb{Z})^\vee \simeq H^1\left(M \setminus N, \mathbb{C}^\times\right) \ . \tag{5.80}$$

So we work relative to the knot on one side, and remove it on the other.

5.3.2 Arithmetic setting

Now we want to do the same thing in number theory, i.e. over curves over finite fields or number fields. We won't try to get the whole story, but just tame ramification.

Over $\mathbb{C}$,

$$D^\times = \operatorname{Spec} \mathbb{C}((t)) \tag{5.81}$$

is some version of a circle, and indeed

$$\pi_1\left(D^\times\right) = \mathbb{Z} \ . \tag{5.82}$$

In algebraic geometry, we should really think about:

$$\pi_1^{\text{ét}}\left(D^\times\right) = \widehat{\mathbb{Z}} \ . \tag{5.83}$$

This means that all extensions of $\mathbb{C}((t))$ correspond to taking roots of the coordinate: $\mathbb{C}((t)) \to \mathbb{C}\left(\left(t^{1/N}\right)\right)$.

In arithmetic, things are much more interesting. Analogues of the punctured disk are:

$$D^\times{}_{\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q((t)) \qquad\qquad \operatorname{Spec} \mathbb{Q}_p \ , \tag{5.84}$$

only now $\pi_1^{\text{ét}}$ is richer. There are two sources of this richness. One is that we have a quotient to the Frobenius:

$$\pi_1^{\text{ét}}\left(D^\times{}_{\mathbb{F}_q}\right) \twoheadrightarrow \widehat{\mathbb{Z}} \ . \tag{5.85}$$

One might object that this is a consequence of working over a non-algebraically-closed field. But even when we pass to $\overline{\mathbb{F}_q}((t))$, this still has a complicated fundamental group. But there is part which mimics what we saw in geometry, called the Tame inertia group. The idea is that $\pi_1^{\text{ét}}\left(D^\times{}_{\mathbb{F}_q}\right)$ and $\operatorname{Gal}(\mathbb{Q}_p)$ both have a quotient given by

$$\Gamma = \left(\left\{F, m \,\middle|\, FmF^{-1} = m^q\right\}\right)^\wedge \ , \tag{5.86}$$

where q is the order of the residue field, and F stands for the Frobenius (and m stands for monodromy). This surjects onto the Frobenius part with the monodromy part sitting inside:

$$\widehat{\mathbb{Z}_m} \to \Gamma \twoheadrightarrow \widehat{\mathbb{Z}_F} \ . \tag{5.87}$$

This is something pretty geometric. We have something going around the point we removed (monodromy), and there is the Frobenius, and they relate via $FmF^{-1} = m^q$. The idea is that, in the arithmetic setting, this étale fundamental group tells us much more, but the tame inertia is the part that looks like the geometric setting.

In number theory our Loc is representations of the Galois group of F (here F is either rational functions on a curve $\mathbb{F}_q(C)$ or a number field):

$$\mathrm{Loc} = \left\{ \mathrm{Gal}_F \to \mathrm{GL}_1\left(\overline{\mathbb{Q}_\ell}\right) \right\} \ . \tag{5.88}$$

For a finite subset $S \subset C$ (or $S \subset$ primes of F) we can look at elements of Loc which are tamely ramified at the points of S, written $\mathrm{Loc}_1(C \setminus S)^{\mathrm{tame}}$.

Remark 25. This is like when we allowed singularities along the knot in section 4.5.1. Only in the physics, we didn't need to control how singular these singularities are. Now we do need to get control, and we do so by asking for the ramification to be tame.

Then you might ask how to match this with some version of Pic. The claim is that the Pontrjagin duality exchanges:

$$\mathrm{Loc}_1(C \setminus S)^{\mathrm{tame}} \qquad \text{and} \qquad \mathrm{Pic}(C, S) \ , \tag{5.89}$$

where $\mathrm{Pic}(C,S)$ consists of line bundles on C equipped with a trivialization on S.

Recall we were thinking of:

$$\mathrm{Pic}\, C = F^\times \backslash {\prod}' \mathbb{Z} \ . \tag{5.90}$$

Recall the basic comparison between Pic and $\pi_1^{\text{ét}}$ in Theorem 2 was based on matching these copies of $\mathbb{Z}$. Specifically these came from the fact that we had a section of Pic $\to \mathbb{Z}$ for every point of the curve, and a copy of $\mathbb{Z}$ in the Galois group for each Frobenius. Now we have this richer description:

$$\mathrm{Pic}\, C = F^\times \backslash \prod_{x \in C}{}' K^\times / \prod_{x\in C} \mathcal{O}^\times \ . \tag{5.91}$$

Recall that this quotient description came from starting with ${\prod}' K^\times$, the collection of line bundles with a generic trivialization, and a trivialization near every point. Then we quotiented out by this extra data. Now we consider only quotienting on the left:

$$F^\times \backslash {\prod}' K^\times \twoheadrightarrow \mathrm{Pic}\, C \ , \tag{5.92}$$

which consists of line bundles with a trivialization near every $x \in C$. I.e. we have quotiented out by the choice of generic trivialization. The group in (5.92) is the *idéle class group*.[3]

Choosing a trivialization around every point is still a huge amount of data. The advantage of this is that it allows us to be flexible about what kind of data we pick where, e.g. we might ask for a finite order trivialization around some finite collection of points. So pick a finite subset $S \subset C$ with multiplicities n_x for $x \in S$. Then we can consider the collection of elements of Pic, equipped with an n_x-order trivialization at all $x \in S$. By trivialize to the order n_x, we mean the following. We have an nth order neighborhood of the point x sitting inside the disk around x:

$$\mathrm{Spec}\, k[t]/t^n \hookrightarrow D_x = \mathrm{Spec}\, k[[t]] \ , \tag{5.93}$$

[3]Before this, it was really the unramified idéle class group.

and instead of asking for an infinite Taylor series section of the bundle, we ask for a section of finite order n_x. This gives us an intermediate quotient Pic_S:

$$\begin{tikzcd} \mathrm{GL}_1(\mathrm{unit}\,F)\backslash\mathrm{GL}_1(\mathbb{A}_{\mathbb{C}}) \arrow[d] \arrow[dr] & \\ & \mathrm{Pic}(C,S) \arrow[dl] \\ \mathrm{Pic} = \mathrm{GL}_1(F^\times)\backslash\mathrm{GL}_1(\mathbb{A}_{\mathbb{C}})\,/\,\mathrm{GL}_1(\mathcal{O}_{\mathbb{A}}) & \end{tikzcd} \tag{5.94}$$

defined by

$$\mathrm{Pic}_S = \mathrm{GL}_1(F)\backslash\mathrm{GL}_1(\mathbb{A}_F)\,/\prod_{x\notin S}\mathrm{GL}_1(\mathcal{O}_X)\times\prod_{x\in S}\mathrm{GL}_1^{(n_x)}(\mathcal{O}_x)\ , \tag{5.95}$$

where $\mathrm{GL}_1^{(n)}(\mathcal{O}_x)$ consists of elements of $\mathrm{GL}_1(\mathcal{O}_x)$ that are congruent to $1 \bmod t^n$. I.e. they consist of changes of trivialization of a line bundle on the disk, constant to order n. So the top is elements of Pic equipped with infinite order trivialization everywhere, the bottom is just Pic, and the middle is elements of Pic equipped with an n_x order trivialization at every $x \in S$. Concretely:

$$\mathcal{O}_x^\times = \left\{a_0 + a_1 t + a_2 t^2 + \ldots \,\middle|\, a_0 \neq 0\right\}\ , \tag{5.96}$$

and

$$\mathrm{GL}_1^{(n)}(\mathcal{O}_x) = \{1 + a_n t^n + \ldots\}\ . \tag{5.97}$$

Example 31. Let S be a finite subset of points of C, and let $n_x = 1$ for all $x \in S$. Then $\mathrm{Pic}(C,S)$ consists of line bundles whose fibers at $x \in S$ are trivialized. This maps down to $\mathrm{Pic}(C)$ with fiber $\mathcal{O}^\times/(\mathcal{O}^\times)^{(1)} = k^\times$.

Recall $\pi_1^{\text{ét}}D^\times$ was complicated, but had this quotient Γ:

$$\pi_1^{\text{ét}}\left(D^\times\right) \twoheadrightarrow \Gamma = \left(\left\{F, m \,\middle|\, FmF^{-1} = m^q\right\}\right)^\wedge\ . \tag{5.98}$$

But we're only considering maps to some abelian group:

$$\begin{tikzcd} \pi_1^{\text{ét}}D^\times \arrow[r, two heads] \arrow[dr] & \Gamma \arrow[d] \\ & \text{abelian group} \end{tikzcd}\ , \tag{5.99}$$

so they factor through the abelianization:

$$\Gamma/[\Gamma,\Gamma] \simeq \mathbb{F}_q^\times \times \widehat{\mathbb{Z}}\ . \tag{5.100}$$

Now we want to compare this to $K^\times$. $K^\times/O^\times = \mathbb{Z}$ was our degree, and then:

$$\mathbb{F}_q^\times \times \mathbb{Z} \simeq K^\times/\mathcal{O}^{\times(1)} \to K^\times/\mathcal{O}^\times = \mathbb{Z}\ , \tag{5.101}$$

and this is the same $\mathbb{F}_q^\times \times \mathbb{Z}$ in the Galois group.

$$\overline{K^\times} \simeq \mathrm{Gal}^{\mathrm{ab}}\, \overline{K_x}/K_x \ . \tag{5.102}$$

This is the first instance of local class field which says that:

$$\widehat{K^\times} \simeq \mathrm{Gal}\, \overline{K_x}/K_x \ . \tag{5.103}$$

The RHS has a quotient to $\widehat{\mathbb{Z}}$ given by the Frobenius, and the LHS has a quotient to $\widehat{\mathbb{Z}}$ given by the degree. We also have quotients down to $\mathbb{F}_q^\times$. So these quotients reveal the part of these groups that we see in the topological context. Exactly as in the physics, this local class field theory gives us a duality between:

$$\mathrm{Pic}\,(C, S) \qquad\qquad \text{and} \qquad\qquad \mathrm{Loc}\,(C \setminus S)^{\mathrm{tame}} \ . \tag{5.104}$$

Chapter 6

Extended TFT

6.1 Ramification

We have seen the notion of ramification in extended TFT, and now we will consider higher codimension versions. This means we're "raising the category level" from vector spaces to categories. Lecture 11; March, 2021

This will be formalizing some notions that we've already seen in certain examples. We had two examples of ramification in physics. The first was the notion of a solenoid in Maxwell theory. In this situation we have some knot in a manifold M^3, and then we considered $H^2\left(\left(M,N\right),\mathbb{Z}\right)$ where N is the boundary of a tubular neighborhood of the knot. The dual of this was:

$$H^2\left(\left(M,N\right),\mathbb{Z}\right)^\vee \simeq H^1\left(M \setminus N, \mathbb{C}^\times\right) \ . \tag{6.1}$$

So the idea is that $H^2\left(\left(M,N\right),\mathbb{Z}\right)$ consists of line bundles equipped with a trivialization on the knot, whereas $H^1\left(M \setminus N, \mathbb{C}^\times\right)$ has flat connections with singularities along the knot.

We also had the Dirac monopole where we consider $H^2\left(M \setminus N, \mathbb{Z}\right)$, where N is the boundary of a small ball. We can think of this as line bundles with a singularity at the point. The dual of this was:

$$H^2\left(M \setminus N, \mathbb{Z}\right)^\vee \simeq H^1\left(\left(M,N\right), \mathbb{C}^\times\right) \ , \tag{6.2}$$

which we can think of as flat connections with a trivialization at this point.

Something similar appeared in the arithmetic setting. We had this Cartier duality between line bundles and flat connections:

$$\operatorname{Pic}\left(C\right)^\vee \simeq \operatorname{Loc}_1 C \ . \tag{6.3}$$

On the left we can add level structure, i.e. we can add trivializations on a subset $S \subset C$. On the right we can add singularities at $S \subset C$. Then ramified class field theory tells us how adding a level structure on one side corresponds to adding singularities on the other.

Now we would like to explain, more systematically, how to add higher codimension data from the physics point of view.

Recall, in field theory, we are studying fields $\mathcal{F}\left(M\right)$ on a manifold M. If M has boundary $\partial M = N$, then we get a map:

$$\mathcal{F}\left(M\right) \xrightarrow{\pi_N} \mathcal{F}\left(N\right) \ . \tag{6.4}$$

In the examples (of a monopole and a solenoid) we then asked for conditions on N (either allowing a singularity or asking for a trivialization). We can say it "all at once" as follows. Let $\mathcal{E}_N$ be any *sheaf* on $\mathcal{F}(\partial M)$. Recall that we attached a vector space to M, given by the vector space of function (or cohomology etc.) on $\mathcal{F}(M)$. Now we want to modify this by the boundary data $\mathcal{E}_N$. Explicitly we can pull $\mathcal{E}_N$ back to M, to get $\pi_N^*\mathcal{E}_N$. This is a sheaf on M, so we can take global sections to get a new vector space:

$$\Gamma\left(M, \pi_N^* \mathcal{E}_N\right) \ . \tag{6.5}$$

This is a modified version of the Hilbert space we attach to M. So we get a vector space attached to the data of M and some $\mathcal{E}_N$, which plays the role of the conditions we were asking for on N (singularity or trivialization).

Remark 26. For any $\pi\colon X \to Y$ and $\mathcal{E} \in \mathbf{Shv}(Y)$ we have:

$$\begin{aligned} \Gamma\left(X, \pi^*\mathcal{E}\right) &= \Gamma\left(Y, \pi_*\pi^*\mathcal{E}\right) && (6.6)\\ &= \Gamma\left(Y, \mathcal{E} \otimes \pi_* \underline{k}_X\right) \ . && (6.7)\end{aligned}$$

So to any X we can attach $\pi_*\underline{k}_X \in \mathbf{Shv}(Y)$ and pairing with that object gives a vector space, or we can just think of this as the vector space attached to X by pulling back $\mathcal{E}$.

6.1.1 Topological Maxwell theory

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In topological Maxwell theory section 4.8.1, the A-side dealt with

$$H^*\left(\mathcal{C}(M)\right) \ , \tag{6.8}$$

where $\mathcal{C}(M)$ is the space of line bundles with connections on M. Now we modify this to recover the relative version, i.e. either

$$H^2\left((M, N), \mathbb{Z}\right) \tag{6.9}$$

or

$$H^2\left(M \setminus N, \mathbb{Z}\right) \ . \tag{6.10}$$

Recall the space of connections had components given by:

$$\pi_0 \mathcal{C}(M) \simeq H^2(M, \mathbb{Z}) \ , \tag{6.11}$$

so we can think of $H^2(M, \mathbb{Z})$ as consisting of locally constant functions on the connected components. The point is that the space of connections is, topologically,

$$\mathcal{C}(M) = \mathrm{Map}\left(M, K(\mathbb{Z}, 2)\right) = \mathrm{Map}\left(M, B\,\mathrm{U}(1)\right) \ , \tag{6.12}$$

so the components are given by $H^2(M, \mathbb{Z})$.

So our Hilbert space in Maxwell theory was cohomology of the constant sheaf on $\mathcal{C}(M)$. Now we want to modify it by taking cohomology of different sheaves. We have a restriction from M to the boundary $\partial M = N$:

$$\mathrm{Map}\left(M, K(\mathbb{Z}, 2)\right) \xrightarrow{\pi_N} \mathrm{Map}\left(N, K(\mathbb{Z}, 2)\right) \ , \tag{6.13}$$

so we can start with a sheaf on the boundary and pull it back to M and take the cohomology.

For example, inside of

$$\operatorname{Map}(N, K(\mathbb{Z}, 2)) = \operatorname{Map}(N, B\,\mathrm{U}(1)) \ , \tag{6.14}$$

we have the constant maps, i.e. there is a point

$$i\colon \{\mathrm{pt}\} \hookrightarrow \operatorname{Map}(N, B\,\mathrm{U}(1)) \tag{6.15}$$

given by maps sending N to $\mathrm{pt} \in B\,\mathrm{U}(1)$. Consider the skyscraper at this point, $i_*\underline{\mathbb{C}}$, and take this as our sheaf:

$$\mathcal{E}_N = i_*\underline{\mathbb{C}} \ . \tag{6.16}$$

Then we pull this back to M to get $\pi^*\mathcal{E}_N$, which has cohomology:

$$H^*(\mathcal{C}(M), \pi^*\mathcal{E}) = H^*(\text{line bundles on } M + \text{trivialization on } N) \ . \tag{6.17}$$

This is the *Dirichlet boundary condition*. In particular, we can recover the relative cohomology $H^2((M,N), \mathbb{Z})$ as π_0 of the following fiber product:

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & \operatorname{Map}(M, K(\mathbb{Z},2)) \arrow[d] \\ \mathrm{pt} \arrow[r] & \operatorname{Maps}(M, K(\mathbb{Z},2)) \end{tikzcd} \ . \tag{6.18}$$

Let $K \subset M$ be a knot, and N the boundary of a tubular neighborhood of K. Write M_0 for the manifold with boundary $\partial M_0 = N$ obtained by removing the knot. So if we want to recover the cohomology of M_0, we just take the constant sheaf on $\mathcal{F}(N)$ and pull it back to $\mathcal{F}(M_0)$. This is the *von Neumann boundary condition.*

To reiterate, given fields on M we can put conditions on them along $K \subset M$ as follows. We have a map:

$$\mathcal{F}(M \setminus K =: M_0) \to \mathcal{F}(\partial M_0) \tag{6.19}$$

and we can take sheaves on the right, pull them back to $\mathcal{F}(M_0)$, and take the cohomology of $\mathcal{F}(M_0)$ with coefficients in this sheaf.

For example, the pullback of the constant sheaf $\underline{\mathbb{C}}$ is still the constant sheaf, and then we take

$$H^*(\mathcal{F}(M \setminus K), \pi^*\underline{\mathbb{C}} = \underline{\mathbb{C}}) = H^*(\mathcal{F}(M \setminus K)) \ . \tag{6.20}$$

I.e. we are allowing arbitrary singularities along K.

We can also consider the sky scraper $i_*\underline{\mathbb{C}}$, where $i\colon \pi \hookrightarrow \partial M_0$. This gives:

$$H^*(\mathcal{F}(M \setminus K), \pi^* i_*\underline{\mathbb{C}}) = H^*(\text{connections} + \text{trivialization on } N) \ . \tag{6.21}$$

In summary, we can think of sheaves on the boundary as being a source of ways to get a new vector space attached to M.

6.2 Extended TFT

The rough idea of an n-dimensional TFT, was to attach a number to a closed n-manifold P. We can think of this as some kind of volume of the space of all field on P. Then a closed $(n-1)$-manifold, we attached a vector space, which we think of as a space of functionals on the space of fields on M. Now we would like to attach some kind of category to a closed $(n-2)$-manifold. Again this is some kind of linearization of the space of fields, but now it's the category of sheaves on $\mathcal{F}(N)$.

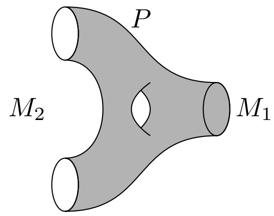


Figure 6.1: A morphism between two objects (closed $(n-1)$-manifolds) M_1 and M_2 in $\mathbf{Bord}_{n-1,m}$ is an n-manifold P with $M_1 \sqcup M_2$ as its boundary.

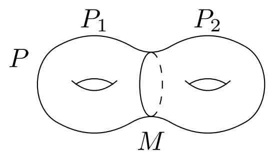


Figure 6.2: Closed n-manifold decomposed into P_1 and P_2 meeting along M.

6.2.1 Atiyah-Segal TFT

An *Atiyah-Segal (non-extended) TFT* is a symmetric monoidal functor

$$(\mathbf{Bord}_{n-1,n}, \sqcup) \xrightarrow{Z} (\mathbf{Vect}, \otimes) \quad . \tag{6.22}$$

The objects of $\mathbf{Bord}_{n-1,n}$ are (oriented[1]) closed $(n-1)$-manifolds. The objects of $\mathbf{Vect}$ are (complex) vector spaces. So a closed $(n-1)$-manifold M goes to some vector space $Z(M)$.

A morphism $P\colon M_1 \to M_2$ in $\mathbf{Bord}_{n-1,n}$ is an n-manifold P with boundary $\partial M = M_1 \sqcup M_2$ as in fig. 6.1.

To see why one would think this way we need to shift our interest away from the vector spaces of states, and briefly refocus on the partition function $Z(P)$ on a closed n-manifold P. The idea is that this Atiyah-Segal formalization of TFT encodes the locality of the partition function. I.e. if we cut P into two pieces P_1 and P_2, glued along M as in fig. 6.2, then $Z(P)$ factors as:

$$\begin{tikzcd} Z(\emptyset) = \mathbb{C} \arrow[r] \arrow[rr, bend left, "Z(P)"] & Z(M) \arrow[r] & \mathbb{C} = Z(\emptyset) \end{tikzcd} \quad . \tag{6.23}$$

We're more interested in the vector space of states on an $(n-1)$-manifold M. The idea is that we can similarly decompose $Z(M)$ into "local pieces" coming from a decomposition of M:

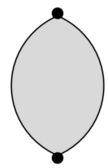
$$M = M_1 \sqcup_N M_2 \tag{6.24}$$

for N of codimension 2. We would like to write something like:

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle_{Z(N)} \quad , \tag{6.25}$$

but the Atiyah-Segal formalism doesn't tell us how to do this. This is the basic idea of extended field theory, due to Freed, Lawrence, Baez-Dolan, ... cite

[1] We can ask for different tangential structures on our bordisms. We're just taking oriented ones.



$$= \quad \begin{tikzcd} M_1 \arrow[r, bend left, "N_1"] \arrow[r, bend right, "N_2"'] \arrow[r, phantom, "\Longrightarrow"] & M_2 \end{tikzcd}$$

Figure 6.3: This manifold (with corners) is a 2-morphism between two 1-morphisms in $\mathbf{Bord}_{0,2}$.

Basically we need to associate some kind of object to spaces of codimension 2. To formalize this, we define some kind of 2-category $\mathbf{Bord}_{n-2,n}$. The objects are closed $(n-2)$-manifold N, morphisms are $(n-1)$-manifolds with boundary $N_1 \sqcup N_2$, and 2-morphisms are n-manifolds with corners. See fig. 6.3 for an example of a 2-morphism between two 1-morphisms in $\mathbf{Bord}_{0,2}$.

A 2-*tier TFT* is a symmetric monoidal functor:

$$Z\colon\ (\mathbf{Bord}_{n-2,n}, \sqcup) \to (\mathcal{C}, \otimes) \tag{6.26}$$

where $(\mathcal{C}, \otimes)$ is some symmetric monoidal 2-category. Beyond being symmetric monoidal, we want this functor to recover the "1-tiered" (i.e. Atiyah-Segal) formalism as follows. The empty $(n-2)$-manifold $\emptyset_{n-2}$ is the unit of $\mathbf{Bord}_{n-2,n}$. This functor being symmetric monoidal implies that this gets sent to the unit $1 \in \mathcal{C}$. The endomorphisms of the empty $(n-2)$-manifold are

$$\mathrm{End}_{\mathbf{Bord}_{n-2,n}}(\emptyset_{n-2}) = \mathbf{Bord}_{n,n-1}\ , \tag{6.27}$$

since bordisms between the empty $\{n-2\}$-manifold and itself comprise closed $(n-1)$-manifolds. So one thing we might require is that $\mathcal{C}$ is a "delooping" of $\mathbf{Vect}$. By this we just mean that:

$$\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = \mathbf{Vect}\ . \tag{6.28}$$

In this case the functor Z determines a map:

$$\mathrm{End}_{\mathbf{Bord}_{n-2,n}}(\emptyset_{n-2}) = \mathbf{Bord}_{n,n-1} \to \mathbf{Vect} = \mathrm{End}(1_{\mathcal{C}})\ . \tag{6.29}$$

Whatever $\mathcal{C}$ is, we get the locality we were interested in: if $M = M_1 \sqcup_N M_2$, then functoriality tells us that

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "Z(M_1)"] \arrow[rr, bend right, "Z(M)"'] & Z(N) \arrow[r, "Z(M_2)"] & 1_{\mathcal{C}} \end{tikzcd}\ . \tag{6.30}$$

A natural choice for $\mathcal{C}$ is $\mathbf{Cat}_{\mathbb{C}}$. The objects are categories enriched over $\mathbf{Vect}_{\mathbb{C}}$, the 1-morphisms are linear functors, and the 2-morphisms are linear natural transformations. There are other choices one can make, but this is somehow the most basic one. Now we want to extend our field theories to codimension 2 to encode ramification. I.e. we want a way to associate N of codimension 2 to some category, which we're thinking of (informally) as $\mathbf{Shv}(\mathcal{F}(N))$.

Let M be a closed 3-manifold, Z a 4-dimensional TFT, and $K \subset M$ a knot. Then the complement of a tubular knot, M_0, has boundary $\partial M_0 = N$. Then this gives us a functor

$$Z(N) \xrightarrow{Z(M_0)} Z(\emptyset) = \mathbf{Vect}\ . \tag{6.31}$$

The idea is that we start with some vector space $Z(M)$, modify it according to some condition along the boundary to get the vector space $Z(M_0)$. The interior of what we removed, i.e. the torus neighborhood of the knot, written M_1, defines a functor $\mathbf{Vect} \to Z(N)$, and functoriality gives us:

$$\begin{tikzcd} \mathbf{Vect} \arrow[r, "Z(M_1)"] \arrow[rr, bend right, "Z(M)"'] & Z(N) \arrow[r, "Z(M_0)"] & \mathbf{Vect} \end{tikzcd} . \tag{6.32}$$

This is the sense in which extended field theory encodes the boundary conditions we have been discussing.

6.3 Abelian duality examples

Recall abelian duality from chapter 3. Specifically the two-dimensional examples from sections 3.7.2, 3.7.3 and 4.6. Now we will discuss these examples with ramification of codimension 2.

6.3.1 Two-dimensions

Recall the first example exchanged the A-side $H^0\left(S^1, \mathbb{Z}\right)$ with the B-side $H^1\left(S^1, \mathbb{C}^\times\right)$.

On the A-side, fields are given by

$$\mathcal{F}(-) = \mathrm{Maps}(-, \mathbb{Z}) \ , \tag{6.33}$$

so on a connected manifold the fields just look like $\mathbb{Z}$. The only submanifold of codimension 2 is the point, and the assignment to the point is:

$$Z(\mathrm{pt}) = \mathbf{Shv}(\mathbb{Z}) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \ , \tag{6.34}$$

i.e. the category of graded vector spaces.

On the B-side, we interpreted $H^1\left(S^1, \mathbb{C}^\times\right)$ as isomorphism classes of $\mathbb{C}^\times$-local systems on S^1. This suggests that our space of fields should be

$$\mathcal{F}(-) = \mathrm{Loc}_{\mathbb{C}^\times}(-) \ . \tag{6.35}$$

If we really just think of the set of isomorphism classes, we won't get a category attached to a point: $\mathrm{Loc}_{\mathbb{C}^\times}(\mathrm{pt})$ is just a single point, so doesn't match the other side. Instead, we have to think of it as a stack. This is still a point, but equipped with an action of $\mathbb{C}^\times$ by automorphisms. I.e. it is the stack:

$$B\mathbb{C}^\times = \mathrm{pt}/\mathbb{C}^\times \ . \tag{6.36}$$

When we take sheaves we get:

$$\mathbf{Shv}(\mathcal{F}(\mathrm{pt})) = \mathbf{Shv}\left(\mathrm{pt}/\mathbb{C}^\times\right) = \mathbf{Rep}\left(\mathbb{C}^\times\right) \simeq \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \ . \tag{6.37}$$

The identification between representations of $\mathbb{C}^\times$ and graded vector spaces is given as follows. Given a representation V of $\mathbb{C}^\times$ we get a grading

$$V = \bigoplus_{n \in \mathbb{Z}} V_n \tag{6.38}$$

where V_n is where $z \in \mathbb{C}^\times$ acts as z^n.

We can think of this as being some version of Pontrjagin duality as follows. We can think of $\mathbf{Shv}(\mathbb{Z})$ as being modules for $\mathbb{C}[\mathbb{Z}]$ equipped with pointwise multiplication:

$$\mathbf{Shv}(\mathbb{Z}) = (\mathbb{C}[\mathbb{Z}], \cdot)\,\text{-}\mathbf{mod}\ . \tag{6.39}$$

Dually, representations of $\mathbb{C}^\times$ can be thought of as modules for $\mathbb{C}[\mathbb{C}^\times]$ with convolution:

$$\mathbf{Rep}\left(\mathbb{C}^\times\right) = \left(\mathbb{C}\mathbb{C}^\times, *\right)\text{-}\mathbf{mod}\ . \tag{6.40}$$

And this duality is asserting that these categories are equivalent. On the B-side we're really looking at comodules over the algebraic functions on $\mathbb{C}^\times$, $\mathbb{C}[\mathbb{C}^\times]$-**comod**, i.e. modules for functions on $\mathbb{Z}$ are comodules for functions on $\mathbb{C}^\times$.

6.3.2 Two-dimensions

Recall sections 3.7.3 and 4.6, where we discuss another example of abelian duality in two dimensions. The A-side is given by functions on $H^1\left(S^1, \mathbb{Z}\right)$, and the B-side is given by functions on $H^0\left(S^1, \mathbb{C}^\times\right)$. This is a basic example of T-duality or mirror symmetry.

On the A-side, we can think of this as:

$$H^1\left(S^1, \mathbb{Z}\right) = H^0\left(S^1, B\mathbb{Z}\right)\ . \tag{6.41}$$

This suggests that the space of fields is given by:

$$\mathcal{F}(-) = \operatorname{Maps}\left(-, B\mathbb{Z} = S^1\right)\ , \tag{6.42}$$

so

$$\mathcal{F}(\mathrm{pt}) = S^1 = B\mathbb{Z}\ . \tag{6.43}$$

So the category we want is some kind of sheaves on this, which we take to be locally constant:

$$\mathbf{Shv}(B\mathbb{Z}) \coloneqq \operatorname{Loc}\left(S^1\right) = \mathbf{Rep}(\mathbb{Z}) = (\mathbb{C}\mathbb{Z}, *)\,\text{-}\mathbf{mod}\ . \tag{6.44}$$

Another name for this category is the (wrapped) Fukaya category of T^*S^1. The punchline is that the symplectic topology of T^*S^1 is the ordinary topology of S^1.

On the B-side, fields on the circle were given by $H^0\left(S^1, \mathbb{C}^\times\right)$, which is telling us that the fields are maps to $\mathbb{C}^\times$ with the discrete topology:

$$\mathcal{F}(-) = \operatorname{Maps}\left(-, {\mathbb{C}^\times}_{\mathrm{disc}}\right)\ . \tag{6.45}$$

In particular:

$$\mathcal{F}(\mathrm{pt}) = \mathbb{C}^\times\ , \tag{6.46}$$

and the natural category of sheaves are going to be quasi-coherent:[2]

$$\mathbf{Shv}_B\left(\mathbb{C}^\times\right) \coloneqq \mathbf{QC}\left(\mathbb{C}^\times\right) = \left(\mathbb{C}\left[\mathbb{C}^\times\right], \cdot\right)\text{-}\mathbf{mod} = \mathbb{C}\left[z, z^{-1}\right]\text{-}\mathbf{mod}\ . \tag{6.47}$$

Again, we see that this is mimicking what we saw in the context of Pontrjagin duality. On the A-side we have modules for $\mathbb{C}[\mathbb{Z}]$ with convolution, and on the B-side we have modules

[2] This is in contrast to the locally-constant sheaves we had on the A-side.

for $\mathbb{C}[\mathbb{C}^\times]$ with pointwise multiplication. In other words, we can think of $\mathbb{C}\left[z, z^{-1}\right]$ as functions on $\mathbb{C}^\times$, or as the group algebra of $\mathbb{Z}$, $\mathbb{C}[\mathbb{Z}]$:

$$\text{f'ns on } \mathbb{C}^\times = \mathbb{C}\left[z, z^{-1}\right] = \mathbb{C}[\mathbb{Z}] \ , \tag{6.48}$$

so a module can be thought of either as a local system on the circle, or as a sheaf on $\mathbb{C}^\times$. This is the first example of homological mirror symmetry:

$$\mathrm{Fuk}^{\mathrm{wr}}\left(T^* S^1\right) = \mathrm{Loc}\left(S^1\right) \simeq \mathbf{QC}\left(\mathbb{C}^\times\right) \ . \tag{6.49}$$

6.4 Four-dimensional Maxwell theory

Recall the four-dimensional Maxwell theory from section 4.8.1. This duality exchanges the topology of line bundles on the A-side with the algebraic geometry of flat line bundles on the B-side. Recall that this duality exchanges the A-side $H^2\left(M^3, \mathbb{Z}\right)$ with the B-side $H^1\left(M^3, \mathbb{C}^\times\right)$. Now we want to extend our defects of codimension 1 from section 4.5.1 down to codimension 2, i.e. down to surfaces.

6.4.1 A-side

Let Σ be a surface. On the A-side we had

$$H^2(-, \mathbb{Z}) = \pi_0\left(\mathrm{Map}\left(-, \underbrace{B\,\mathrm{U}(1)}_{K(\mathbb{Z},2)}\right)\right) \ , \tag{6.50}$$

i.e. we were detecting the connected components of some larger thing. This suggests our fields should be something like:

$$\mathcal{F}(-) = \mathrm{Map}(-, B\,\mathrm{U}(1)) \ . \tag{6.51}$$

We will write $\mathrm{Pic}\,\Sigma$ for these:

$$\mathcal{F}(\Sigma) = \mathrm{Pic}\,\Sigma \coloneqq \mathrm{Map}(\Sigma, B\,\mathrm{U}(1)) \ . \tag{6.52}$$

To see the identification with line bundles more clearly, choose a complex structure on Σ (so it is now a Riemann surface). Then we can consider the space $\mathrm{Pic}\,\Sigma$ of holomorphic/algebraic line bundles on Σ. This is an object of algebraic geometry, but we're only concerning ourselves with the topology. So this space doesn't depend on the complex structure, but choosing one lets us realize this space explicitly. This fits into the short exact sequence:

$$0 \to \mathrm{Jac}\,\Sigma \to \mathrm{Pic}\,\Sigma \xrightarrow{\deg} \mathbb{Z} \to 0 \ , \tag{6.53}$$

where $\mathrm{Jac}\,\Sigma$ consists of degree 0 line bundles. The claim is that $\mathrm{Jac}\,\Sigma$ is a torus of real dimension $2g$, so topologically it is just:

$$\mathrm{Jac}\,\Sigma \simeq \mathbb{C}^g / \mathbb{Z}^{2g} \ . \tag{6.54}$$

Remark 27. Again we're ignoring the automorphisms of these line bundles, which would contribute an extra (stacky) factor of $B\,\mathrm{U}(1)$.

We get an explicit description as follows. The exponential short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\times \longrightarrow 0 \tag{6.55}$$

gives us:

$$0 \longrightarrow H^1(\Sigma, \mathcal{O}) / H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}^\times) \longrightarrow H^2(\Sigma, \mathbb{Z}) = \mathbb{Z} \quad , \tag{6.56}$$

so

$$\mathrm{Jac} \simeq K\left(H^1(\Sigma, \mathbb{Z}), 1\right) \quad , \tag{6.57}$$

i.e. a torus with $\pi_1 = H^1(\Sigma, \mathbb{Z})$.

As a result, this abelian category can be written as:

$$\begin{aligned} \mathrm{Loc}(\mathrm{Pic}\,\Sigma) &= \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \otimes \mathrm{Loc}(\mathrm{Jac}\,\Sigma) && (6.58)\\ &= \mathbf{Rep}_{\mathbb{Z}\text{-gr}}\left(\Lambda = H^1(\Sigma, \mathbb{Z})\right) && (6.59)\\ &= \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \otimes \left(\mathbb{C}\left[H^1(\Sigma, \mathbb{Z})\right], *\right) \ . && (6.60) \end{aligned}$$

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6.4.2 B-side

Now we will check what we do on the B-side, where we are studying the algebraic geometry of $\mathbb{C}^\times$-local systems. I.e. for Σ a surface (not necessarily with a complex structure) we consider:

$$\begin{aligned} \mathrm{Loc}_{\mathbb{C}^\times}(\Sigma) &= \mathrm{Hom}\left(\pi_1(\Sigma), \mathbb{C}^\times\right) / \mathbb{C}^\times && (6.61)\\ &= \mathrm{Hom}\left(H_1(\Sigma, \mathbb{Z}), \mathbb{C}^\times\right) / \mathbb{C}^\times && (6.62)\\ &\simeq \left(H^1(\Sigma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times\right) / \mathbb{C}^\times \ . && (6.63) \end{aligned}$$

Now we're supposed to attach some kind of category (of sheaves). We're doing algebraic geometry, so the natural category of sheaves is:

$$\mathbf{QC}(\mathrm{Loc}_{\mathbb{C}^\times}\Sigma) = (\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)], \cdot)\text{-}\mathbf{mod} \otimes \mathbf{Rep}\left(\mathbb{C}^\times\right) \ . \tag{6.64}$$

This ring of functions can be written as:

$$\mathbb{C}[\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)] = \mathbb{C}\left[H^1(\Sigma, \mathbb{Z})\right] \ . \tag{6.65}$$

Now by Poincaré duality we have $H^1(\Sigma, \mathbb{Z}) \simeq H_1(\Sigma, \mathbb{Z})$. Also, as we have seen, $\mathbf{Rep}\mathbb{C}^\times = \mathbf{Vect}^{\mathbb{Z}\text{-gr}}$.

Now we can compare this to what we had on the A-side in eq. (6.60). The group algebra of a lattice with convolution $(\mathbb{C}\Lambda, *)$ gets identified, via Fourier series, with $(\mathbb{C}[T^\vee], \cdot)$. This means that modules over these two algebras are the same, i.e.

$$\mathrm{Loc}\,K(\Lambda, 1) \simeq \mathbf{QC}(T^\vee_{\mathbb{C}}) \ . \tag{6.66}$$

The LHS here is $\mathrm{Loc}\,\mathrm{Jac}(\Sigma)$, and the RHS is $\mathbf{QC}(\mathrm{Loc})$ but ignoring the extra action of $\mathbb{C}^\times$. So the conclusion is that:

$$\mathrm{Loc}\,\mathrm{Pic}(\Sigma) \simeq \mathbf{QC}(\mathrm{Loc}) \ . \tag{6.67}$$

This is our abelian duality for topological maxwell theory for codimension 2.

Remark 28. This is a first example of a *Fourier-Mukai* transform, i.e. a version of a Fourier transform for categories of sheaves.

Remark 29. This is the statement of the Betti geometric[3] Langlands correspondence for $G = \mathrm{GL}_1$.

6.4.3 Ramified E-M duality

We want to work out a general E-M duality with ramification. This will include Dirac monopoles (section 4.5.1) and solenoids (section 4.8.2). Recall the surface Σ comes to us as the boundary of the 3-manifold $M_0 = M \setminus K$ for $K \subset M$ a knot.

On the A-side, we have the restriction map:

$$\mathrm{Pic}\,(M_0) \xrightarrow{\pi} \mathrm{Pic}\,(\Sigma) \ , \tag{6.68}$$

where again we're just thinking of Pic topologically. Suppose $\mathcal{E} \in \mathrm{Loc}\,(\mathrm{Pic}\,\Sigma)$. Then we get:

$$\pi^* \mathcal{E} \in \mathrm{Loc}\,(\mathrm{Pic}\,(M_0)) \tag{6.69}$$

and we can attach the following vector space to this:

$$\Gamma\,(\mathrm{Pic}\,(M_0)\,, \pi^* \mathcal{E}) \ . \tag{6.70}$$

This was our way of saying that we're modifying $Z\,(M)$ by some ramification data.

On the B-side, we can also restrict:

$$\mathrm{Loc}_{\mathbb{C}^\times} M_0 \xrightarrow{\pi^\vee} \mathrm{Loc}_{\mathbb{C}^\times} \Sigma \tag{6.71}$$

and then we can pull $\mathcal{E}^\vee \in \mathbf{QC}\,(\mathrm{Loc}_{\mathbb{C}^\times} \Sigma)$ back to get:

$$\pi^{\vee *} \mathcal{E}^\vee \in \mathbf{QC}\,(\mathrm{Loc}_{\mathbb{C}^\times} M_0) \tag{6.72}$$

and we can attach the following vector space to this:

$$\Gamma\left(M_0, \pi^{\vee *} \mathcal{E}^\vee\right) \ . \tag{6.73}$$

Then ramified E-M duality exchanges $\mathcal{E}$ and $\mathcal{E}^\vee$ by a Fourier-Mukai transform. In particular it tells us that:

$$\Gamma\left(\mathrm{Pic}\, M_0^3, \pi^* \mathcal{E}\right) \simeq \Gamma\left(\mathrm{Loc}_{\mathbb{C}^\times} M, \pi^{\vee *} \mathcal{E}^\vee\right) \ . \tag{6.74}$$

The sheaves $\mathcal{E}$ and $\mathcal{E}^\vee$ prescribe boundary conditions, e.g. Dirichlet or Neumann. For example $\mathcal{E}^\vee$ might be the structure sheaf of some locus in $\mathrm{Loc}_{\mathbb{C}^\times}(\Sigma)$. And then we're looking at local systems with some particular singularity along that locus. I.e. elements of $\mathrm{Loc}\, M_0$ such that after restricting to the boundary Σ we land in a particular subset.

[3] The "geometric" just means we're looking at the level of categories.

6.5 Ramified class field theory

Let $C/\mathbb{F}_q$ be a curve over a finite field. For a point $x \in C$, we want to "allow ramification" at this point. Consider the punctured disk around x inside the punctured curve:

$$\operatorname{Spec} K_x = D_x^\times \hookrightarrow C \setminus x \ . \tag{6.75}$$

Note that $K_x \simeq \mathbb{F}_{q^n}((t))$. This picture is our analogue of the boundary of a tubular neighborhood of a knot sitting inside of our 3-manifold:

$$N = \partial\left(\text{neighborhood}\left(K\right)\right) \hookrightarrow M_0 = M \setminus K \ . \tag{6.76}$$

So now we're excising a point (instead of a knot), and we're trying to understand this punctured disk rather than the boundary of a tubular neighborhood of the knot.

On the B/Galois side, we were looking at. $\operatorname{Loc}_1(C \setminus x)$. Whatever these are, they can be restricted to the punctured disk:

$$\operatorname{Loc}_1(C \setminus x) \to \operatorname{Loc}_1 D^\times = \operatorname{Hom}_{\text{cts}}\left(\operatorname{Gal} \overline{K}_x/K_x, \overline{\mathbb{Q}_\ell^\times}\right) \tag{6.77}$$

$$= \operatorname{Hom}_{\text{cts}}\left(\operatorname{Gal}^{\text{ab}} \overline{K}_x/K_x, \overline{\mathbb{Q}_\ell^\times}\right) \ . \tag{6.78}$$

I.e. we're studying characters of the abelianized Galois group, but they're arising as local systems on $D^\times$. So now the idea is that we're going to study local systems on $C \setminus x$ which are equipped with particular ramification data around the puncture.

On the A-side we are looking at $\operatorname{Pic} C$. Recall we had the adélic description of this from section 5.2.2:

$$\operatorname{Pic} C = F^\times \backslash {\prod_{y\in C}}' K_y^\times / \prod_{y \in C} \mathcal{O}_Y^\times \tag{6.79}$$

$$= F^\times \backslash {\prod}' \mathbb{Z} \ . \tag{6.80}$$

We have a similar description for the punctured curve:

$$\operatorname{Pic}(C \setminus x) = F^\times \backslash {\prod_{y\in C\setminus x}}' K_y^\times / \prod_{y \in C\setminus x} \mathcal{O}_y^\times \ . \tag{6.81}$$

We can write this as:

$$\operatorname{Pic}(C \setminus x) = F^\times \backslash {\prod_{y\in C\setminus x}}' K_y^\times / \prod_{y \in C\setminus x} \mathcal{O}_y^\times \tag{6.82}$$

$$= \operatorname{Pic} C / \left(\mathbb{Z}_x = K_x^\times / \mathcal{O}_x^\times\right) \ , \tag{6.83}$$

where we're modding out by "adding or subtracting copies of this point x". Then we can also write this as:

$$\operatorname{Pic}(C \setminus x) = F^\times \backslash {\prod_{y\in C\setminus x}}' K_y^\times / \prod_{y \in C\setminus x} \mathcal{O}_y^\times \tag{6.84}$$

$$= \operatorname{Pic} C / \left(\mathbb{Z}_x = K_x^\times / \mathcal{O}_x^\times\right) \tag{6.85}$$

$$= \operatorname{Pic}(C, \widehat{x}) / K_x^\times \tag{6.86}$$

where $\mathrm{Pic}\,(C,\widehat{x})$ is defined to be line bundles on C equipped with a trivialization on D_x. So we're modding out by this trivialization on the right.

$\mathrm{Pic}\,(C,\widehat{x})$ maps to $\mathrm{Pic}\,C$ with fiber given by changes of trivialization, and with intermediates $\mathrm{Pic}\,(C,nx)$ for $n\in\mathbb{Z}^+$:

$$\begin{tikzcd} \mathrm{Pic}\,(C,\widehat{c}) \arrow[rr, "\mathcal{O}_X^\times"] \arrow[dr] & & \mathrm{Pic}\,(C) \\ & \mathrm{Pic}\,(C,nx) \arrow[ur] & \end{tikzcd} , \tag{6.87}$$

where $\mathrm{Pic}\,(C,nx)$ consists of objects of $\mathrm{Pic}\,(C)$ with trivialization of order n around x. This story is telling us that to study the punctured curve, we don't need to puncture. We can study bundles over C equipped with a trivialization, as long as we understand how $K_x^\times$ acts.

In any case, $\mathrm{Pic}\,(C,\widehat{x})$ has an action of $K_x^\times$. So we're going to look at functions on $\mathrm{Pic}\,(C,\widehat{x})$ and decompose them under this action. From the TFT point of view we said this a little differently. Namely, we have a restriction:

$$\begin{tikzcd} \mathrm{Pic}\,(C\setminus x) \arrow[r, equal] \arrow[d] & \mathrm{Pic}\,(C,\widehat{x})\,/K^\times \arrow[d] \\ \mathrm{Pic}\,(D_x^\times) \arrow[r, equal] & \mathrm{pt}/K^\times \end{tikzcd} , \tag{6.88}$$

and we can take a sheaf and take global sections on $\mathrm{Pic}\,(C,\widehat{x})$, i.e. push forward to $\mathrm{pt}/K_x^\times$. Studying this picture is equivalent to studying the action of $K^\times$ on $\mathrm{Pic}\,(C,\widehat{x})$.

Number theorists will tell you that you don't want "infinite level structure". We have $K=\mathbb{F}_q\left((t)\right)$. Then we have:

$$K^\times=\mathbb{F}_q\left((t)\right)^\times\supset\mathcal{O}^\times=\mathbb{F}_q\left[[t]\right] \ . \tag{6.89}$$

This also contains these subgroups:

$$K^\times\supset\mathcal{O}^\times\supset\mathcal{O}^{\times(1)}\supset\cdots\supset\mathcal{O}^{\times(n)}\supset\cdots \tag{6.90}$$

where being in $\mathcal{O}^{\times(n)}$ just means you're of the form $(1+t^n)(-)$. Before, the quotient $K^\times/\mathcal{O}^\times=\mathbb{Z}$ gave us divisors. So in the unramified situation we had $\mathbb{Z}$ acting on $\mathrm{Pic}\,(C)$. Now when we add level structure, i.e. we pass from $\mathrm{Pic}\,(C)$ to $\mathrm{Pic}\,(C,nx)$, we have an action of $K^\times/\mathcal{O}^{\times(n)}$.

So what we do is look at *smooth* representations of $K^\times$, which means every vector is fixed by $\mathcal{O}^{\times(N)}$ for $N\gg0$. This is equivalent to topological continuity if we give the thing it's acting on the discrete topology.

Now we want to establish an analogue to the picture that for a 3-manifold M_0 with boundary $\partial M_0=\Sigma$, we get

$$Z\left(M_0\right)\in Z\left(\Sigma\right) \ . \tag{6.91}$$

The analogue here is as follows. To $C/\mathbb{F}_q$ and $x\in C$ we can attach

$$\mathrm{Pic}\,(C,\widehat{x}) \ . \tag{6.92}$$

Now we want to consider

$$\mathbb{C}\left[\mathrm{Pic}\,(C,\widehat{x})\right] \ , \tag{6.93}$$

specifically thought of as a representation of $K_x^\times$. This can be thought of as taking the constant sheaf $\underline{k}_{\mathrm{Pic}(C\backslash x)}$ and pushing it forward to get

$$\pi_* \underline{k} \in \mathbf{Shv}\left(\mathrm{pt}/K^\times\right) = \mathrm{Pic}\left(D^\times\right) \ , \tag{6.94}$$

i.e. a representation of $K^\times$.

So on the A-side we have a representation of $K^\times$ coming from Pic with infinite level structure, $\mathrm{Pic}\,(C, \widehat{x})$ (or some finite level structure $\mathrm{Pic}\,(C, nx)$). On the B-side we have a sheaf $\mathcal{O}_{\mathrm{Loc}(C\backslash x)}$ which we pushforward to get a sheaf on $\mathrm{Loc}_1\,(D^\times)$, i.e. a module for functions on $D^\times$. Now we need the analogue of the Fourier-Mukai transform from remark 28. This is the content of local class field theory.

Global class field theory says that:

$$\mathrm{Gal}^{\mathrm{ab,\ ur}} \simeq \widehat{\mathrm{Pic}} \ , \tag{6.95}$$

where $\widehat{\mathrm{Pic}}$ denotes the profinite completion. The statement of local class field theory says that there is a canonical map:

$$K_x^\times \to \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K_x}/K_x\right) \ , \tag{6.96}$$

which is an isomorphism after profinitely completing:

$$\widehat{K_x^\times} \xrightarrow{\sim} \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K_x}/K_x\right) \ . \tag{6.97}$$

This tells us that the group ring of $K^\times$ is Pontrjagin dual to:

$$\mathrm{Hom}\left(\mathrm{Gal}^{\mathrm{ab}}\,\overline{K_x}/K_x, \overline{\mathbb{Q}_\ell}^\times\right) = \mathrm{Hom}\left(\mathrm{Gal}\,\overline{K_x}/K_x, \overline{\mathbb{Q}_\ell}^\times\right) = \mathrm{Loc}_1\,(C) \ . \tag{6.98}$$

In other words:

$$\left(\overline{\mathbb{Q}_\ell}^\times\left[K^\times\right], *\right) \simeq \left(\overline{\mathbb{Q}_\ell}^\times\left[\mathrm{Loc}_1\right], \cdot\right) \ . \tag{6.99}$$

This is analogous to when we exchanged $\pi_1\,\mathrm{Jac} = \Lambda$ with $\mathbb{C}\left[T^\vee\right]$ in the case of a Riemann surface. Now passing to modules we get:

$$\mathbf{Rep}\left(K^\times\right) = \mathbf{QC}\left(\mathrm{Loc}_1\, C\right) \ . \tag{6.100}$$

To make a statement like this technically precise requires a lot of work. So the local statement is that local systems on the punctured disk have to do with representation of $K^\times$. And the global statement is that the vector space attached to a punctured surface, i.e. a representation of $K^\times$, is identified with local systems with a prescribed ramification.

6.6 Ramification for number fields

The statements are very similar for number fields. Local class field theory says that for a non-archimedean place v of F, i.e. a prime in $\mathcal{O}_F$, there is a natural map:

$$K_v^\times \to \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K}/K\right) \tag{6.101}$$

which is an isomorphism on the profinite completion:

$$\widehat{K^\times} \xrightarrow{\sim} \mathrm{Gal}^{\mathrm{ab}}\left(\overline{K}/K\right) \ . \tag{6.102}$$

E.g. think $K_v = \mathbb{Q}_p$. We can interpret this as identifying representations of $K^\times$ with modules for functions on GL_1 Galois representations.

The global class field theory says the following. The analogue to $\mathrm{Pic}\,(C, \widehat{x})$ is:

$$F^\times \backslash \prod_v K^\times / \prod_{v \notin S} \mathcal{O}^\times \ , \tag{6.103}$$

and functions on this has an action of:

$$\prod_{v \in S} K_v^\times \ . \tag{6.104}$$

So we can decompose them under this action. This will match with functions on characters of Galois groups ramified at S. Lecture 13; March 9, 2021

Chapter 7

Geometric class field theory

Let Σ be a Riemann surface. Then $\operatorname{Pic}\Sigma$ fits into the following short exact sequence

$$0 \to \operatorname{Jac}\Sigma \to \operatorname{Pic}\Sigma \to \mathbb{Z} \to 0 \tag{7.1}$$

and the Jacobian is a lattice, i.e. it is

$$V/H^1(\Sigma,\mathbb{Z}) \tag{7.2}$$

for some vector space V. This means that, by Poincaré duality:

$$\pi_1(\operatorname{Jac}) \simeq H_1(\Sigma,\mathbb{Z}) = \pi_1^{\mathrm{ab}}(\Sigma) \ , \tag{7.3}$$

so $\pi_1(\operatorname{Jac})$ is just the abelianization of $\pi_1(\Sigma)$.

The upshot is that for an abelian group A we have:

$$\operatorname{Loc}_A(\Sigma) = \{\pi_1(\Sigma) \to A\} / \sim \tag{7.4}$$

which is the same as

$$\operatorname{Loc}_A(\operatorname{Jac}\Sigma) = \{\pi_1(\operatorname{Jac}) \to A\} / \sim \tag{7.5}$$

since maps to an abelian group do not detect if the source is abelianized or not. I.e. abelian A-covers of Σ and abelian A-covers of $\operatorname{Jac}(\Sigma)$ are in bijection.

We can understand a finite abelian group via its characters, so we might as well take $A = \mathrm{GL}_1\,\mathbb{C} = \mathbb{C}^\times$. E.g. we can write

$$\operatorname{Loc}_A\Sigma = \operatorname{Hom}(A^\vee, \operatorname{Loc}_1\Sigma) \ . \tag{7.6}$$

This is just to say that the main content here is that:

$$\operatorname{Loc}_1\Sigma \simeq \operatorname{Loc}_1\left(\operatorname{Jac} = \operatorname{Pic}^0\right) \ . \tag{7.7}$$

We can express this in a more canonical way. The most important ingredient is the Abel-Jacobi map:

$$\Sigma \to \operatorname{Pic}^1\Sigma \ . \tag{7.8}$$

A point $x \in \Sigma$ goes to the divisor $1 \cdot x \in \operatorname{Pic}^1\Sigma$. Of course $\operatorname{Pic}^1 \simeq \operatorname{Pic}^0$. The way Abel and Jacobi would have written this is: given a basepoint $e \in \Sigma$ we get a map (which is also commonly called the Abel-Jacobi map)

$$\Sigma \to \operatorname{Jac} \tag{7.9}$$

where x maps to $x - e$.

The pullback is:

$$\mathrm{AJ}^* \colon \mathrm{Loc}_1\left(\mathrm{Pic}^1\right) \xrightarrow{\sim} \mathrm{Loc}_1(\Sigma) \ , \tag{7.10}$$

so AJ^* identifies A-covers of Pic^1 with A-covers of Σ. So this is a more canonical way to write this bijection from before.

7.1 Albanese property

What we have seen so far is really expressing something called the *Albanese property* of the Jacobian. For higher dimensional varieties, there are two different abelian varieties we can attach to them: the Picard and the Albanese, and they are dual. The idea is that this is not fundamentally a fact about divisors/line bundles but rather about 0-dimensional cycles. They just turn out to coincide here in dimension 1.

Let A be an abelian group scheme.[1] Given a map of schemes $\Sigma \to A$ for A an abelian group, the *Albanese property* says that this factors as:

$$\begin{tikzcd} \Sigma \arrow[rr] \arrow[dr, "\mathrm{AJ}_e"'] & & A \\ & \mathrm{Pic} \arrow[ur, dashed] & \end{tikzcd} \ . \tag{7.11}$$

In other words

$$\mathrm{AJ}^* \colon \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Pic}\,\Sigma, A) \xrightarrow{\sim} \mathrm{Map}(\Sigma, A) \ . \tag{7.12}$$

We can think of this as saying that Pic is the abelian group scheme "freely generated" by Σ.

Remark 30. For X smooth and projective there is an abelian variety $\mathrm{Alb}(X)$, called the *Albanese*, satisfying the above universal property. (Usually this is defines as 0-cycles modulo some equivalence relation.) Then the classical statement is that there is a canonical map $X \to \mathrm{Alb}(X)$. For a curve the Albanese happens to be the same as the Jacobian. The Albanese is a torus whose fundamental group is H_1. This is the dual abelian variety to the Picard. Accordingly, Pic^0 has fundamental group H^1.

7.2 Dold-Thom theorem

The Dold-Thom theorem relates homology of a pointed space (X, x) to $\mathrm{Sym}^\infty(X, x)$. We can think of $\mathrm{Sym}^\infty(X, x)$ as the "free commutative monoid" built out of X. Specifically the theorem says that the homotopy groups of this are the homology of X. Now we want to state the algebro-geometric analogue of this.

[1] Classically one might have taken A to be an abelian variety. The most modern version might be to take A to be an abelian group stack.

We have the following maps:

$$
\begin{aligned}
\Sigma &\hookrightarrow \mathrm{Pic}^1 \\
\mathrm{Sym}^2 \Sigma &\longrightarrow \mathrm{Pic}^2 \\
x, y &\longmapsto x + y \\
\dots & \\
\mathrm{Sym}^d \Sigma &\longrightarrow \mathrm{Pic}^d
\end{aligned}
\tag{7.13}
$$

which assemble to give a map:

$$
\mathrm{AJ}^\bullet \colon \mathrm{Sym}^\bullet (\Sigma) \to \mathrm{Pic}^\bullet (\Sigma) \ . \tag{7.14}
$$

An element of $\mathrm{Sym}^\bullet \Sigma$ is what is known as an *effective divisor*. I.e. it's some linear combination of points with positive coefficients. This is the same as a line bundle $\mathcal{L}$ and a holomorphic section $\sigma \in \Gamma(\mathcal{L})$. I.e. it has no poles, but it does have some zeros determining a weighted combination of points. From this point of view, the $\mathrm{AJ}^\bullet$ map is just forgetting the section. In other words, $\mathrm{AJ}^\bullet$ has fibers given by:

$$
\left(\mathrm{AJ}^\bullet\right)^{-1} (\mathcal{L}) = \mathbb{P}\left(H^0 (\Sigma, \mathcal{L})\right) \ , \tag{7.15}
$$

where we take the projectivization since a divisor only determines a section up to a scalar.

The Riemann-Roch theorem says that for $d > 2g - 2$ then $H^1(\Sigma, \mathcal{L}) = 0$, so we just get a projective space bundle $\mathbb{P}^{d-g}$. So if we look at high enough degrees, these spaces $\mathrm{Sym}^\bullet$ and Pic look more similar. The point is that these projective spaces are simple-connected, so $\mathrm{AJ}^\bullet$ is an isomorphism on π_1 for $d \gg 0$.

The idea is that we want to form a better understanding of the relationship between $\mathrm{Loc}(\mathrm{Pic})$ and $\mathrm{Loc}_1 \Sigma$. Recall that:

$$
\mathrm{Loc}(\mathrm{Pic}) \simeq \mathbf{QC}(\mathrm{Loc}_1 \Sigma) \ . \tag{7.16}
$$

This is what we want to wrap our heads around. The problem was that Σ only saw a single component of Pic. But to understand $\mathrm{Loc}_1 \Sigma$, we have to relate the different components together. This is what the statement of geometric class field theory will say.

7.3 Recall: unramified CFT

Recall the unramified CFT for a curve $C/\mathbb{F}_q$. The statement was that rank 1 local systems on C correspond to characters of Pic. Explicitly we started with $E \in \mathrm{Loc}_1(C)$ and we attached χ_E, a character of Pic. Recall that being a character means:

$$
\chi_E (\mathcal{L} \otimes \mathcal{M}) \simeq \chi_E (\mathcal{L}) \cdot \chi_E (\mathcal{M}) \ . \tag{7.17}
$$

But how does this relate to E? Recall Pic is generated by $\mathbb{Z}x$ for $x \in C$. Then we can characterize χ_E as satisfying the following. We know:

$$
\chi_E (\mathcal{L}(x)) = \chi_E (\mathcal{L} \otimes \mathcal{O}(x)) \tag{7.18}
$$

where $\mathcal{O}(x)$ is the line bundle with associated divisor $1 \cdot x$. By the character property:

$$\chi_E(\mathcal{L}(x)) = \chi_E(\mathcal{L} \otimes \mathcal{O}(x)) \tag{7.19}$$
$$= \chi_E(\mathcal{O}(x)) \cdot \chi_E(\mathcal{L}) \ , \tag{7.20}$$

and this number $\chi_E(\mathcal{O}(x))$ should be the value of the local system E on Fr_x, the Frobenius element at x. Note that $\mathcal{O}(x) = \mathrm{AJ}(x)$, so this value $\chi_E(\mathcal{O}(x))$ is really something about the curve.

7.4 Character sheaves

Now we want to state a geometric[2] version. Let C/k be a curve over any field (or Σ a Riemann surface if we just want $k = \mathbb{C}$). We don't have Frobenius elements anymore, so we can't automatically form this kind of statement, but we can formulate something else. Consider *character local systems* on $\mathrm{Pic}(C)$. These satisfy a multiplicative property, just as ordinary characters did. Specifically a *character local system* is a local system $\chi \in \mathrm{Loc}(\mathrm{Pic})$ equipped with an isomorphism:

$$\mu^*\chi \xrightarrow{\sim} \chi \boxtimes \chi \tag{7.21}$$

where $\mu\colon \mathrm{Pic} \times \mathrm{Pic} \to \mathrm{Pic}$. This also has to satisfy an associativity relation.

Digression 4. We can rewrite the definition (7.17) of a character χ_E as insisting that:

$$\mu^*\chi_E = \chi_E \boxtimes \chi_E \ . \tag{7.22}$$

So we can insist on the same condition on sheaves. I.e. that

$$\mu^*\chi \xrightarrow{\sim} \chi \boxtimes \chi \tag{7.23}$$

or equivalently

$$\chi|_{\mathcal{L}\otimes\mathcal{M}} \simeq \chi|_{\mathcal{L}} \otimes \chi|_{\mathcal{M}} \ . \tag{7.24}$$

We can make this definition of a character sheaf on anything with sheaves living on it. The claim is that they are always rank 1. The idea is that

$$\chi(0 + a) \cong \chi(a) \ , \tag{7.25}$$

so $\chi(0)$ is 1-dimensional, and there is a unital property that identifies it with k. Then $\chi(a - a) \cong \chi(0) \cong \chi(a) \to \chi(-a)$. This means the total space is a $k^\times$ torsor over A:

$$\begin{array}{c} A_\chi = \mathrm{Tot}(\chi)^\times \\ \big\downarrow{\scriptstyle k^\times} \\ A \end{array} \ . \tag{7.26}$$

So the statement is that χ is a character iff A_χ is an abelian group, i.e. it defines an extension:

$$0 \to k^\times \to A_\chi \to A \to 0 \ . \tag{7.27}$$

So really we're saying $\chi \in \mathrm{Ext}^1(\mathrm{Pic}, \mathbb{G}_m)$, i.e. χ is a character iff it is a homomorphism

$$\chi\colon \mathrm{Pic} \to B\mathbb{G}_m \ . \tag{7.28}$$

[2] This means we're looking at sheaves.

Let χ be a character local system on $\operatorname{Pic} C$. Then

$$\chi\left(\mathcal{L}\otimes\mathcal{M}\right)\simeq\chi\left(\mathcal{L}\right)\otimes\chi\left(\mathcal{M}\right)\ . \tag{7.29}$$

Just like with ordinary characters, we can look at $\mathcal{L}$ modified at a single point $x\in C$:

$$\chi\left(\mathcal{L}\left(x\right)\right)=\chi\left(\underbrace{\mathcal{O}\left(x\right)}_{=\mathrm{AJ}(x)}\right)\otimes\chi\left(\mathcal{L}\right)\ . \tag{7.30}$$

Let $L_\chi=\mathrm{AJ}^*\chi\in\mathrm{Loc}_1 C$. Then this is saying that

$$\chi\left(\mathcal{L}\left(x\right)\right)\simeq L_\chi\big|_x\to\chi\left(\mathcal{L}\right)\ . \tag{7.31}$$

We can say this in families. We have a diagram:

$$\begin{tikzcd}
\mathrm{Pic}\times\mathrm{Pic} \arrow[r, "\otimes"] & \mathrm{Pic} & \mathcal{L}\left(x\right) \\
C\times\mathrm{Pic} \arrow[u, hook, "\mathrm{AJ}\times\mathrm{id}"] \arrow[ur, "\mu_C"] & & \\
\left(x,\mathcal{L}\right) \arrow[uurr, mapsto] & &
\end{tikzcd}\ , \tag{7.32}$$

and the statement is that we have an identification

$$\mu_C^*\chi\simeq L\boxtimes\chi\ . \tag{7.33}$$

This is called the *Hecke eigenproperty*. This should be enough to completely determine the multiplicativity of χ.

7.5 Geometric CFT

Geometric CFT says that the map

$$\chi\mapsto L_\chi=\mathrm{AJ}^*\chi \tag{7.34}$$

gives an equivalence between:

$$\{\text{character local systems on }\ \mathrm{Pic}\}\simeq\{\text{rank 1 local systems on } C\}\ . \tag{7.35}$$

Before we used some π_1 calculations to learn that local systems on one component of Pic and local systems on C are the same. But now we're looking at local systems on all of Pic which have this very strong property of being characters. And the claim is that if we take such a character, then it is completely determined by the rank 1 local system on C we get by restricting.

We already have a map one way. Inside of Pic we have:

$$\mathrm{AJ}\colon C\hookrightarrow\mathrm{Pic}^1\subset\mathrm{Pic}\ . \tag{7.36}$$

But we should imagine that we have all of $\operatorname{Sym}^\bullet C \to \operatorname{Pic}$:

$$\begin{tikzcd} \operatorname{Sym}^\bullet C \arrow[r] & \operatorname{Pic} \\ C \arrow[u, hook] \arrow[r, hook, "\mathrm{AJ}"] & \operatorname{Pic}^1 \arrow[u, hook] \end{tikzcd} . \tag{7.37}$$

$\operatorname{Sym}^\bullet C$ is like the free monoid generated by C, whereas $\operatorname{Pic}(C)$ is like the free abelian group generated by C. So this map is a kind of group completion. The claim is going to be that any rank 1 local system L on C extends canonically to $\operatorname{Sym}^\bullet C$, and then this descends to Pic, the "group completion".

Proof (Deligne). Start with a local system L on C. First we extend this to $\operatorname{Sym}^\bullet C$. Write a local system $L^{(d)}$ on $\operatorname{Sym}^d C$ defined by:

$$L^{(d)}\left(\sum_1^d x_i\right) = \bigotimes_i L(x_i) \ . \tag{7.38}$$

This defines a rank 1 local system on $\operatorname{Sym}^d X$ for any d. More formally, we start L and construct $L^{\boxtimes d}$ on C^d. Then we can project down to $\pi_* L^{\boxtimes d}$ where $\pi\colon C^d \to \operatorname{Sym}^d$. Then we take invariants under the symmetric group:

$$\left(\pi_* L^{\boxtimes d}\right)^{S_d} \ . \tag{7.39}$$

Now we need to descend down to $\operatorname{Pic} C$.

The key point is that $\mathbb{P}^{d-g}$ (the fibers of the map $\operatorname{Sym}^d \to \operatorname{Pic}^d$ for $d \gg 0$) are simply connected, so $\operatorname{Sym}^\bullet C \to \operatorname{Pic} C$ is an isomorphism in high enough degree. Therefore $L^{(d)}$ descends to Pic^d, i.e.

$$L^{(d)} \simeq \mathrm{AJ}^{d^*}\left(\chi_L^d\right) \tag{7.40}$$

for χ_L^d a local system on Pic^d.

So starting with L we constructed $\chi_L^\bullet$ on Pic^d (for $d \gg 0$) which satisfies the multiplicative property. Then there exists a unique extension of $\chi_L^\bullet$ to all of Pic satisfying the multiplicative property. Then this gives the formula:

$$\chi(D) = \chi(D_1) \otimes \chi(D_2)^{-1} \ , \tag{7.41}$$

for $D = D_1 - D_2$ where D_1 and D_2 are effective divisors of degree $d \gg 0$. ☐

Remark 31. We followed [Bha] for this proof. There he also mentions that the same proof works for *ramified* geometric CFT. So for a subset of points $S \subset \sigma$:

$$\mathrm{AJ}\colon C \setminus S \hookrightarrow \operatorname{Pic}(C, S) \tag{7.42}$$

where $\operatorname{Pic}(C,S)$ consists of line bundles trivialized near S.

We want to think of this geometric CFT as a spectral decomposition of $\operatorname{Loc}(\operatorname{Pic})$. Lecture 14; March 11, 2021

Chapter 8

Spectral decomposition 2.0: Tannakian reconstruction

The basic theme is: if you see commutativity then this indicates geometry. Now we are taking it up a notch. If R is commutative, we can consider the category R-**mod**. This category is commutative itself under the operation:

$$(M, N) \mapsto M \otimes_R N \ . \tag{8.1}$$

This makes R-**mod** into a symmetric monoidal category (or $\otimes$-category). In brief this is a category $\mathcal{C}$ which is abelian and has some "niceness properties", e.g. we might insist on including arbitrary direct sums. The main example we will have in mind, and in particular satisfies any niceness properties, is R-**mod**. So we have a tensor product:

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \tag{8.2}$$

and a unit $1 \in \mathcal{C}$, and then they have to satisfy some axioms: the identity axiom,

$$1 \otimes (-) \simeq (-) \otimes 1 \ , \tag{8.3}$$

if you switch twice you come back to the identity:

$$\begin{tikzcd} M \otimes N \arrow[r, "\sim"] \arrow[rr, bend right, "\mathrm{id}"'] & N \otimes M \arrow[r, "\sim"] & M \otimes N \end{tikzcd} \ , \tag{8.4}$$

and then we have associativity (pentagon identity) and commutativity (hexagon identity) conditions on $\otimes$.

Before we just used R-modules for spectral decomposition. Now we're viewing R-**mod** itself as some kind of (categorified) commutative ring. For an affine scheme $X = \operatorname{Spec} R$, we can basically define:

$$\mathbf{QC}\,(X) = R\text{-}\mathbf{mod} \ , \tag{8.5}$$

and the tensor products match too. But not many schemes are affine. This is, in some sense, the main difference between algebraic geometry and differential topology: in differential topology C^∞ functions can capture any space, but in algebraic geometry (algebraic) functions only capture a small collection of varieties. However, in algebraic geometry, when we replace functions by sheaves we get a lot more flexibility. For any scheme X we have

$$X = \varinjlim \operatorname{Spec} R_i \ , \tag{8.6}$$

i.e. X is built out of affines. Similarly:

$$\mathbf{QC}(X) = \varprojlim R_i\text{-}\mathbf{mod} \ . \tag{8.7}$$

The key point is that if we have a map:

$$\operatorname{Spec} S = Y \xrightarrow{f} X = \operatorname{Spec} R, \tag{8.8}$$

i.e. a map $R \to S$, then we get a functor:

$$(-) \otimes_R S \colon R\text{-}\mathbf{mod} \xrightarrow{f^*} S\text{-}\mathbf{mod} \ , \tag{8.9}$$

i.e. a functor:

$$f^* \colon \mathbf{QC}(X) \to \mathbf{QC}(Y) \ . \tag{8.10}$$

The key fact is that f^* is actually a *symmetric monoidal functor*. I.e. the tensor product goes to the tensor product, the unit R goes to the unit S, preserving all the extra structure. I.e. we can build $\mathbf{QC}(X)$ out of the local pieces R_i-**mod** as a *symmtric monoidal* category, i.e. the tensor products for R_i-**mod** induce the correct tensor product globally.

This is all to say that $\mathbf{QC}(X)$ has the universal property that whenever I have an affine $f \colon \operatorname{Spec} R \to X$, we get a functor $f^* \colon \mathbf{QC}(X) \to R$-**mod**, and this preserves tensor the product. In other words we have an operation:

$$\mathbf{QC} \colon \mathbf{Sch} \to (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \ . \tag{8.11}$$

This is the categorified analogue of:

$$\mathcal{O} \colon \mathbf{Spaces} \to (\mathbf{cRing})^{\mathrm{op}} \ . \tag{8.12}$$

Remark 32. We can define $\mathbf{QC}$ on a bigger source category, e.g. a category of stacks, **Stacks**, or even prestacks. The point is just that the objects can be formally built as colimits of affines.

So we have a big source of $\otimes$-categories from algebraic geometry. We can think of this as the *geometric* source of $\otimes$-categories. This is our home base. For us, we will "understand" a $\otimes$-category when it is sheaves on some kind of space. We will test arbitrary "commutative rings" by comparing them to these "basic" geometric examples of the form R-**mod**. This means that the "questions" we ask will be geometrically biased. We have this functor:

$$\mathbf{QC} \colon \mathbf{Stacks} \to (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \ , \tag{8.13}$$

and we want to construct a right adjoint:

$$\operatorname{Spec} \colon (\otimes\text{-}\mathbf{Cat})^{\mathrm{op}} \to \mathbf{Stacks} \ . \tag{8.14}$$

So we start with a $\otimes$-category $\mathcal{C}$, and we want to construct some algebro-geometric space (e.g. a stack) called $\operatorname{Spec} \mathcal{C}$. This is the best approximation of $\mathcal{C}$ within the world of algebraic geometry. Technically the adjunction says that $\operatorname{Spec}(\mathcal{C}, \otimes, 1)$ is characterized by the property that:

$$\operatorname{Hom}_{\mathbf{Stacks}}(X, \operatorname{Spec}\mathcal{C}) = \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, \mathbf{QC}(X)) \ . \tag{8.15}$$

This is analogous to how we characterized Spec of a ring in eq. (2.1) in section 2.1. This boils down to:

$$\operatorname{Hom}_{\mathbf{Stacks}}(\operatorname{Spec} R, \operatorname{Spec}\mathcal{C}) = \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, R\text{-}\mathbf{mod}) \ , \tag{8.16}$$

so we define this as a functor on rings:

$$\mathbf{cRing} \to \mathbf{Grpd} \ , \tag{8.17}$$

by sending

$$R \mapsto \operatorname{Spec}\mathcal{C}(R) \tag{8.18}$$

where

$$\operatorname{Spec}\mathcal{C}(R) \coloneqq \operatorname{Hom}_{\otimes\text{-}\mathbf{Cat}}(\mathcal{C}, R\text{-}\mathbf{mod}) \ . \tag{8.19}$$

For references see [Lur04] and [BHL17]. cite

The adjunction implies that there is a canonical map for any X:

$$X \to \operatorname{Spec}\mathbf{QC}(X) \ . \tag{8.20}$$

We can think of this as a sort of a 1-affinization (or "Tannakization") of X. This is just like how we always have a map to the usual affinization:

$$X \to \operatorname{Spec}\mathcal{O}(X) \ , \tag{8.21}$$

which was the best affine approximation of X.

Now let's check that this recovers our old notion of Spec. Suppose $\mathcal{C} = S\text{-}\mathbf{mod}$ for $S \in \mathbf{cRing}$, and consider a $\otimes$-functor:

$$F \colon \mathcal{C} \to R\text{-}\mathbf{mod} \ . \tag{8.22}$$

This tells us that $1 \in \mathcal{C}$ goes to $R \in R\text{-}\mathbf{mod}$. Because this is a functor we get a map

$$\operatorname{End}(1) \to \operatorname{End}_{R\text{-}\mathbf{mod}}(R) = R \ . \tag{8.23}$$

We can think of $\operatorname{End}(1)$ as being a commutative ring attached to $\mathcal{C}$. I.e. we have a map:

$$\operatorname{Hom}_{\otimes}(\mathcal{C}, R\text{-}\mathbf{mod}) \to \operatorname{Hom}_{\mathbf{cRing}}(\operatorname{End}(1), R) \ , \tag{8.24}$$

and elements of the target are easier to understand. If $\mathcal{C} = S\text{-}\mathbf{mod}$, then $\mathcal{C}$ is *generated* by $1 = S$. This means we can build (resolve) any S-modules by resolutions starting with S. Another way to think about this is that: $\operatorname{Hom}_{S\text{-}\mathbf{mod}}(S, -)$ is the underlying abelian group. The upshot is that any $\otimes$-functor $S\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$ comes from a ring map $S \to R$, so

$$\operatorname{Spec} S\text{-}\mathbf{mod} = \operatorname{Spec} S \ . \tag{8.25}$$

I.e. on $\mathcal{C} = S\text{-}\mathbf{mod}$ our new Spec recovers our old notion of Spec.

The key point is that most schemes/stacks don't have enough functions, so

$$\operatorname{Spec} \mathcal{O}(X) \neq X \ . \tag{8.26}$$

So we want to see if we are any luckier a categorical level up. And it turns out we are: we *do* have enough quasi-coherent sheaves to see most stacks. The main issue is that functions don't push, only distributions/measures do. But once we have categorified we do have a pushforward functor.

Theorem 4. *If X is a* geometric stack *then*

$$X \xrightarrow{\sim} \operatorname{Spec} \mathbf{QC}(X) \tag{8.27}$$

is an isomorphism. I.e. X is Tannakian/categorically affine*.*

The notion of a geometric stack includes anything we encounter in practice. In particular it includes any quasicompact scheme. The definition we will use is that a *geometric stack* is a quasicompact stack with an affine diagonal. In practice this means that we can explicitly build X out of quotients of some affine U by the action of some affine groupoid $\mathcal{G}$. The idea is that we can build it out of affines as long as we insist on some relations coming to us from the groupoids $\mathcal{G}$.

Example 32. If X is a scheme then maps $f\colon \operatorname{Spec} R \to X$ are the same as maps $f^*\colon \mathbf{QC}(X) \to R$-**mod**.

Example 33. Let $f\colon \operatorname{Spec} k \to X$ be a point of X. Then this corresponds to a map

$$f^*\colon \mathbf{QC}(X) \to \mathbf{Vect}_k \ . \tag{8.28}$$

8.1 Category of representations

Besides $\mathcal{C} = \mathbf{QC}(X)$, the category of representations of a group is the quintessential example of Tannakian duality. Let G be a group. Then the category $\mathbf{Rep}_k(G)$ is a $\otimes$-category. I.e. the objects are k-vector spaces with a G-action, and morphisms are linear maps respecting the G-action. The unit is the trivial rep $1 = k$, and the tensor product is the usual one of vector spaces with the diagonal G-action.

To calculate $\operatorname{Spec}(\mathbf{Rep}(G))$ we will consider some extra structure on this category. We have a *faithful $\otimes$-functor*

$$\mathbf{Rep}_k G \to \mathbf{Vect}_k \ , \tag{8.29}$$

i.e. maps in the source inject into maps in the target. This is called a *fiber functor*, i.e. for any monoidal category $(\mathcal{C}, \otimes, 1)$ a faithful $\otimes$-functor from $\mathcal{C} \to \mathbf{Vect}_k$ is called a *fiber functor*. Having a fiber functor is actually saying that $\operatorname{Spec} \mathcal{C}$ is covered by a point, i.e. we have a faithfully flat map

$$\mathrm{pt} \to \operatorname{Spec} \mathcal{C} \ , \tag{8.30}$$

i.e. as a space this is a point modulo the action of some groupoid.

Remark 33. Having a fiber functor is kind of orthogonal to the example $\mathcal{C} = \mathbf{QC}(X)$. Here we get functors

$$i^*\colon \mathcal{C} = \mathbf{QC}(X) \to \mathbf{Vect}_k \tag{8.31}$$

where $i\colon \operatorname{Spec} k \to X$, but they kill most things. But this makes sense: $\operatorname{Spec}(\mathbf{QC}(X))$ should not be covered by a point in general.

Let G be an affine algebraic group, i.e. G is an affine variety. This implies:

$$\mathrm{Spec}\left(\mathbf{Rep}\left(G\right)\right) = BG = \mathrm{pt}/G \ . \tag{8.32}$$

This is indeed covered by a point

$$G \rightrightarrows \mathrm{pt} \longrightarrow \mathrm{pt}/G \quad . \tag{8.33}$$

For general $\mathcal{C}$ equipped with some fiber functor F, we get a group $G = \mathrm{Aut}\left(F\right)$, and

$$\mathrm{Spec}\,\mathcal{C} = BG^{\mathrm{proalg}} \ , \tag{8.34}$$

where G^{proalg} is the pro-algebraic completion of G. This is the thing we see out of G by studying algebraic representations.

Now we're going to look for $\otimes$-categories (in nature) which naturally have a fiber functor. From this we can produce a group. This sounds a bit abstract until we realize that this comes up all over the place.

Example 34. Again, we want to consider vector spaces with some extra structure. So consider $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$, the category of $\mathbb{Z}$-graded vector spaces. The tensor product is given by:

$$\left(M^{\bullet} \otimes N^{\bullet}\right)_k = \bigoplus M^i \otimes N^{k-i} \ . \tag{8.35}$$

This has a functor:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \xrightarrow{F} \mathbf{Vect} \ , \tag{8.36}$$

which is a fiber functor. This means we get a group G such that $\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Rep}G$. In this case $G = \mathbb{G}_m$:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Rep}\mathbb{G}_m \simeq \mathbf{QC}\left(B\mathbb{G}_m\right) \ , \tag{8.37}$$

which is an equivalence we have seen before.

This is a categorical version of the Fourier duality between $\mathbb{Z}$ and $\mathbb{G}_m$. Similarly we have an equivalence between lattice-graded representations and representations of $T^{\vee}$:

$$\mathbf{Vect}^{\Lambda\text{-gr}} \simeq \mathbf{QC}\left(BT^{\vee}\right) \ . \tag{8.38}$$

Pontrjagin duality told us that G being compact was dual to $G^{\vee}$ being discrete. Now the discreteness of $\mathbb{Z}$ is manifested in the semisimplicity of $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$. This is equivalent to G being reductive. In characteristic 0, being reductive can even be defined to mean that $\mathbf{QC}\left(\mathrm{pt}/G\right) = \mathbf{Rep}G$ is semisimple.

Example 35. This is where the name "fiber functor" comes from. Let X be a connected topological space. Out of this we will construct a $\otimes$-category with a fiber functor. The category is $\mathrm{Loc}\left(X\right)$, the category of local systems on X. This has a tensor structure given by pointwise tensoring. Now choose a basepoint $x \in X$. This gives a functor:

$$F_x \colon \ \mathrm{Loc}\left(X\right) \xrightarrow{\text{fiber at } x} \mathbf{Vect}_k \ . \tag{8.39}$$

The way we defined the tensor product makes this a tensor functor. X being (reasonable and) connected means F_x is faithful (i.e. if a local system is 0 at x then it is 0 everywhere). This tells us that we get a Tannakian group G so that $\mathrm{Loc}\left(X\right) = \mathbf{Rep}\left(G\right)$. G is essentially the fundamental group based at x, $\pi_1\left(X, x\right)$. The idea is that representations of π_1 on a

k-vector space are the same as k-local systems on X where we send a local system to its monodromy. The formal statement is that:

$$G = \pi_1^{\text{pro-alg}}(X, x) \ , \tag{8.40}$$

i.e. the part of π_1 you see by studying only its finite-dimensional representations.

Example 36. Let $\mathcal{C} = \mathbf{Vect}_k$ for $k \neq \overline{k}$. Then we can tensor with $\overline{k}$ to get a functor:

$$\mathbf{Vect}_k \xrightarrow{(-)\otimes_k \overline{k}} \mathbf{Vect}_{\overline{k}} \ . \tag{8.41}$$

This is a fiber functor and:

$$\mathbf{Vect}_k \simeq \mathbf{Rep}\left(\mathrm{Gal}\,\overline{k}/k\right) \ . \tag{8.42}$$

8.2 Fourier-Mukai theory

Our next source of examples will come from "Fourier-Mukai" theory. We will follow the same logic from Pontrjagin duality. The point was that a nice source of commutative rings is group algebras of abelian groups. Likewise we can play the Tannakian game for particular tensor-categories. Let G be commutative with product $\mu \colon G \times G \to G$. Then we can always push sheaves, so we get a functor:

$$\mathbf{Shv}(G) \times \mathbf{Shv}(G) \xrightarrow{\mu_*} \mathbf{Shv}(G) \ . \tag{8.43}$$

This sends a pair of sheaves $\mathcal{F}$ and $\mathcal{G}$ to the convolution product $\mathcal{F} * \mathcal{G}$. This operation makes sense for any group, but for G abelian this makes $\mathbf{Shv}(G)$ a $\otimes$-category (i.e. convolution is commutative).

Remark 34. We're being intentionally vague about what we mean by **Shv**. E.g. in topology we might take local systems, in algebraic geometry we might take **QC**, etc. but in any setting this will still be a $\otimes$-category.

We can define the 1-*shifted Cartier/Fourier-Mukai dual* of G to be:

$$G^\vee[1] \coloneqq \mathrm{Spec}\left(\mathbf{Shv}(G), *\right) \ . \tag{8.44}$$

Now we're in more of a "Fourier Transform-like setting". We start with G, we pass to $\mathbf{Shv}(G)$, which is a kind of categorified group algebra, and we get a new space/group out of this Tannakian machinery.

Example 37. Let $G = \mathbb{Z}$. Then $\mathbf{Shv}(\mathbb{Z}) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}}$, and the convolution operation is the tensor product of graded vector spaces:

$$(M \otimes N)_n = \bigoplus M_i \otimes N_{n-i} \ . \tag{8.45}$$

Then

$$\mathrm{Spec}\left(\mathbf{Shv}(\mathbb{Z}), *\right) = B\mathbb{G}_m = \mathbb{Z}^\vee[1] \ . \tag{8.46}$$

Recall that the ordinary dual was $\mathbb{Z}^\vee \simeq \mathbb{G}_m$.

Lecture 15;
March 23, 2021

8.2.1 Cartier

Consider a setting where Cartier duality works, e.g. G a finite group scheme. Then recall from section 3.6 that the Cartier dual of G is

$$G^\vee = \mathrm{Spec}\left(\mathbb{C}G, *\right) \ . \tag{8.47}$$

The point being that functions on G with $*$ are identified with functions on $G^\vee$ with $\cdot$. Passing to modules we get:

$$\mathbf{Rep}\left(G\right) \simeq \mathbf{QC}\left(G^\vee\right) \ . \tag{8.48}$$

But we can understand $\mathbf{Rep}\left(G\right)$ as sheaves on the classifying space:

$$\mathbf{Rep}\left(G\right) \simeq \mathbf{QC}\left(BG\right) \ . \tag{8.49}$$

If we really want an equivalence of sheaves (like we have an equivalence of functions), then we get: The upshot is that we get an equivalence of sheaves:

$$\mathbf{QC}\left(G^\vee\right) \simeq \mathbf{QC}\left(BG\right) \ , \tag{8.50}$$

so this Fourier-Mukai duality exchanges G with B of its ordinary Cartier dual, so this is why we also called the Fourier-Mukai dual the 1-shifted Cartier dual. This is exactly what we saw in example 37 where $\mathbb{Z}$ got exchanged with $B\mathbb{G}_m$.

Let's try to say this a bit more methodically. We want to form a better understanding of $G^\vee[1]$. The k-points are $\otimes$-functors from sheaves on G (with convolution) to k-vector spaces:

$$\left(\mathbf{QC}\left(G\right), *\right) \to \left(\mathbf{Vect}_k, \otimes\right) \ . \tag{8.51}$$

Now we want to see what this tells us about the group itself. We're thinking about $\mathbf{Vect}_k$ as:

$$\mathbf{Vect}_k = \mathrm{End}\left(\mathbf{Vect}_k\right) \ . \tag{8.52}$$

The idea is that this map (8.51) is a "categorical 1-dimensional representation" of $\left(\mathbf{QC}\left(G\right), *\right)$, in analogy with a 1-dimensional representation $G \to \mathrm{GL}\left(k\right) \simeq k^\times$, or equivalently:

$$\mathbb{C}G \to \mathrm{End}\left(k\right) \simeq k \ . \tag{8.53}$$

So a k-point:

$$\mathbf{QC}\left(G\right) \to \mathrm{End}\left(\mathbf{Vect}_k\right) \ , \tag{8.54}$$

and we can restrict this to the invertible elements G to get a map:

$$G \to \mathrm{Aut}\left(\mathbf{Vect}_k\right) \ . \tag{8.55}$$

Now notice that these automorphisms are just given by tensoring with lines:

$$\mathrm{Aut}\left(\mathbf{Vect}_k\right) = \left(\text{lines}, \otimes\right) = B\mathbb{G}_m \ . \tag{8.56}$$

$B\mathbb{G}_m$ is a group object in the world of stacks, i.e. as a functor it is valued in Picard groupoids (symmetric monoidal groupoids with invertible objects). It has R-points given by:

$$B\mathbb{G}_m\left(R\right) = \left(R\text{-lines}, \otimes\right) \ . \tag{8.57}$$

The upshot of this discussion is that a k-point of $G^\vee[1]$ amounts to a map:

$$G \to B\mathbb{G}_m\ . \tag{8.58}$$

Recall $B\mathbb{G}_m$ showed up in example 37 as the 1-shifted Cartier dual of $\mathbb{Z}$. Now we're saying that $B\mathbb{G}_m$ plays in important role in general.

Recall the points of $\operatorname{Spec}(\mathbb{C}G, *)$ are *characters* of G:

$$G \to \mathbb{G}_m = \operatorname{Aut}(k)\ . \tag{8.59}$$

The claim is that the same analysis works a level up. I.e. the points of $\operatorname{Spec}(\mathbf{Shv}(G), *)$ are categorical characters/character sheaves:

$$G \to B\mathbb{G}_m = (\text{lines}, \otimes)\ . \tag{8.60}$$

This is the same notion that we discussed in section 7.4. The idea is that it is a consistent attachment of lines $\mathcal{L}_g$ to the elements $g \in G$. I.e. this is a line bundle $\mathcal{L}$ over G. This map is actually a homomorphism so this line bundle is multiplicative, which we wrote before as the condition that:

$$\mu^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}\ . \tag{8.61}$$

This is just writing $\mathcal{L}_{gh} = \mathcal{L}_g \otimes \mathcal{L}_h$ in families. So the claim is that:

$$G^\vee[1] \cong \operatorname{Hom}_{\mathbf{Grp}}(G, B\mathbb{G}_m) \tag{8.62}$$

$$= \text{mult. line bundles.} \tag{8.63}$$

Then multiplicative line bundles are given by Ext^1 to $\mathbb{G}_m$:

$$G^\vee[1] = \operatorname{Ext}^1_{\mathbf{Grp}_{\mathrm{Ab}}}(G, \mathbb{G}_m) \tag{8.64}$$

$$= \text{extensions of } G \text{ by } \mathbb{G}_m\ . \tag{8.65}$$

Remark 35. Usual commutative group schemes and commutative group stacks live in the derived category of sheaves of abelian groups on the category $\mathbf{Sch}^{\mathrm{Aff}} = \mathbf{Ring}^{\mathrm{op}}$. This shift in the notation for the Fourier-Mukai dual is literally a shift in this derived category.

8.2.2 Fourier

With this setup we can now write down a version of the Fourier transform. On the product $G \times G^\vee[1]$, there is a tautological line bundle $\mathcal{L}$, because a point in $G^\vee[1]$ specifies a line bundle on G:

$$\begin{tikzcd} & \mathcal{L} \arrow[d] & \\ & G \times G^\vee[1] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & G^\vee[1] \end{tikzcd}\ . \tag{8.66}$$

Now, for decent[1] G, we can write the *Fourier-Mukai transform*, which is:

$$\begin{aligned} \mathbf{QC}(G) &\longrightarrow \mathbf{QC}(G^\vee[1]) \\ F &\longmapsto \pi_{2*}(\pi_1^*\mathcal{F} \otimes \mathcal{L}) \end{aligned}\ . \tag{8.67}$$

[1] See remark 36.

This is a sheaf-theoretic analogue of the Fourier transform from sections 3.2 and 3.4.

Remark 36 (What do we mean by decent G?)*.* Recall, from section 3.6, that we needed really strict conditions for ordinary Cartier duality. E.g. we might ask for a finite group scheme (Spec of a finite-dimensional k-algebra) in order to really make it really work. Now this duality works in much greater generality. This is because sheaves are much more robust than functions: there are a lot more sheaves and they behave much better than functions.

8.2.3 Mukai

The original example introduced of this, by Mukai, took $G = A$ to be an abelian variety, i.e. an abelian, connected, projective algebraic group. So something like an elliptic curve (compact complex torus with an algebraic structure). For example, we might take the Jacobian of a Riemann surface, $\mathrm{Jac}(C)$. The *dual abelian variety*, written A^*, is the dual compact torus. It turns out that:

$$A^* \simeq A^\vee [1] \ , \tag{8.68}$$

i.e. A^* is the collection of multiplicative line bundles on A. This means we have a Fourier-Mukai transform which is an equivalence of derived categories

$$D(A) \xrightarrow{\sim} D(A^*) \ , \tag{8.69}$$

but from now on we will just write $\mathbf{QC}$ for the derived category of quasi-coherent sheaves. The basic problem is that the pushforward in the formula eq. (8.67) is not an exact functor because the fibers are tori, so not affine. So the formula eq. (8.67) is not good to write naively if G is not affine, but we can fix it by deriving everything.

So we have an equivalence:

$$\mathbf{QC}(A) \xrightarrow{\sim} \mathbf{QC}(A^\vee [1]) \ . \tag{8.70}$$

So this is much more robust than down a level. We don't need to be finite, or affine, or anything. This works for everything we will run into. There is a nice article by Laumon, [Lau91], where a lot of examples are worked out from this point of view in great generality.

Remark 37*.* Note that we're just saying Fourier-Mukai transforms are a certain special type of integral transforms. In some places in the literature, "Fourier-Mukai transform" is used to refer to arbitrary integral transforms. The problem is that *all* reasonable functors on derived categories can be expressed as integral transforms, so it doesn't make sense to call them all Fourier-Mukai. We will only use Fourier-Mukai transform to refer to a transform as above. In particular it should exchange multiplication with convolution, as the ordinary Fourier transform does.

Remark 38 (Spectral decomposition)*.* For $(\mathcal{C}, *)$ acting on $\mathcal{M}$, we get a module category for

$$\mathbf{QC}(\mathrm{Spec}\,\mathcal{C}) \ . \tag{8.71}$$

This gives rise to a *sheaf of categories* over $\mathrm{Spec}\,\mathcal{C}$. By Gaitsgory's 1-affineness, this is actually an equivalence between modules over $\mathbf{QC}(\mathrm{Spec}\,\mathcal{C})$ and sheaves of categories over $\mathrm{Spec}\,\mathcal{C}$.

Example 38. Let C be a Riemann surface. Then Riemann, Abel, and Jacobi showed that Jac is self-dual:

$$\mathrm{Jac}\,(C) \simeq (\mathrm{Jac}\,(C))^* \ . \tag{8.72}$$

This is a version of Poincaré duality between H_1 and H^1.

We will use $\mathrm{Pic}\,(C)$ to denote the stack of line bundles on C, i.e.

$$\mathrm{Pic}\,(C) = |\mathrm{Pic}\,(C)| \times B\mathbb{G}_m \ , \tag{8.73}$$

where $|\mathrm{Pic}\,(C)|$ is the usual Picard scheme. We can describe $\mathrm{Pic}\,(C)$ as consisting of three parts: the degree, the degree 0 line bundles, and the automorphisms:

$$\mathrm{Pic}\,C \cong \mathbb{Z} \times \mathrm{Jac} \times B\mathbb{G}_m \ . \tag{8.74}$$

When we take the dual we get:

$$(\mathrm{Pic}\,(C))^* = \mathrm{Pic}\,(C)^\vee\,[1] \simeq B\mathbb{G}_m \times \mathrm{Jac} \times \mathbb{Z} \ , \tag{8.75}$$

i.e. Pic is self-dual just like Jac, but in this more interesting way where the automorphisms are exchanged with the degrees. So there is a Fourier-Mukai transform between sheaves on $\mathrm{Pic}\,C$ and itself, where different degrees correspond to a different action of this stabilizer $\mathbb{G}_m$. This example has to do with class field theory. The discussion continues in section 8.2.4.

8.2.4 Betti CFT

The previous example is related to class field theory. So far, e.g. in chapter 7, we have discussed what one might call *Betti class field theory (CFT)*. We studied Pic, as a commutative group, and we attached the category of local systems with convolution:

Starting here, **Loc** denotes the category and Loc denotes the stack. Need to fix earlier.

$$(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \ , \tag{8.76}$$

which is a $\otimes$-category. Taking Spec we get:

$$\mathrm{Spec}\,(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \simeq \mathrm{Loc}_1\,C \ , \tag{8.77}$$

the stack of rank 1 local systems on C.

Maybe more suggestively, this is saying that:

$$(\mathbf{Loc}\,(\mathrm{Pic})\,, *) \simeq (\mathbf{QC}\,(\mathrm{Loc}_1\,C)\,, \otimes) \ . \tag{8.78}$$

We didn't state this, but what we did state, in chapter 7, was that given L a local system, it corresponds to a character local system χ_L. This extends to a Fourier-Mukai transform, which is the equivalence we have above. So we can identify this as saying that the 1-shifted Cartier dual of Pic is Loc.

We can write this more suggestively as follows. Over $\mathrm{Pic} \times \mathrm{Loc}_1$, there is a universal character local system χ:

$$\begin{tikzcd} & \chi \arrow[d] & \\ & \mathrm{Pic} \times \mathrm{Loc}_1 \arrow[dl] \arrow[dr] & \\ \mathrm{Pic} & & \mathrm{Loc}_1 \end{tikzcd} \ . \tag{8.79}$$

This means χ is a local system in the Pic "direction" and a quasi-coherent sheaf in the Loc_1 direction.

Remark 39. This is the sense in which the Betti picture is less symmetric than the de Rham one. In the Betti world, we're treating Pic as something purely topological, and Loc_1 as an object of algebraic geometry. In the de Rham version, the two sides will be much more symmetric, but both sides will be more complicated.

In any case, we get an identification:

$$\mathbf{Loc}\,(\mathrm{Pic}) \simeq \mathbf{QC}\,(\mathrm{Loc}_1)\ , \tag{8.80}$$

by the same integral transform formalism. One of the features of this is as follows. For $x \in C$ we get a homomorphism $\mathbb{Z} \to \mathrm{Pic}$ given by multiples of that point. Correspondingly we have a homomorphism on the other side: $\mathrm{Loc}_1\,C \to B\mathbb{G}_m$ given by $L \mapsto L|_x$. So the above identification is realised in such a way that a map from $\mathbb{Z}$ is dual to a map to $B\mathbb{G}_m$. More generally we can act by $\mathbb{Z}x$ on local systems, i.e. the Hecke modifications $\mathcal{L} \mapsto \mathcal{L}\,(x)$ for $\mathcal{L}$ a line bundle. These are identified under the duality with tensoring with the line bundle W_x on Loc_1. This line bundle is exactly the map $W_x\colon \mathrm{Loc}_1 \to B\mathbb{G}_m$, since a map to $B\mathbb{G}_m$ is the same as a line bundle over the source.

8.2.5 de Rham CFT

Now we will do some cool tricks with this 1-shifted Cartier duality, based on Laumon [Lau91] and Rothstein [Rot96]. The idea is that we will start from this self-duality of Jac, and build up to something that looks like class field theory.

The de Rham space

To do this, we need to find new abelian varieties, which we will do by introducing something called the de Rham space/functor. For X a variety (or scheme or stack, etc.) there is an associated object X_{dR}, called the de Rham space. We can think of it as the quotient of X by the equivalence relation of being infinitesimally close. More formally:

$$X_{\mathrm{dR}} = X/\widehat{\Delta} \tag{8.81}$$

where $\widehat{\Delta}$ is a formal neighborhood of the diagonal. This is not representable as a scheme, but it still makes sense as a functor of points, i.e. we can at least do algebraic geometry on it. But for us, it's really just a placeholder for $\mathcal{D}$-modules in the following sense. Since this is an object of algebraic geometry, we can consider sheaves on it, i.e. sheaves on X which are equivariant for this equivalence relation:

$$\mathbf{QC}\,(X_{\mathrm{dR}}) = \mathbf{QC}\,(X)^{\widehat{\Delta}}\ . \tag{8.82}$$

In other words it has objects given by $\mathcal{F} \in \mathbf{QC}\,(X)$, and an identification

$$\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_y \tag{8.83}$$

for all x and y which are infinitesimally close. This is reminiscent of a flat connection: it gives you exactly this kind of data. So this is a version of sheaves on X equipped with a flat connection. This turns out to be identified with the category of *$\mathcal{D}$-modules on X*:

$$\mathbf{QC}\,(X_{\mathrm{dR}}) = \mathcal{D}_X\text{-}\mathbf{mod}\ . \tag{8.84}$$

$\mathcal{D}_X$**-mod** is the category of quasicoherent sheaves $\mathcal{F} \in \mathbf{QC}(X)$ with an action of rings:

$$\mathcal{D}(U) \circlearrowright \mathcal{F}(U) \ , \tag{8.85}$$

for affine opens U. This is just saying that we have a map:

$$T_X \to \mathrm{End}(\mathcal{F}) \tag{8.86}$$

satisfying the Leibniz rule: for any two vector fields ξ_1 and ξ_2, $[\xi_1, \xi_2]$ acts by the commutator of ξ_1 and ξ_2 (flatness), and they satisfy:

$$\xi f = f\xi + f' \ . \tag{8.87}$$

In other words $\mathcal{F}$ is a quasi-coherent sheaf with a flat connection:

$$\nabla \colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1 \tag{8.88}$$

where $\nabla^2 = 0$.

Back to Fourier-Mukai theory

Let G be an abelian group. Then G_{dR} can be written as a quotient:

$$G_{\mathrm{dR}} = G/\widehat{G} \tag{8.89}$$

where $\widehat{G}$ is the *formal group of* G, i.e. the formal neighborhood of the identity. This is saying that, in the case of a group, x is infinitesimally close to y iff $xy^{-1} \in \widehat{G}$, i.e. xy^{-1} is infinitesimally close to the identity. This allows us to add some "spice" (de Rham) to all of our favorite examples of Cartier duality from section 3.6. Now we can add some spice to all of our favorite examples of Cartier duality.

Recall from example 19 that the Cartier dual of $\mathbb{A}^1$ is given by the formal completion of $\mathbb{A}^1$:

$$\left(\mathbb{A}^1\right)^\vee = \widehat{\mathbb{A}^1} \tag{8.90}$$

and more generally:

$$V^\vee = \widehat{(V^*)} \ . \tag{8.91}$$

This means the Fourier-Mukai dual is:

$$V^\vee[1] = B\widehat{(V^*)} \ . \tag{8.92}$$

Likewise:

$$\widehat{V}^\vee[1] = BV^* \ . \tag{8.93}$$

This becomes interesting when we do the following trick. Let V be some vector space. V is an abelian group, so V_{dR} is the quotient:

$$V_{\mathrm{dR}} = V/\widehat{V} \tag{8.94}$$

or we can think of this as a chain complex:

$$V_{\mathrm{dR}} = \left[\widehat{V} \to V\right] \ . \tag{8.95}$$

Now we get something beautifully self-dual:

$$V_{\mathrm{dR}}^{\vee}[1] = \left[\widehat{V} \to V\right]^{\vee}[1] \tag{8.96}$$

$$= \left[\widehat{V^*} \to V^*\right] \tag{8.97}$$

$$= V_{\mathrm{dR}}^{*} \ . \tag{8.98}$$

Writing this another way, we have:

$$\left(V/\widehat{V}\right)^{\vee}[1] = V^*/\widehat{V^*} \ . \tag{8.99}$$

This tells us that $\mathcal{D}$-modules on V and V^* are identified:

$$\mathcal{D}(V) \simeq \mathcal{D}(V^*) \ . \tag{8.100}$$

Example 39. For example:

$$\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x\rangle / (\partial x - x\partial = 1) \ , \tag{8.101}$$

and on the other hand

$$\mathcal{D}_{\mathbb{A}^{1*}} = \mathbb{C}\langle y, \partial_y\rangle / (\partial y - y\partial = 1) \ . \tag{8.102}$$

But $\mathcal{D}_{\mathbb{A}^1}$ has an interesting automorphism:

$$\begin{aligned} x &\longmapsto \partial_x \\ \partial_x &\longmapsto -x \end{aligned} \ , \tag{8.103}$$

which is exactly what happens under the Fourier transform. This automorphism gives an equivalence between the categories of modules:

$$\mathbb{F}\colon \mathcal{D}_{\mathbb{A}^1}\textbf{-mod} \to \mathcal{D}_{\mathbb{A}^1}\textbf{-mod} \ , \tag{8.104}$$

called the Fourier transform, which is the identity on objects as vector spaces, but changes the action of x and ∂_x by switching them (with a sign).

In general we get a Fourier transform:

$$\mathcal{D}_V\textbf{-mod} \simeq \mathcal{D}_{V^*}\textbf{-mod} \ . \tag{8.105}$$

Example 40 (Relation to usual Fourier transform on $\mathbb{R}$.)**.** For φ a generalized function, it defines a module for $\mathcal{A}_{\mathbb{A}^1}$, $\mathcal{M}_\varphi \in \mathcal{D}_{\mathbb{A}^1}\textbf{-mod}$, which is

$$\mathcal{D}_{\mathbb{A}^1}\textbf{-mod} = \{L\varphi \mid L \in \mathcal{D}_{\mathbb{A}^1}\} \subset \text{generalized functions on } \mathbb{R} \ . \tag{8.106}$$

The claim is:

$$\mathbb{F}(\mathcal{M}_\varphi) = \mathcal{M}_{\mathbb{F}(\varphi)} \ . \tag{8.107}$$

So this really is the usual Fourier transform of vector spaces.

Laumon and Rothstein, [Lau91, Rot96], pointed out that you can do the same thing on any abelian variety, so in particular on $\mathrm{Jac}(C)$. So let $A = \mathrm{Jac}(C)$. Then, as usual, $A_{\mathrm{dR}} = A/\widehat{A}$, and the claim is that:

$$A^{\vee}_{\mathrm{dR}}[1] \simeq \mathrm{Conn}_1(C) \ , \tag{8.108}$$

the space of rank 1 flat connections on C. In fact, what we have already seen is enough to show that:

$$A_{\mathrm{dR}} = \left(\mathrm{Jac}/\widehat{\mathbb{A}^1}\right) \ , \tag{8.109}$$

so the dual is an $\mathbb{A}^1$-bundle over Jac, which is what $\mathrm{Conn}_1(C)$ is. So starting from the self-duality of the Jacobian, and formally applying this Cartier duality, we get a statement, $A^{\vee}_{\mathrm{dR}}[1] \simeq \mathrm{Conn}_1(C)$, which looks a lot like in the Betti setting. This implies that (after taking **QC**) we have

$$\mathcal{D}_{\mathrm{Jac}}\text{-}\mathbf{mod} \simeq \mathbf{QC}(\mathrm{Conn}_1(C)) \ , \tag{8.110}$$

where again there is some χ_L on the left corresponding to any L on the right. This is de Rham geometric CFT.

Lecture 16; March 25, 2021

8.3 Betti vs. de Rham

transition

Recall our general paradigm is a duality between two types of field theories, or two types of "calculations". One side, called the $\mathcal{A}$-model, studies topology, and the other side, called the $\mathcal{B}$-model, studies algebraic geometry. There are a lot of choices involved in making this precise, and the choices appear very differently on the $\mathcal{A}$-side versus the $\mathcal{B}$-side. To something of codimension 1 we attach functions on the space of fields. On the $\mathcal{B}$-side these will be algebraic functions, so we need to choose how to view the space of fields as an algebraic variety (or something else in the world of algebraic geometry). On the $\mathcal{A}$-side we're dealing with *locally-constant* functions, so we need to choose some way of formalizing the notion of a locally-constant function. We can formalize this notion in three ways:

1. Betti: functions constant along paths. When we pass to the derived version we get H^*_B, i.e. usual topological cohomology.
2. de Rham: functions whose derivative is zero. When we pass to the derived version we get H^*_{dR}, i.e. usual de Rham cohomology. This is equivalent to functions on the associated de Rham space.
3. Symplectic: rather than studying the topology of X, we can study the symplectic topology of T^*X (Floer theory, etc.).

The point is that on the $\mathcal{A}$-side we don't need to choose a structure on the space of fields, but we do need to choose the type of functions we consider. On the $\mathcal{B}$-side we *do* need to choose a structure on the space of fields, but there is no ambiguity in the notion of functions.

In codimension 2 we attach some category of sheaves on the space of fields. Again, on the $\mathcal{B}$-side, we need to view the space of fields as an algebraic variety. So again this is a choice, but there is no ambiguity[2] in the notion of sheaves we consider: we just take **QC** of the space of fields. On the $\mathcal{A}$-side we don't need to choose any structure on the space of

[2] There is also **IndCoh**, but this is really just a version of **QC** where we're being more careful about singularities of the space.

fields, but we do need to specify what type of sheaf theory we're considering. Again there are three versions of sheaf theory on the $\mathcal{A}$-side:

3. Symplectic: some version of the Fukaya category of T^*X.

1. Betti: locally constant finite rank sheaves, i.e. it locally looks like $U \times \mathbb{C}^r$. This says we can parallel transform along paths.

2. de Rham: flat vector bundles, i.e. they are locally $V \simeq U \times \mathbb{C}^r$ with the de Rham connection. I.e. in the Betti case we're looking at constant sections, but now we're looking at sections with the de Rham differential. Flatness is a way of saying that the derivative is 0. I.e. equivariance for the equivalence relation of infinitesimal nearness.

There is a correspondence between locally constant finite rank sheaves and flat vector bundles: to a flat vector bundle we can attach the monodromy. This is a trivial case of the Riemann-Hilbert correspondence. We will write the category of finite-dimensional local systems and the category of flat connections respectively as:

$$\mathbf{Loc}^{\mathrm{fd}}(X) \simeq \mathbf{Conn}^{\flat}(X) \ . \tag{8.111}$$

A key point here is that these two notions of sheaves sit inside of larger categories which are very different. On the de Rham side this sits inside of:

$$\mathbf{Conn}^{\flat}(X) \subset \mathcal{D}_X\text{-}\mathbf{mod} = \mathbf{QC}(X_{\mathrm{dR}}) \ , \tag{8.112}$$

which we recall is the category of quasi-coherent sheaves on X with a flat connection. This is a way of asking for the derivative to vanish. On the Betti side, this sits inside of

$$\mathbf{Loc}^{\mathrm{fd}}(X) \hookrightarrow \mathbf{Loc}(X) = \mathbf{Rep}(\pi_1(X)) \ , \tag{8.113}$$

i.e. we're asking for things which are constant along paths.

Remark 40. **Loc** sits inside of an even larger category of constructible sheaves, which are local systems with "singularities".

8.3.1 Betti vs. de Rham for $\mathbb{C}^\times$

Betti version

To demonstrate how different $\mathcal{D}$-**mod** is from **Loc**, we will consider the simplest example: $X = \mathbb{C}^\times$. The Betti version of the $\mathcal{A}$-side is:

$$\begin{aligned} \mathbf{Loc}\left(\mathbb{C}^\times\right) &= \mathbf{Loc}\left(S^1\right) && (8.114)\\ &= \mathbf{Loc}(B\mathbb{Z}) && (8.115)\\ &= \mathbf{Rep}(\mathbb{Z}) \ . && (8.116)\end{aligned}$$

Notice we're not imposing any finiteness conditions. Now we can perform a Cartier duality by writing:

$$\begin{aligned} \mathbf{Rep}(\mathbb{Z}) &= \mathbb{C}\mathbb{Z}\text{-}\mathbf{mod} && (8.117)\\ &= \mathbb{C}\left[z, z^{-1}\right]\text{-}\mathbf{mod} && (8.118)\\ &= \mathbf{QC}(\mathbb{G}_m) && (8.119)\end{aligned}$$

where we write $\mathbb{G}_m$ because this is really the dual torus to the $\mathbb{C}^\times$ we started with, i.e. this is really the Cartier duality between $\mathbb{Z}$ and $\mathbb{G}_m$ appearing. So the Betti $\mathcal{A}$-side is this category $\mathbf{Loc}(\mathbb{C}^\times)$, and the $\mathcal{B}$-side is the category $\mathbf{QC}(\mathbb{G}_m)$. Now there are some basic objects we can write down here: for any $\mu \in \mathbb{G}_m$ we get a skyscraper $\mathcal{O}_\mu \in \mathbf{QC}(\mathbb{G}_m)$. This corresponds to the rank one local system with monodromy μ, written $L_\mu \in \mathbf{Loc}(\mathbb{C}^\times)$. These are the character sheaves on $\mathbb{C}^\times$. So these categories are equivalent in such a way that these "small" objects are matched:

$$\begin{array}{ccc} \mathbf{Loc}(\mathbb{C}^\times) & \xrightarrow{\simeq} & \mathbf{QC}(\mathbb{G}_m) \\ \{L_\mu\} & \longleftrightarrow & \{\mathcal{O}_\mu\} \end{array} . \tag{8.120}$$

There are also very different objects that are kind of "large". For example we can also consider the "universal cover" sheaf. Consider the universal cover:

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times , \tag{8.121}$$

and pushforward the constant sheaf to get: $\exp_* \underline{\mathbb{C}}$, which is the free rank 1 $\mathbb{Z}$-module corresponding to $\mathcal{O}_{\mathbb{G}_m}$ on the other side:

$$\exp_* \underline{\mathbb{C}} \leftrightarrow \mathcal{O}_{\mathbb{G}_m} . \tag{8.122}$$

8.3.2 de Rham version

The de Rham story is a version of the *Mellin transform.* The classical Mellin transform is the Fourier transform for the group $\mathbb{R}_+$. This is of course isomorphic to $\mathbb{R}$ under exp, so we can think of this as doing the Fourier transform for $\mathbb{R}$ in additive notation. So for f a function on $\mathbb{R}_+$, the Mellin transform is:

$$\widehat{f}(s) = \int f(z)\, z^s \frac{dz}{z} . \tag{8.123}$$

The idea is that z^s is a character of $\mathbb{R}_+$ for $s \in \mathbb{C}$: if we had $z = e^x$ then $z^s = e^{sx}$, which is a character of $\mathbb{R}$ that appeared in the Fourier transform for $\mathbb{R}$.

Just like we rewrote the Fourier transform in terms of $\mathcal{D}$-modules, we want to do the same with the Mellin transform. Consider the ring of differential operators on $\mathbb{C}^\times$:

$$\mathcal{D}_{\mathbb{C}^\times} = \mathbb{C}\left\langle t, t^{-1}, \partial \right\rangle / \left(\partial t = t\partial + t\right) \tag{8.124}$$

where we can think that $\partial = t\partial/\partial t$. We can switch the generators and write this as:

$$\mathcal{D}_{\mathbb{C}^\times} \simeq \mathbb{C}\left\langle s, \sigma, \sigma^{-1} \right\rangle / \left(s\sigma = \sigma s + \sigma\right) . \tag{8.125}$$

We can think of $\mathbb{C}\left\langle t, t^{-1}, \partial \right\rangle$ as being made up of functions on $\mathbb{C}^\times$ with differentiation, i.e. $\mathbb{C}\left[t, t^{-1}\right]$ with ∂ adjoined. We can think of $\mathbb{C}\left\langle s, \sigma, \sigma^{-1} \right\rangle$ as being functions on $\mathbb{C}$ with a shift, i.e. $\mathbb{C}[s]$ with σ and σ^{-1} adjoined. So in the second case we're studying an action of $\mathbb{Z}$ on $\mathbb{C}$ by translation. These are referred to as *difference operators* on $\mathbb{C}$.

Now when you pass to modules:

$$\mathcal{D}_{\mathbb{C}^\times}\textbf{-mod} \simeq \Delta_{\mathbb{C}}\textbf{-mod} \tag{8.126}$$

$$= \mathbf{QC}(\mathbb{C})^{\mathbb{Z}} \tag{8.127}$$

$$= \mathbf{QC}(\mathbb{C}/\mathbb{Z}) , \tag{8.128}$$

where $\Delta_{\mathbb{C}}$-**mod** denotes difference operators. So in conclusion, the de Rham version of the $\mathcal{A}$-side fits into the equivalence:

$$\mathcal{D}_{\mathbb{C}^\times}\text{-}\mathbf{mod} \simeq \mathbf{QC}\left(\mathbb{C}/\mathbb{Z}\right) \ , \tag{8.129}$$

in contrast with the Betti version which was the equivalence:

$$\mathbf{Loc}\mathbb{C}^\times \simeq \mathbf{QC}\left(\mathbb{C}^\times\right) \ . \tag{8.130}$$

In complex analysis, $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times$ are the same, but in algebraic geometry these are of a completely different nature (because the exponential map is not algebraic).

There is a common piece consisting of finite rank local systems and skyscrapers at points:

$$\begin{tikzcd} \mathcal{D}_{\mathbb{C}^\times}\text{-}\mathbf{mod} \arrow[r, phantom, "\simeq"] & \mathbf{QC}\left(\mathbb{C}/\mathbb{Z}\right) \\ \text{finite rank} \arrow[u, hook] \arrow[d, hook] & \text{points} \arrow[u, hook] \arrow[d, hook] \\ \mathbf{Loc}\left(\mathbb{C}^\times\right) \arrow[r, phantom, "\simeq"] & \mathbf{QC}\left(\mathbb{C}^\times\right) \end{tikzcd} \ . \tag{8.131}$$

We saw this correspondence in the Betti version, now we identify a de Rham version. For $\lambda \in \mathbb{C}^\times$, we can try to solve the differential equation $(t\partial - \lambda) f = 0$. Now from the point of view of $\mathcal{D}$-modules, this defines the following:

$$M_\lambda = \mathcal{D}_{\mathbb{C}^\times} / \left(t\partial - \lambda\right) \in \mathcal{D}_{\mathbb{C}^\times}\text{-}\mathbf{mod} \ . \tag{8.132}$$

M_λ is the trivial line bundle on $\mathbb{C}^\times$, but it has flat connection:

$$\nabla = d - \lambda \frac{dt}{t} \tag{8.133}$$

where $\partial = t\partial/\partial t$ acts by λ. $t\partial/\partial t f = \lambda f$ is saying that f is homogeneous of degree λ, i.e. z^λ is an analytic function solving this.

So for any λ we have this flat connection, but as it turns out $M_\lambda \simeq M_{\lambda+n}$ for any $n \in \mathbb{Z}$. In fact, up to isomorphism, this M_λ only depends on $\mu = \exp(\lambda)$, i.e. the monodromy. For $[\lambda] \in \mathbb{C}/\mathbb{Z}$, or equivalently $\mu \in \mathbb{C}^\times$, we had this skyscraper $\mathcal{O}_{[\lambda]}$, and now that gets identified with this M_λ. As it turns out $M_\lambda \simeq L_\mu$ from before.

Recall we had this universal cover sheaf in the Betti story. There isn't anything like this in the world of $\mathcal{D}$-modules. For example:

$$\mathcal{O}_{\mathbb{C}/\mathbb{Z}} = \mathbb{C}\left[s\right] \circlearrowleft \mathbb{Z} \text{ by shifts} \tag{8.134}$$

corresponds to the $\mathcal{D}$-module of delta functions at $1 \in \mathbb{C}^\times$:

$$\mathbb{C}\left[\partial\right] \cdot \delta_1 \ . \tag{8.135}$$

This looks nothing like the universal cover sheaf. This is just to illustrate that the small pieces, the skyscrapers, are the same in both stories. However the big objects don't get matched. $\mathcal{D}$-modules on $\mathbb{C}^\times$ has all these weird objects supported at finitely many points, or with singularities. On the other hand, local systems on $\mathbb{C}^\times$ might have infinite rank, but can't see the difference between two points because they're locally constant.

These are just two flavors of this duality. On the $\mathcal{B}$-side we had $\mathbb{C}/\mathbb{Z}$ versus $\mathbb{C}^\times$, which were the same analytically, but different algebraically. So we changed the variety. On the $\mathcal{A}$-side, we change the sheaf theory, i.e. this change corresponds to changing from $\mathcal{D}$ to **Loc** on $\mathbb{C}^\times$.

8.4 Geometric CFT

The main statement was that abelian local systems on C and Jac correspond. Or the stronger version: rank one local systems on C correspond to character local systems on Pic.

On the $\mathcal{A}$-side, the underlying space is always $\mathrm{Pic}(C)$. We're studying its topology so this is something like $\mathbb{Z} \times \mathrm{Jac}$, with a factor of $B\mathbb{G}_m$ that we're ignoring. But we have two ways of linearizing it:

$$\mathbf{Loc}(\mathrm{Pic}) \qquad \text{vs} \qquad \mathcal{D}_{\mathrm{Pic}}\text{-}\mathbf{mod} \ . \tag{8.136}$$

The Betti side is $\mathbf{Loc}(\mathrm{Pic})$, which consists of $\mathbb{Z}$-graded representation of

$$\pi_1(\mathrm{Jac}) = H_1(C, \mathbb{Z}) \simeq \mathbb{Z}^{2g} \ . \tag{8.137}$$

Remark 41. There is a slight derived correction to this, coming from a factor of $B\mathbb{G}_m$ in Pic, which isn't very interesting.

Now we will do the de Rham version. Since Jac is not affine, it is harder to write this side down. $\mathcal{D}_{\mathrm{Pic}}$ is $\mathbf{QC}(\mathrm{Pic})$ together with an action of vector fields. This is an abelian group, so this is the same as an action of the tangent space at the identity:

$$T_e \,\mathrm{Jac} = H^1(C, \mathcal{O}) \ . \tag{8.138}$$

These categories look different, but contain the same basic objects. We get two corresponding statements of geometric CFT, i.e. two Fourier-Mukai dualities. The first one is the Betti CFT:

$$\mathbf{Loc}(\mathrm{Pic}) \simeq \mathbf{QC}(\mathrm{Loc}_1 C) \ . \tag{8.139}$$

$\mathbf{QC}(\mathrm{Loc}_1 C)$ consists of $\mathbf{Rep}(\mathbb{G}_m)$-valued modules for $\mathbb{C}[\mathrm{Loc}_1]$.

$$\mathrm{Loc}_1 = \left\{\pi_1 \to \mathbb{C}^\times\right\} = H^1(C, \mathbb{Z}) \otimes \mathbb{C}^\times \ , \tag{8.140}$$

so functions on this are:

$$H_1(C, \mathbb{Z}) \otimes \mathbb{C}\left[z, z^{-1}\right] \ . \tag{8.141}$$

We already saw that on the $\mathcal{A}$-side we have representations of $H_1(X, \mathbb{Z})$, so these abelian categories match. Inside of $\mathbf{QC}(\mathrm{Loc}_1 C)$ we have the skyscrapers $\mathcal{O}_L$ at L. These corresponded to the character sheaves/geometric characters χ_L.

The de Rham CFT looks similar, but in some sense it is much harder. Now the equivalence is only a *derived* one:

$$\mathcal{D}_{\mathrm{Pic}}\text{-}\mathbf{mod} \simeq \mathbf{QC}\left(\mathrm{Conn}_1^\flat C\right) \ . \tag{8.142}$$

$\mathrm{Conn}_1^\flat C$ is not affine, and $\mathcal{D}_{\mathrm{Pic}}$-**mod** is not globally generated, so we can't specific the isomorphism as an isomorphism of rings like we did in the Betti case.

Recall Jac self-dual so we get a derived equivalence:

$$(\mathbf{QC}(\mathrm{Jac}), *) \simeq (\mathbf{QC}(\mathrm{Jac}), \otimes) \ . \tag{8.143}$$

Then we want to enhance both sides to go from $\mathbf{QC}$ to $\mathcal{D}$-modules on the left, and to go from Jac (i.e. line bundles) to flat line bundles on the right. Luckily this can be done pretty easily by abstract nonsense. There is a faithful forgetful functor:

$$\mathcal{D}_{\mathrm{Jac}}\text{-}\mathbf{mod} \to \mathbf{QC}(\mathrm{Jac}) \ , \tag{8.144}$$

since a $\mathcal{D}$-module is a quasi-coherent sheaf which is equivariant for $\widehat{\mathrm{Jac}}$, i.e. equivariant for the action of vector fields on Jac. On the other side we have a forgetful functor:

$$\mathbf{QC}\left(\mathrm{Jac}^\flat = \mathrm{Conn}_1^\flat\right) \to \mathbf{QC}\left(\mathrm{Jac}\right) \ , \tag{8.145}$$

where we just forget the connection, since an element of $\mathbf{QC}\left(\mathrm{Conn}_1^\flat\right)$ is a quasi-coherent sheaf on Jac with an action of functions on the fibers. So we just need to check that, under this equivalence, these two get identified.

So we have this map $\mathrm{Conn}_1^\flat C \to \mathrm{Jac}$ with fibers given by $\Gamma\left(\mathrm{Jac}, \Omega^1\right)$:

$$\begin{tikzcd}
\Gamma\left(\mathrm{Jac}, \Omega^1\right) \arrow[r] & \mathrm{Conn}_1^\flat C \arrow[d, "\text{Forget}"] \\
& \mathrm{Jac} \simeq \left(S^1\right)^{2g}
\end{tikzcd} \ , \tag{8.146}$$

and

$$\Gamma\left(\mathrm{Jac}, \Omega^1\right) = T_e^* \mathrm{Jac} \simeq H^0\left(C, \Omega^1\right) \ , \tag{8.147}$$

so this is an affine-bundle with fibers of dimension g (genus of C). This looks very different from $\mathrm{Loc}_1\left(C\right) \simeq \left(\mathbb{C}^\times\right)^g$. Here $\mathrm{Conn}_1^\flat$ is an affine bundle over $\mathrm{Jac} \simeq \left(S^1\right)^{2g}$ with fibers that look like $\mathbb{C}^g$. I.e. it is an affine bundle over a torus, but $\mathrm{Loc}_1\left(C\right) \simeq \left(\mathbb{C}^\times\right)^g$. is an affine variety.

Remark 42. The Betti story is more elementary in some ways, e.g. it is an abelian equivalence, but you lose the beautiful thing in the de Rham story which is that both sides are in the world of algebraic geometry.

Again recall that Jac is self-dual so we get a derived equivalence:

$$\left(\mathbf{QC}\left(\mathrm{Jac}\right), *\right) \simeq \left(\mathbf{QC}\left(\mathrm{Jac}\right), \otimes\right) \ . \tag{8.148}$$

We can enhance this by passing from Jac to

$$T^* \mathrm{Jac} = \mathrm{Jac} \times H^0\left(C, \Omega\right) \ . \tag{8.149}$$

Call

$$H^0\left(C, \Omega\right) = \mathrm{Hitch}_1 \tag{8.150}$$

the rank 1 Hitchin space. This gives us what is called the *self-duality of the Hitchin integral system* (in the rank 1 case):

$$\mathbf{QC}\left(T^* \mathrm{Jac}\right) \simeq \mathbf{QC}\left(T^* \mathrm{Jac}\right) \ . \tag{8.151}$$

From here we can deform the two sides in a way which preserves the duality. On the left, we will deformation quantize. There is a natural deformation from functions on T^*X to differential operators on X:

$$\begin{array}{cc}
\hbar = 0 & \mathcal{O}\left(T^* X\right) = \mathbb{C}\left[x_i, \xi_i\right] \\
& \rightsquigarrow \\
\hbar = 1 & \mathcal{D}_X = \mathbb{C}\left\langle x_i, \partial_{x_i}\right\rangle / \left(\partial_{x_i} x_i - x_i \partial_{x_i} = \delta_{ij} \hbar\right)
\end{array} \ . \tag{8.152}$$

In other words we can pass from $\mathbf{QC}(T^* \operatorname{Jac})$ to

$$\mathcal{D}_{\operatorname{Jac}}\text{-}\mathbf{mod} \ . \tag{8.153}$$

On the right we will instead deform the *space* to pass from:

$$\mathbf{QC}(T^* \operatorname{Jac}) \rightsquigarrow \mathbf{QC}\left(\operatorname{Conn}_1^\flat\right) \ . \tag{8.154}$$

So we start with $\mathbf{QC}(T^* \operatorname{Jac})$, and can deform it two different ways:

$$\begin{tikzcd} \mathbf{QC}(T^* \operatorname{Jac}) \arrow[d, rightsquigarrow, "\text{deformation quantize}"] & \simeq & \mathbf{QC}(T^* \operatorname{Jac}) \arrow[d, rightsquigarrow, "\text{deform space}"] \\ \mathcal{D}_{\operatorname{Jac}}\text{-}\mathbf{mod} & & \mathbf{QC}\left(\operatorname{Conn}_1^\flat\right) \end{tikzcd} \ . \tag{8.155}$$

Remark 43. After deforming the space, we can then deformation quantize. This gives us *twisted $\mathcal{D}$-modules on* Jac. After deformation quantizing, we can deform $\mathcal{D}$-**mod** to twisted $\mathcal{D}$-modules as well, so if we go one step further we return to an equivalence.

Remark 44. A $\mathcal{D}$-module is a quasi-coherent sheaf with extra structure: there is a monad, on the category of quasicoherent sheaves, whose modules are $\mathcal{D}$-modules. This extra structure is really given by some algebra action with respect to *convolution*. Once we say it this way, it becomes clear that on the other side of the duality we will get quasi-coherent sheaves with an action of some algebra with respect to *tensor product*. This just means we're considering sheaves on something affine over my base, and indeed $\operatorname{Conn}_1^\flat$ was an affine bundle over Jac.

Chapter 9

Nonabelian

Our goal is to develop an analogue of abelian duality for nonabelian groups. The basic problem is that if we start with G nonabelian, then the group algebra $\mathbb{C}G$ is noncommutative so there isn't immediately a theory of spectral decomposition, so there isn't an obvious notion of "dual", since before this was $\operatorname{Spec}(\mathbb{C}G, *)$. So we need to find some commutativity somewhere in order to describe some sort of dual. This commutativity will come to us, in a natural way, from TFT. Lecture 17; March 30, 2021

One could start with the field theory, but we will instead start with the "problem" we're trying to eventually solve. Basically the first problem is that we need to replace $\operatorname{Pic}(C)$ by some nonabelian counterpart, which will be the theory of automorphic forms.

9.1 Automorphic forms

9.1.1 G-bundles

Let C be a smooth projective curve over a field k. We're mainly interested in the finite field case $k = \mathbb{F}_q$, or $k = \mathbb{C}$ in which case we're studying compact Riemann surfaces. Then in geometric CFT, we studied either functions (in the $k = \mathbb{F}_q$ case) or sheaves (in the general case) on $\operatorname{Pic}(C)$, which consists of line bundles on C up to isomorphism.

We get a nonabelian analogue by replacing line bundles by vector bundles, or something more general called a *principal G-bundle*. So let G be an algebraic group over k. We will usually want it to be affine, but later we will even assume it is *reductive*.[1] Then a *principal G-bundle over C* is a bundle:

$$\begin{tikzcd} \mathcal{P} \arrow[d] \\ C \end{tikzcd} \tag{9.1}$$

with a G action such that locally[2] on an open U the bundle looks trivial:

$$\mathcal{P}|_U \simeq U \times G \ . \tag{9.2}$$

[1]For now one can just think of reductive as meaning G is one of the familiar Lie groups: GL_n, SO_n, Sp_n, ...

[2]In general this should really be étale locally, but on curves it is sufficient to work with the Zariski topology.

Example 41. If $G = \mathrm{GL}_n$, then principal G-bundles over C are just rank n vector bundles over C.

Now we can consider the moduli space $\mathrm{Bun}_G(C)$, which is really the moduli space (or stack) of principal G-bundles on C. This has R-points given by

$$\mathrm{Bun}_G(C)(R) = \{G\text{-bundles on } C \times \operatorname{Spec} R\} / \sim \;, \tag{9.3}$$

i.e. R-families of G-bundles over C.

Example 42. The case $G = \mathrm{GL}_1 = \mathbb{G}_m$ recovers the Picard stack:

$$\begin{aligned} \mathrm{Bun}_{\mathrm{GL}_1}(C) &= \mathrm{Pic}(C) & (9.4)\\ &= |\mathrm{Pic}(C)| \times \mathrm{pt}/\mathbb{G}_m \;. & (9.5) \end{aligned}$$

So to a curve C, we attach this space $\mathrm{Bun}_G(C)$, and then we will do some function theory on this space and this will be nonabelian geometric CFT.

9.1.2 Automorphic functions

Suppose $k = \mathbb{F}_q$ is a finite field and consider $\mathrm{Bun}_g(C)(\mathbb{F}_q)$. We can forget about the stack structure to get the discrete set of isomorphism classes of G-bundles $|\mathrm{Bun}_G(C)(\mathbb{F}_q)|$.

Definition 2. An *(unramified) automorphic function on* C for G is a function on

$$|\mathrm{Bun}_G(C)(\mathbb{F}_q)| \;, \tag{9.6}$$

possibly with a growth condition, e.g. compactly supported.

9.1.3 Weil's description

This description works for any field. The first statement to make is that a G-bundle on C is generically trivial, i.e. any $\mathcal{P} \to C$ has a *rational section*, i.e. it can be trivialized away from some finite collection of points. The other statement is that any G-bundle is trivial *formally* near any $x \in C$, i.e. it can be trivialized near any point. From all of this data, we get an element of

$$\prod_{x \in C}{}' G(K_x) \tag{9.7}$$

as follows. For $x \in C$ we get a complete local ring of Taylor series $\mathcal{O}_x$, which sits inside of the local field of Laurent series: $\mathcal{O}_x \subset K_x$. So we have a bundle, we have a trivialization away from a point, and a trivialization near a point and, on the overlap, the difference between the two trivializations defines an element of $G(K_x)$. The restricted product $\prod'$ means that for all but finitely many x, we have an element of $G(\mathcal{O}_x)$ rather than $G(K_x)$. The theorem is as follows.

Theorem 5 (Weil)**.** *For* $k = \mathbb{F}_q$*:*

$$\begin{aligned} |\mathrm{Bun}_G(C)(k)| &= G(F_C) \backslash {\prod}' G(K_x) / \prod G(\mathcal{O}_x) & (9.8)\\ &= G(F_C) \backslash G(\mathbb{A}_C) / G(\mathcal{O}_{\mathbb{A}_C}) \;. & (9.9) \end{aligned}$$

Recall this is the same statement that we had in section 5.2.2. The Beauville-Laszlo theorem says that we can actually glue a G-bundle from formal transition data. So if we have a bundle on a disk around a point, and a bundle away from the point, and transition data between them, then we get an honest bundle on the whole thing. This is not automatic because this isn't an actually an open cover (Spec of formal power series is not actually open).

Recall that in the abelian case we had this local factor for a point $x \in C$ given by:

$$\left|\mathrm{GL}_1\left(K_x\right) / \mathrm{GL}_1\left(\mathcal{O}_x\right)\right| = \left|K_x^\times / \mathcal{O}_x^\times\right| \simeq \mathbb{Z} \ , \tag{9.10}$$

i.e. these quotients are still groups. This was the source of the divisor description of line bundles. In the nonabelian case we instead have something called the *affine Grassmannian*

$$\mathrm{Gr}_G = G\left(K_x\right) / G\left(\mathcal{O}_x\right) \ . \tag{9.11}$$

This is not a group, since we're not modding out by a normal subgroup anymore. We can think of this as consisting of a G-bundle on C and a trivialization on $C \setminus x$.

This Weil theorem has an obvious analogue in the number field setting.

Number field version

The first thing to identify is a G-analogue of the idéle class group. Let $F/\mathbb{Q}$ be a finite extension, e.g. $F = \mathbb{Q}$. Then we want to write a double quotient analogous to what we had before:

$$G\left(\mathbb{Q}\right) \backslash G\left(\mathbb{A}_\mathbb{Q}\right) / G\left(\mathcal{O}_{\mathbb{A}_\mathbb{Q}}\right) \ . \tag{9.12}$$

Recall the adéle ring for a field F is:

$$\mathbb{A}_F = \prod_v{}' F_v \ . \tag{9.13}$$

In the case of $\mathbb{Q}$:

$$\mathbb{A}_\mathbb{Q} = \prod_p{}' \mathbb{Q}_p \times \mathbb{R} \tag{9.14}$$

where

$$\prod{}' \mathbb{Q}_p = \mathbb{A}_{\mathrm{fin}} = \widehat{\mathbb{Z}} \otimes_\mathbb{Z} \mathbb{Q} \tag{9.15}$$

is the finite adèles. Therefore we have

$$G\left(\mathbb{A}_\mathbb{Q}\right) = G\left(\mathbb{A}_{\mathrm{fin}}\right) \times G\left(\mathbb{R}\right) \ . \tag{9.16}$$

Notice that

$$G\left(\mathbb{Q}\right) \subset G\left(\mathbb{A}_\mathbb{Q}\right) \tag{9.17}$$

forms a discrete subgroup. Then we can form:

$$G\left(\mathbb{Q}\right) \backslash G\left(\mathbb{A}_\mathbb{Q}\right) / G\left(\widehat{\mathbb{Z}}\right) \times K_\infty \tag{9.18}$$

where $K_\infty \subset G\left(\mathbb{R}\right)$ is compact. We can think of this as playing the role of $\mathrm{Bun}_G\left(\mathbb{Q}\right)$. There is a natural subgroup to consider here. First notice that the real Lie group is a subgroup $G\left(\mathbb{R}\right) \hookrightarrow G\left(\mathbb{A}_\mathbb{Q}\right)$. On the left we will quotient by:

$$\prod{}' G\left(\mathbb{Q}_p\right) \supset G\left(\mathbb{Q}\right) \cap \prod G\left(\mathbb{Z}_p\right) = G\left(\mathbb{Z}\right) \ . \tag{9.19}$$

So by basic algebra we get:

$$G(\mathbb{Z})\backslash G(\mathbb{R})/K_\infty \hookrightarrow G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/G\left(\widehat{\mathbb{Z}}\right)\times K_\infty \ . \tag{9.20}$$

The claim is that this is a connected component, and for a decent class of groups (simply-connected, semisimple) this will be a bijection.

cite: M. Emerton: classical and p-adic modular forms.

The most basic case here is $G=\mathrm{GL}_2$. Then we're asking for something like $\mathrm{Bun}_2\left(\overline{\operatorname{Spec}\mathbb{Z}}\right)$, which is just poetic notation for rank 2 $\mathbb{Z}$-modules (trivialized at ∞).

Let $\mathcal{B}$ consist of the $\mathbb{R}$-bases of $\mathbb{R}^2=\mathbb{C}$, i.e.

$$\mathcal{B}\simeq \mathrm{Iso}_\mathbb{R}\left(\mathbb{Z}^2\otimes\mathbb{R},\mathbb{R}^2\right)=\mathrm{GL}_2\,\mathbb{R} \tag{9.21}$$
$$\simeq \mathbb{C}^\times\times\mathbb{C}\setminus\mathbb{R}\ . \tag{9.22}$$

The isomorphism with the last line is given by a basis (z_1,z_2) going to the pair $(z_1,z_2/z_1)$.

Now we can consider the (more interesting) space:

$$\mathcal{L}=\text{ lattices in }\mathbb{C}\simeq\mathbb{R}^2\ . \tag{9.23}$$

Such a lattice is a free rank 2 $\mathbb{Z}$-module with an isomorphism $\Lambda\otimes_\mathbb{Z}\mathbb{R}\xrightarrow{\sim}\mathbb{C}$. This is the same as:

$$\mathcal{L}=\mathrm{GL}_2\,\mathbb{Z}\backslash\mathcal{B}\ , \tag{9.24}$$

i.e. we're modding out by the action of $\mathrm{GL}_2\,\mathbb{Z}$ on $\Lambda\simeq\mathbb{Z}^2$. One should think of $\mathcal{L}$ as a version of rank 2 bundles on $\operatorname{Spec}\mathbb{Z}$, which are trivialized at infinity.

There are various variants of $\mathcal{L}$, for example $\mathcal{L}(N)$ which consists of lattices along with a fixed basis modulo N:

$$\mathcal{L}/N\simeq(\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\ , \tag{9.25}$$

i.e. a rank 2 bundle and a trivialization on $(N)\subset\operatorname{Spec}\mathbb{Z}$. We also have $\overline{\mathcal{L}}$, which consists of lattices up to $\mathbb{C}$-homothety, so we have:

$$\overline{\mathcal{L}}=\mathrm{GL}_2\,\mathbb{Z}\backslash\mathcal{B}/\mathbb{C}^\times=\mathrm{PSL}_2\,\mathbb{Z}\backslash\mathbb{H}\ . \tag{9.26}$$

Lemma 6. $\mathcal{L}\simeq\mathrm{GL}_2\,\mathbb{Q}\backslash\mathrm{GL}_2\,\mathbb{A}/\mathrm{GL}_2\,\mathcal{O}_\mathbb{A}$.

This lemma relies on two facts:

- $\mathrm{GL}_2\,\mathbb{Z}=\mathrm{GL}_2\,\mathbb{Q}\cap\mathrm{GL}_2\,\widehat{\mathbb{Z}}$, and
- $\mathrm{GL}_2(\mathbb{A}_{\mathrm{fin}})=\mathrm{GL}_2(\mathcal{O}_\mathbb{A})\,\mathrm{GL}_2\,\mathbb{Q}$.

The idea is that we start with $\mathcal{L}\simeq\mathrm{GL}_2\,\mathbb{Z}\backslash\mathrm{GL}_2\,\mathbb{R}$, and then we want to quotient out by another subgroup on the right. One choice is the maximal compact $\mathrm{O}_2\subset\mathrm{GL}_2\,\mathbb{R}$. We could also take the center $\mathbb{R}^\times\subset\mathrm{GL}_2\,\mathbb{R}$. Together they space this copy of $\mathbb{C}^\times$ that we were modding out by before. This is where classical modular forms come in. These are holomorphic sections of a line bundle on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ plus a growth condition. These sit inside the space of all functions on

$$\mathrm{GL}_2(\mathbb{Z})\backslash\mathcal{L}\ . \tag{9.27}$$

An automorphic form for $\mathrm{GL}_2/\mathbb{Q}$ is a function φ on

$$\mathrm{GL}_2\,\mathbb{Q}\backslash\mathrm{GL}_2\,\mathbb{A}/\mathrm{GL}_2\,\mathcal{O}_\mathbb{A}\ , \tag{9.28}$$

which is smooth on $\mathrm{GL}_2\mathbb{R}$ and satisfies a growth condition. This growth condition is a condition on the infinite place $\mathbb{R}$. We can also do a level N version where we instead take functions on

$$\mathrm{GL}_2\mathbb{Q}\backslash\mathrm{GL}_2\mathbb{A}/\mathrm{GL}_2\mathcal{O}_{\mathbb{A}}^{(N)}\ . \tag{9.29}$$

Example 43. This works for other groups too. Consider the moduli of orthogonal bundles. Fix an even lattice. This is a free $\mathbb{Z}$-module V and quadratic form

$$q\colon V\to\mathbb{Z}\ . \tag{9.30}$$

Then we can define the orthogonal group:

$$\mathrm{Aut}\left(V,q\right)=\mathcal{O}_q\left(\mathbb{Q}\right)\backslash\mathcal{O}_q\left(\mathbb{A}\right)/\mathcal{O}_q\left(\mathcal{O}_{\mathbb{A}}\right)\ . \tag{9.31}$$

This consists of isomorphism classes of quadratic forms in the genus of q. This means that we're considering quadratic forms that identify with q modulo every N.

So for any G and F we can write down Bun_G over F which is:

$$G\left(F\right)\backslash G\left(\mathbb{A}_F\right)/G\left(\mathcal{O}_{\mathbb{A}}\right)\times K_\infty\ . \tag{9.32}$$

In the function field setting over a finite field this was a discrete set, so we didn't have to worry about what kind of functions we consider on this space. Now this looks like a real Lie group modulo a maximal compact and a lattice. Explicitly this is as follows. The Archimedean completion of F is

$$F\otimes_{\mathbb{Q}}\mathbb{R}=\mathbb{R}^{\times s_1}\times\mathbb{C}^{\times s_2}\ . \tag{9.33}$$

So the Archimedean part of G is the following real Lie group, which we write as $G_{\mathbb{R}}$:

$$G_{\mathbb{R}}=G\left(F\otimes_{\mathbb{Q}}\mathbb{R}\right)=G\left(\mathbb{R}\right)^{\times s_1}\times G\left(\mathbb{C}\right)^{\times s_2}\ . \tag{9.34}$$

So we're looking at the space:

$$[G]_F\coloneqq\Gamma\backslash G_{\mathbb{R}}/K\ , \tag{9.35}$$

where K is maximal compact, and $\Gamma=G\left(\mathcal{O}_F\right)$ is a lattice. This is called an *arithmetic locally symmetric space*. To avoid specifying the type of functions, we can just pass to cohomology.

9.1.4 Cohomological automorphic forms

Now we will pass from functions to cohomology, so that we don't have to concern ourselves with what kind of functions we're considering. This results in the *cohomological automorphic forms*:

$$H^*\left(G\left(F\right)\backslash G\left(\mathbb{A}\right)/G\left(\mathcal{O}\right)\times K\right)\ . \tag{9.36}$$

Up to being careful about components, this is:

$$H^*\left(G\left(F\right)\backslash G\left(\mathbb{A}\right)/G\left(\mathcal{O}\right)\times K\right)=H^*\left([G]_F=G\left(\mathbb{Z}\right)\backslash G_{\mathbb{R}}/K\right)\ . \tag{9.37}$$

This locally symmetric space is contractible so:

$$H^*\left([G]_F\right)=H^*\left(G\left(\mathbb{Z}\right),\mathbb{C}\right) \tag{9.38}$$

is just group cohomology. More generally, for $G(Z) \circlearrowright V_{\mathbb{C}}$, we can replace this by:

$$H^*(G(\mathbb{Z}), V_{\mathbb{C}}) = H^*([G]_F, \text{local system}) \ . \tag{9.39}$$

There is something called the *Eichler-Shimura* isomorphism (Hodge theory on $\mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H}$) which says that classical modular forms sit inside of cohomological forms for SL_2. So instead of talking about holomorphic functions with growth conditions, we can realize everything in cohomology.

Recall in the function field case the analogue of $[G]_F$ was $\mathrm{Bun}_G(C)(\mathbb{F}_q)$, and then we took functions. But cohomology of a discrete set is the same as functions on it:

$$\mathrm{Funct}(\mathrm{Bun}_G(C)(\mathbb{F}_q)) = H^*(\mathrm{Bun}_G(C)(\mathbb{F}_q)) \ . \tag{9.40}$$

9.1.5 Geometric automorphic forms

More generally we can consider "geometric" automorphic forms. This means the same thing as geometric CFT: we're studying sheaves instead of functions. So for k any field and C/k, geometric automorphic forms are

$$\mathbf{Shv}(\mathrm{Bun}_G(C)) \ . \tag{9.41}$$

These are what one might call *automorphic sheaves*. We hope that this appears in TFT at the level of a surface, since it is analogous to what we had in the abelian story:

$$\Sigma \mapsto \mathbf{Shv}(\mathrm{Pic}\,\Sigma) \ . \tag{9.42}$$

Remark 45. This category of automorphic sheaves will become one of our basic objects of study. One source of motivation, is that it directly generalizes what we had before: if replace GL_1 by GL_n then you get these moduli spaces of bundles, and we can try to do the same thing we did in CFT to them. Another source of motivation comes from the case of number fields, where these spaces of bundles are the locally-symmetric spaces that we see from the number theory.

The basic problem is that, unlike for $\mathrm{Pic} = \mathrm{Bun}_{\mathrm{GL}_1}$, Bun_G are not groups, and there is not even a natural group acting on them: when we form

$$\mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H} = \mathrm{SL}_2 \mathbb{Z} \backslash \mathrm{SL}_2 \mathbb{R} / \mathrm{SO}_2 \ , \tag{9.43}$$

we mod out by groups on both sides, so there isn't room for anything else to act. We will find that there are things called *Hecke operators* which do act, but they don't commute, so we can't quite do spectral decomposition. This is where we will use ideas from TFT. transition

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Recall we have this adélic uniformization picture:

$$|\mathrm{Bun}_G(C)(\mathbb{F}_q)| = G(F_C) \backslash \prod\nolimits' G(K_x) / \prod G(\mathcal{O}_x) \tag{9.44}$$
$$= G(F_C) \backslash G(\mathbb{A}_C) / G(\mathcal{O}_{\mathbb{A}}) \ . \tag{9.45}$$

Then we want to study functions on this. This is a discrete set, so there is no ambiguity in the notion functions we're taking. But we want to decompose these under some action, so we need to answer the following question.

Question 2. What acts on automorphic forms?

The problem is that Bun_G is not a group, whereas when $G = \mathrm{GL}_1$, $\mathrm{Bun}_G = \mathrm{Pic}$ was naturally a group, so functions on it had a notion of convolution. Luckily the space has the form X/K where X is a space with an action of a group G with a subgroup $K \subset G$. So we start with a space X with G-symmetry, and now the question is: what is the leftover symmetry on X/K? In our situation, we start out with this restricted product $\prod'_{x\in C} G(K_x)$, we quotient out by this group of Taylor series $G(\mathcal{O}_X)$ on the right, and then you quotient out by this global field $G(F_C)$ on the left. So we have this huge amount of symmetry: at every point in the curve we have $G(K_x)$, and then we've only quotient out by "half" of it, i.e. by Taylor series at every point. We could instead form the analogue of the idéle class group:

$$G(F_c) \backslash \prod{}' G(K_x) \ , \tag{9.46}$$

where we don't quotient out on the right, and we have an action of a giant group:

$$\prod G(K_x) \ , \tag{9.47}$$

and in fact they all commute. I.e. the group $G(K_x)$ itself is not commutative, but the copies commute. So we have a huge amount of symmetry. Then when we quotient out by the Taylor series part, $\prod G(\mathcal{O}_x)$, we can hope some of that symmetry remains. This is exactly the idea of Hecke algebras: they capture this residual symmetry.

9.2 Hecke algebras

Assume G is a finite group acting on a finite set X, and let $K \subset G$ be a subgroup. These assumptions are just for convenience; the features we will see will generalize to other settings. The question we want to answer is:

Question 3. What acts on $\mathbb{C}[X/K]$?

The kind of answer that you first learn in group theory, is the normalizer of K in G, $N_G(K)$. But sometimes $N_G(K) = K$, which acts trivially. So this is really only the right answer if we think in terms of elements of the group. Instead we should think in terms of the group algebra $\mathbb{C}G$. $\mathbb{C}G$ acts on functions on X, and functions on X/K are the same as K-invariant ones on X:

$$\mathbb{C}[X/K] = \mathbb{C}[X]^K \ . \tag{9.48}$$

Now we want to understand the K-invariants for G-rep, i.e. we're thinking of:

$$(-)^K : \mathbf{Rep}G \to \mathbf{Vect} \tag{9.49}$$

as a functor, and we want to understand its endomorphisms $\mathrm{End}\left((-)^K\right)$. Whenever you want to calculate endomorphisms of a functor, you should ask if it's representable (nice functors are representable), i.e. if there is a G-representation $V_{G,K}$ such that:

$$(-)^K = \mathrm{Hom}(V_{G,K}, -) \ . \tag{9.50}$$

The point is that the Hom-$\otimes$ adjunction gives us:

$$V^K = \mathrm{Hom}_{\mathbb{C}K}(\mathbb{C}, V|_K) = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}K} \otimes \mathbb{C}, V) \ . \tag{9.51}$$

This is a version of Frobenius reciprocity:

$$\mathrm{Hom}_{\mathbb{C}K}\left(\mathbb{C}, \mathrm{Res}_G^K V\right) = \mathrm{Hom}_{\mathbb{C}G}\left(\mathrm{Ind}_K^G \mathbb{C}, V\right) \ . \tag{9.52}$$

The upshot is that the object which represents $(-)^K$ is:

$$V_{G,K} = \mathrm{Ind}_K^G \mathbb{C} = \mathbb{C}G \otimes_{\mathbb{C}K} \mathbb{C} \ . \tag{9.53}$$

We can think of this as:

$$V_{G,K} = \mathrm{Ind}_K^G \mathbb{C} = \mathbb{C}G \otimes_{\mathbb{C}K} \mathbb{C} \tag{9.54}$$
$$= \{f\colon G \to \mathbb{C} \mid f(gk) = f(g)\} \tag{9.55}$$
$$= \mathbb{C}[G/K] = \mathbb{C}[G]^K \ , \tag{9.56}$$

where we're implicitly using the fact that functions and measures are the same on a finite group, i.e. we really should end up with K-invariant *measures* on G.

The *Hecke algebra* $\mathcal{H}_{G,K}$ is:

$$\mathcal{H}_{G,K} \coloneqq \mathrm{End}\left((-)^K\right) = \mathrm{End}_{\mathbf{Rep}(G)} V_{G,K} = \mathrm{Hom}_{\mathbb{C}G}\left(V_{G,K}, V_{G,K}\right) \tag{9.57}$$
$$= \left(V_{G,K}\right)^K = \mathbb{C}[G/K]^K \tag{9.58}$$
$$= \mathbb{C}[K \backslash G/K] \ . \tag{9.59}$$

This tells us that functions on the double coset has a canonical algebra structure.

There is a particular brand of abstract nonsense (Barr-Beck) which says the following. What we have found, is that functor $(-)^K$ factors as:

$$\begin{tikzcd} \mathbf{Rep}(G) \arrow[rr, "(-)^K"] \arrow[dr] & & \mathbf{Vect} \\ & \mathcal{H}_{G,K}\text{-}\mathbf{mod} \arrow[ur, "\text{Forget}"] & \end{tikzcd} \ . \tag{9.60}$$

We can think of $\mathcal{H}_{G,K}$-**mod** as being not very far from $\mathbf{Rep}(G)$, i.e. it might not be an equivalence, but it might be on some class of representations. For example we might consider the subcategory of $\mathbf{Rep}(G)$ consisting of G-representations that are generated by their K-invariants. E.g. irreducible representations V are generated by their K-invariants if V^K is nontrivial. Call these *spherical representations*. Then the abstract nonsense tells us that this is an equivalence:

$$\begin{tikzcd} \mathbf{Rep}(G) \arrow[rr, "(-)^K"] & & \mathcal{H}_{G,K}\text{-}\mathbf{mod} \\ & \{\text{generated by } K\text{-invariants}\} \arrow[ul, hook] \arrow[ur, "\sim"] & \end{tikzcd} \ . \tag{9.61}$$

Another way of saying this is that this subcategory of spherical representations is generated by $V_{G,K}$. I.e. irreducible representation V is spherical if and only if it appears in the decomposition of $V_{G,K}$ into irreducibles.

Remark 46. The word *spherical* goes back to the study of spherical harmonics. L^2 functions on the 2-sphere, $L^2\left(S^2\right)$, is acted on by SO_3. Then we can decompose $L^2\left(S^2\right)$ under this action. This is sometimes taught in high school chemistry: the different types of atomic orbitals have to do with different irreducible representations of SO_3. The connection with what we're doing is that $G = \mathrm{SO}_3$, $K = \mathrm{SO}_2$, and $S^2 = \mathrm{SO}_3 / \mathrm{SO}_2$.

Now we will write the algebra structure on $\mathbb{C}\left[K\backslash G/K\right]$ explicitly. First let's rewrite the group algebra $\mathbb{C}G$ as follows. We have the multiplication and projection maps:

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[d, "\mu"] \arrow[dr, "\pi_2"] & \\ G & G & G \end{tikzcd} \tag{9.62}$$

and then convolution is given by:

$$f * h = \mu_* \left(\pi_1^* f \cdot \pi_2^* h\right) \ . \tag{9.63}$$

We can take the same diagram and mod out by K to get:

$$\begin{tikzcd} & K\backslash G \times G/K \arrow[dl, "\pi_1"'] \arrow[d, "\mu"] \arrow[dr, "\pi_2"] & \\ K\backslash G & K\backslash G/K & G/K \end{tikzcd} \ . \tag{9.64}$$

We can go even further, and take a balanced product with respect to the K-action on the middle. Then we get a diagram:

$$\begin{tikzcd} & K\backslash G \times^K G/K \arrow[dl, "\pi_1"'] \arrow[d, "\mu"] \arrow[dr, "\pi_2"] & \\ K\backslash G/K & K\backslash G/K & K\backslash G/K \end{tikzcd} \ . \tag{9.65}$$

This diagram demonstrates that we have a subalgebra of bi-invariant functions:

$$\mathbb{C}\left[G\right]^{K \times K} \subset \mathbb{C}\left[G\right] \ . \tag{9.66}$$

Group theory tells us that:

$$K\backslash G/K = G\backslash \left(G/K \times G/K\right) \ , \tag{9.67}$$

where we're modding out by the diagonal action. Then functions on this are:

$$\mathbb{C}\left[K\backslash G/K\right] = \mathbb{C}\left[G/K \times G/K\right]^G \tag{9.68}$$

$$\subset \mathbb{C}\left[G/K \times G/K\right] \tag{9.69}$$

$$= G/K \times G/K\text{-matrices} \ . \tag{9.70}$$

We can motivate calling these $G/K \times G/K$-matrices as follows. If X is a finite set with n-elements, $\mathbb{C}[X \times X]$ is just $n \times n$ matrices, i.e. $\operatorname{End}\mathbb{C}[X]$. Then matrix multiplication has a push-pull description as above:

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X \times X & & X \times X \\ & X \times X & \end{tikzcd} \quad . \tag{9.71}$$

This is all to say that:

$$(\mathbb{C}G, *) \subset \mathbb{C}[G \times G], \text{matrices} , \tag{9.72}$$

and in fact it is equal to:

$$\mathbb{C}G = \mathbb{C}[G \times G]^{\operatorname{diag}G} \quad . \tag{9.73}$$

9.2.1 Groupoid version

Recall we have a groupoid pt/G, which is the groupoid of G-bundles on a point. This has a map from pt, so we can form the fiber product:

$$\begin{tikzcd} & \mathrm{pt} \times_{\mathrm{pt}/G} \mathrm{pt} \arrow[dl] \arrow[dr] & \\ \mathrm{pt} \arrow[dr] & & \mathrm{pt} \arrow[dl] \\ & \mathrm{pt}/G & \end{tikzcd} \quad . \tag{9.74}$$

We can think of this as a way of saying what G is since:

$$\mathrm{pt} \times_{\mathrm{pt}/G} \mathrm{pt} = \operatorname{Aut}(\text{trivial } G\text{-bundle}) = G \ . \tag{9.75}$$

More generally, for K a subgroup, we have a map $\mathrm{pt}/K \to \mathrm{pt}/G$, and we can form the fiber product:

$$\begin{tikzcd} & \mathrm{pt}/K \times_{\mathrm{pt}/G} \mathrm{pt}/K \arrow[dl] \arrow[dr] & \\ \mathrm{pt}/K \arrow[dr] & & \mathrm{pt}/K \arrow[dl] \\ & \mathrm{pt}/G & \end{tikzcd} \quad , \tag{9.76}$$

which is:

$$\mathrm{pt}/K \times_{\mathrm{pt}/G} \mathrm{pt}/K \simeq K \backslash G/K \simeq G \backslash (G/K \times G/K) \ . \tag{9.77}$$

We can think of these as matrices, but where the base isn't a point. The general idea is that we have $X \to Y$ (for us $\mathrm{pt}/K \to \mathrm{pt}/G$) and we're considering functions on $X \times_Y X$, i.e. if

X is a finite set, then these are some kind of block diagonal matrices where you cluster the points of X according to their image in Y.

This tells us that $K\backslash G/K$ is the groupoid acting on pt/K with quotient pt/G. The conclusion is that $\mathcal{H}_{G,K}$ is the groupoid algebra of $K\backslash G/K$:

$$\mathcal{H}_{G,K} = \mathbb{C}\ K\backslash G/K\ . \tag{9.78}$$

The data of a G action on X is the same as a map:

$$X/G \to \mathrm{pt}/G\ . \tag{9.79}$$

Given an action we just take the quotient, and given a map to pt/G, we can form X as

$$X = (X/G) \times_{\mathrm{pt}/G} \mathrm{pt}\ . \tag{9.80}$$

Recall we started by asking: what acts on (functions on) X/K for X a G-space? We can also ask it on the level of a space: once we have quotiented out by a subgroup, is there any symmetry left? Again, the normalizer acts, but even if there is no nontrivial normalizer there is always a canonical groupoid which acts on X/K. We get this by essentially combining all of the diagrams. X/K is the pullback of X/G along the map $\mathrm{pt}/K \to \mathrm{pt}/G$:

$$\begin{tikzcd} & & K\backslash G/K \arrow[dl] \arrow[dr] & \\ X/K \arrow[r] \arrow[d] & \mathrm{pt}/K \arrow[dr] & & \mathrm{pt}/K \arrow[dl] \\ X/G \arrow[rr] & & \mathrm{pt}/G & \end{tikzcd} \ . \tag{9.81}$$

And whenever something is pulled back, this descent groupoid $K\backslash G/K$ will act on it.

More concretely, we have the following. Recall we have the projection p and the action a:

$$\begin{tikzcd} & X \times G \arrow[dl, "p"'] \arrow[dr, "a"] & \\ X & & X \end{tikzcd} \ . \tag{9.82}$$

As before, we can mod out by K to get:

$$\begin{tikzcd} & X/K \times_{\mathrm{pt}/K} K\backslash G/K \arrow[dl, "p"'] \arrow[dr, "a"] & \\ X/K & & X/K \end{tikzcd} \tag{9.83}$$

where the middle term is the balanced product:

$$X/K \times_{\mathrm{pt}/K} K\backslash G/K \simeq X \times^{K} G/K\ . \tag{9.84}$$

The point is the following. We started out with a group action, and modded out by a subgroup. Rather than a map, we have this correspondence, and this is the remnant left of the group action we started with.

Another way of saying this is as follows. Over X/K we have a principal K-bundle given by X. K acts on G/K, so we can take the associated G/K-bundle over X, and that is what this construction is.

9.2.2 Examples

Example 44. If $K = 1 \subset G$, then $K\backslash G/K = G$, so $\mathcal{H}_{G,K} = \mathbb{C}G$.

Example 45. If $K = G \subset G$, then $G\backslash G/G = \mathrm{pt}/G$, so $\mathcal{H}_{G,K} = \mathbb{C}$.

Example 46. Let $G = G(\mathbb{F}_q)$ be the $\mathbb{F}_q$ points of any Lie group, and $K = B(\mathbb{F}_q)$ the Borel. E.g. $G = \mathrm{GL}_2$ and B is upper triangular matrices. Then we get the *finite Hecke algebra*:

$$\mathcal{H}_{G,K} = H_q = H_{G,q} \ , \tag{9.85}$$

which is

$$H_q = \mathbb{C}[K\backslash G/K] \ . \tag{9.86}$$

Note that

$$\mathcal{B} := G/K = G(\mathbb{F}_q)/B(\mathbb{F}_q) \tag{9.87}$$

consists of flags in $\mathbb{F}_q^n$, and once we have quotiented out by $B(\mathbb{F}_q)$ on the left, we have the collection of Bruhat cells, which are in bijection with the Weyl group, so we have:

$$\mathcal{H}_{G,K} = \mathbb{C}[W] \ . \tag{9.88}$$

If $G = \mathrm{GL}_n$ then $W = S_n$ is the symmetric group. However, $\mathbb{C}[W]$ is not the group algebra of the Weyl group as an algebra.

Another way of writing double cosets is:

$$G(\mathbb{F}_q)\backslash \mathcal{B} \times \mathcal{B} \ , \tag{9.89}$$

where we're modding out by the diagonal action. I.e. we take two flags, and we ask what invariants they have, if I'm allowed to apply any matrix (element of $G(\mathbb{F}_q)$), i.e. relative positions of flags.

If $G = \mathrm{SL}_2$ then $\mathcal{B} = \mathbb{P}^1(\mathbb{F}_q)$. Given two points in $\mathbb{P}^1$, there's only one thing we can ask in a coordinate independent way: they're either equal or not equal. Then

$$\mathrm{SL}_2(\mathbb{F}_q)\backslash \mathcal{B} \times \mathcal{B} \tag{9.90}$$

consists of the orbits of this $\mathrm{SL}_2(\mathbb{F}_q)$-action on $\mathbb{P}^1$, i.e. the unequal, and equal parts. So there are two basic functions: T, the characteristic function for the "unequal" subset, and 1, the characteristic function for the "equal" subset. The only relation to calculate is:

$$T * T(\text{equal}) = q \qquad\qquad T * T(\text{unequal}) = q - 1 \ . \tag{9.91}$$

These come from:

$$\#\left(\mathbb{A}^1\right) = q \tag{9.92}$$

$$\#(\mathbb{G}_m) = q - 1 \ . \tag{9.93}$$

So we find the relation $T^2 = (q-1)T + q$, which we can write as:

$$(T+1)(T-q) = 0 \ . \tag{9.94}$$

So in this case, the Hecke algebra has a single generator subject to this quadratic relation.

When $q = 1$ this is just saying that $T^2 = 1$, so we get the group algebra of the symmetric group S_2, which is the Weyl group for SL_2.

Hecke algebras usually aren't commutative, but they *can* be.

Example 47. Let $G = K \times K$ with subgroup the diagonal copy of $K \subset K \times K = G$. The claim is that $\mathcal{H}_{K\times K,K}$ is commutative for all K. First note that:

$$K\backslash G/K = K\backslash K \times K/K \tag{9.95}$$
$$= K/K \tag{9.96}$$

where we're quotienting by the conjugation action of K on itself. So $\mathcal{H}_{K\times K,K}$ consists of conjugation-invariant functions on K, i.e. class functions on K, which do have a commutative multiplication. In fact, more abstractly:

$$\mathcal{H}_{K\times K,K} = \operatorname{End}_{\mathbb{C}K\otimes\mathbb{C}K^{\mathrm{op}}} \mathbb{C}K = Z\left(\mathbb{C}K\right) \ . \tag{9.97}$$

Remark 47. This is a general fact having nothing to do with groups. For any algebra A, endomorphisms of A as a module over itself on the left and the right comprise the center:

$$Z\left(A\right) = \operatorname{End}_{A\otimes A^{\mathrm{op}}} A \ . \tag{9.98}$$

The point of this is that we will see that the Hecke algebras acting on automorphic forms are commutative. This is really the birth of the Langlands program. What will take some time to build up, is to see that they're commutative for structural reasons even though just writing something in terms of double cosets doesn't give you commutativity in general.

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9.2.3 Hecke algebras in our setting

The main example we're interested in is either $\operatorname{Bun}_G C$ or its analog for a number field. Then we're interested in the symmetries of these spaces that come from their presentation as double quotients, e.g. in the case of a number field we're considering spaces of the form:

$$\Gamma\backslash G_{\mathbb{R}}/K \ , \tag{9.99}$$

where $G_{\mathbb{R}}$ is a real Lie group, K is a maximal compact and Γ is a lattice. For example, if $G_{\mathbb{R}} = \mathrm{SL}_2$ then $K = \mathrm{SO}_2$, and $G_{\mathbb{R}}/\mathrm{SO}_2 = \mathbb{H}$ is hyperbolic space. Then we can take $\Gamma = \mathrm{SL}_2 \mathbb{Z}$, and the locally-symmetric space is:

$$\mathrm{SL}_2 \mathbb{Z}\backslash\mathbb{H} = \mathrm{SL}_2 \mathbb{Z}\backslash \mathrm{SL}_2 \mathbb{R}/\mathrm{SO}_2 \ . \tag{9.100}$$

We're thinking of $X = \Gamma\backslash G_{\mathbb{R}}$ so that our space is of the form:

$$X/K = \Gamma\backslash G_{\mathbb{R}}/K \ . \tag{9.101}$$

So we get a symmetry by:

$$K\backslash G_{\mathbb{R}}/K \ . \tag{9.102}$$

Let's get a feeling for what this kind of symmetry looks like in the case where $G_{\mathbb{R}} = \mathrm{SL}_2$, $K = \mathrm{SO}_2$, and $\Gamma = \mathrm{SL}_2 \mathbb{Z}$. Then we get a symmetry by:

$$\mathrm{SO}_2 \backslash \mathrm{SL}_2 \mathbb{R}/\mathrm{SO}_2 = \mathrm{SO}_2 \backslash\mathbb{H} \ . \tag{9.103}$$

Recall we can think of the Hyperbolic space as the upper-half space, and then this right copy of SO_2 is the stabilizer of $i \in \mathbb{H}$ where it is acting by rotations. These don't look very much like rotations in the upper-half space, but if we instead consider the disk model of $\mathbb{H}$, then they do look like rotations. The set of orbits looks like $\mathbb{R}_+$, the set of all distances (in the hyperbolic metric) from $i \in \mathbb{H}$. In the disk model it looks like an interval $[a, b)$, but in the half-space model it looks more like a ray. Then the claim is that we get operators on functions on $G_{\mathbb{R}}/K$ commuting with $G_{\mathbb{R}}$ from double cosets.

If we fix $r \in \mathbb{R}_+$ then we get an operator which takes a function f on $\mathbb{H}$ and maps it to the function:

$$(\mathcal{O}_r * f)(z) = \int_{|w-z|=r} f(w) \ . \tag{9.104}$$

I.e. given a function, we get a new function which takes $z \in \mathbb{H}$ and averages the old function along the points of distance r from z.

Claim 1. These operators all commute and descend to $\Gamma\backslash\mathbb{H}$.

Another manifestation of the same phenomenon is as follows. We can think about double cosets as pairs of cosets invariant under the diagonal group action:

$$K\backslash G/K = G\backslash (G/K \times G/K) \ . \tag{9.105}$$

I.e. the symmetry is realized by operators on functions on G/K commuting with the left G-action. Inside of this we can look at G-invariant differential operators on G/K:

$$\mathrm{Diff}^G (G/K) \ . \tag{9.106}$$

A *differential operator* on a space X is an endomorphism of functions on X which is *local* in the sense that $(\mathcal{O} * f)(x)$ depends only on the Taylor series of f at x. I.e. this is saying that we're looking at operators whose "matrix" is supported at the diagonal.

Remark 48. This definition works in the context of algebraic geometry, and was in fact Grothendieck's definition.

Theorem 7 (Harish-Chandra)**.** *Differential operators on the symmetric space $G_{\mathbb{R}}/K$ (for K maximal compact) which are $G_{\mathbb{R}}$-invariant,* $\mathrm{Diff}_{G_{\mathbb{R}}}(G_{\mathbb{R}}/K)$*, is a commutative ring. In fact, it is a polynomial ring $\mathbb{C}[\mathfrak{a}^*]^W$, where $\mathfrak{a}$ is the Cartan of G/K and W is the Weyl group of G/K.*

Example 48. Let $G_{\mathbb{R}}/K = \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2 = \mathbb{H}$. Then we get $\mathbb{C}[\Delta_{\mathbb{H}}]$, where $\Delta_{\mathbb{H}}$ is the hyperbolic Laplacian.

For general symmetric spaces we get higher order "Laplacians" which all commute. So this is the differential operator version of the Hecke algebra, and Harish-Chandra tells us that this is just a commutative algebra generated by higher order analogues of the Laplace operator.

Example 49 (The "usual"/group Harish-Chandra isomorphism)**.** Let G be a real reductive Lie group, i.e. a complexification of a compact Lie group. Consider differential operators on G which are invariant under both left and right multiplication: $\mathrm{Diff}_{G\times G}(G)$.

What are these? First of all, left-invariant differential operators are the universal enveloping algebra of $\mathfrak{g} = \mathbf{Vect}_G(G)$:

$$\mathrm{Diff}_G(G) = U\mathfrak{g} \ . \tag{9.107}$$

So if we ask for them to be invariant on both sides we get G-invariant elements of $U\mathfrak{g}$, i.e. the center:

$$\mathrm{Diff}_{G\times G}(G) = (U\mathfrak{g})^G = Z(U\mathfrak{g}) \ . \tag{9.108}$$

Then the Harish-Chandra isomorphism says that this is W-invariant functions on $\mathfrak{h}^*$:

$$Z(U\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^W \ , \tag{9.109}$$

where $\mathfrak{h}$ is the Cartan subalgebra, and W is the Weyl group. I.e. we're looking at some vector space $\mathfrak{h}^*$, we're asking for functions on it, and we're asking that they're invariant under a reflection group W. There is a theorem of Chevalley which says that in this general situation we just get a polynomial algebra:

$$\mathbb{C}[\mathfrak{h}^*]^W \simeq \mathbb{C}[p_1, \ldots, p_\ell] \tag{9.110}$$

where $\ell = \dim \mathfrak{h} = \mathrm{rank}\, G$.

So for us, the idea is that, for G compact, we have $G\times G$ acting on $G = G\times G/G_{\mathrm{diag}}$ (so G is playing the role of the compact subgroup K from before).

There is a more general Harish-Chandra isomorphism due to Knop which says that when we have a spherical variety $G \circlearrowright X$, then:

cite

$$\mathcal{D}_G(X) \simeq \mathbb{C}[\mathfrak{a}_X]^{W_X} \tag{9.111}$$

where $\mathfrak{a}$ is some kind of Cartan associated to X and W_X is some kind of Weyl group associated to X.

This theorem suggests that if we're looking at functions on $\Gamma\backslash G_{\mathbb{R}}/K$, then this space carries a natural commutative algebra of symmetries (by thinking about it as a quotient of $G_{\mathbb{R}}$) given by $\mathrm{Diff}_{G_{\mathbb{R}}}(G_{\mathbb{R}}/K)$. But in our case, a whole lot more acts. This nice presentation $\Gamma\backslash G_{\mathbb{R}}/K$ hides a lot of the symmetry. We also had the presentation:

$$\Gamma\backslash G_{\mathbb{R}}/K \simeq G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_A) \tag{9.112}$$

$$= G(F)\backslash {\prod_x}' G(K_x)/G(\mathcal{O}_x) \ . \tag{9.113}$$

When we write it like this we see a huge amount of Hecke symmetry appearing. At each point we have a factor $G(K_x)/G(\mathcal{O}_x)$. Then there is this "secret sauce" or "glue" given by quotienting out by this global group $G(F)$. In any case, each of these factors will have their own Hecke symmetry commuting with the action of $G(F)$. I.e. functions on

$$G(F)\backslash {\prod_x}' G(K_x)/G(\mathcal{O}_x) \ . \tag{9.114}$$

carries an action of this giant algebra given by the tensor product over all $x \in C$ of "local" Hecke algebras which are associated to

$$G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x) \ , \tag{9.115}$$

which all commute with each other.

Theorem 8 (Satake)**.** *The Hecke algebra of compactly supported functions on this double coset:*

$$\mathbb{C}_c[G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)] \tag{9.116}$$

is also a polynomial algebra.

In the number field setting we were looking at $K \simeq \mathbb{Q}_p$ and our ring of integers $K \supset \mathcal{O} = \mathbb{Z}_p \subset \mathbb{Q}_p$. Then we formed:

$$G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p) \ . \tag{9.117}$$

In the function field setting we had Laurent series $K \cong \mathbb{F}_q((t))$, and Taylor series $\mathcal{O} = \mathbb{F}_q[[t]]$. Then we formed:

$$G(\mathbb{F}_q[[t]]) \backslash G(\mathbb{F}_q((t))) / G(\mathbb{F}_q[[t]]) \ . \tag{9.118}$$

Now we will consider the example $G = \mathrm{PGL}_2$, and try to illustrate what these Hecke operators look like. So we want to draw the space $G(K)/G(\mathcal{O})$ where K is either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Just to fix notation we will set:

$$G(K)/G(\mathcal{O}) = \mathrm{PGL}_2(\mathbb{Q}_p) / \mathrm{PGL}_2(\mathbb{Z}_p) \ . \tag{9.119}$$

This consists of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$ up to homothety. What does this mean? We have $\Lambda \subset \mathbb{Q}_p^2$, which is a $\mathbb{Z}_p$-submodule. Being a lattice means that if we invert p, we get $\mathbb{Q}_p^2$:

$$\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p^2 \ , \tag{9.120}$$

for example $\Lambda_0 = \mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$ is the standard lattice. Some linear algebra tells us that any lattice is taken to the standard one by the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. Another example of a lattice is:

$$p\mathbb{Z}_p^2 \subset \mathbb{Q}_p^2 \ . \tag{9.121}$$

If Λ_0 is spanned by e_1 and e_2, then $p\mathbb{Z}_p^2$ is spanned by $\{pe_1, pe_2\}$. But in between Λ_0 and $p\Lambda$ we have some intermediate lattices:

$$\Lambda_0 \subset \{e_1, pe_2\} \subset p\Lambda_0 \ . \tag{9.122}$$

Consider a graph where the vertices correspond to lattices in $\mathbb{Q}_p^2$ up to homothety, and we will connect two vertices Λ and Λ' by an edge if, up to homothety, we have:

$$\Lambda \subset \Lambda' \subset p\Lambda \ . \tag{9.123}$$

One can check this is an equivalence relation because:

$$p^{-1}\Lambda \subset \Lambda \subset \Lambda' \subset p\Lambda \ . \tag{9.124}$$

The resulting graph is called the *building* of $\mathrm{PGL}_2/\mathbb{Q}_p$, but this graph is actually just a tree. In fact, for $\mathbb{Q}_2$, it turns out that we get a 3-regular tree. Our first vertex will correspond to the lattice Λ_0. Then we ask how many lattices Λ' can be sandwich between: $\Lambda_0 \subset \Lambda' \subset p\Lambda_0$. The data of Λ' is equivalent to a line in:

$$\Lambda_0 / p\Lambda_0 \simeq \mathbb{F}_p^2 \ . \tag{9.125}$$

In other words we get a point in $\mathbb{P}^1(\mathbb{F}_p)$. The number of points is $\left|\mathbb{P}^1(\mathbb{F}_p)\right| = p+1$. The upshot is that every vertex Λ has three neighbors, i.e. we get a 3-regular tree as in fig. 9.1.

In general, the vertices of this graph are given by:

$$G(K)/G(\mathcal{O}) \ , \tag{9.126}$$

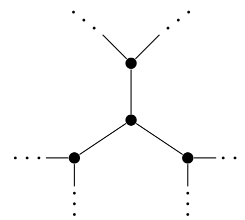


Figure 9.1: The building of PGL_2 over $\mathbb{Q}_2$ is a 3-regular tree. Here we show 4 vertices of the infinite 3-regular tree comprising the building of $\mathrm{PGL}_2/\mathbb{Q}_2$. The full building is in fig. 9.2.

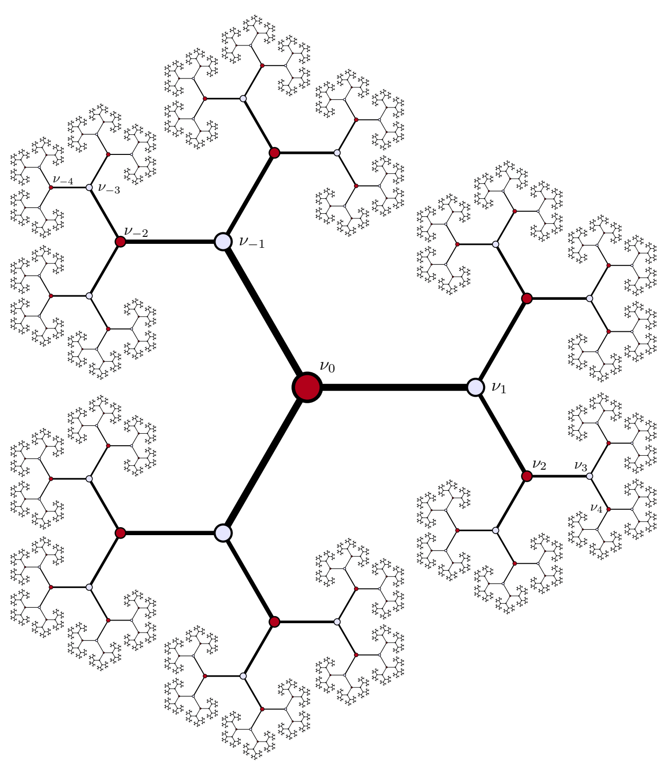


Figure 9.2: The building of PGL_2 over $\mathbb{Q}_2$. The $G(\mathcal{O})$-orbits are concentric "circles" of vertices around the center. They are color-coded. Figure from [Cas14].

which is also called the affine Grassmannian for G. This is the p-adic analogue of hyperbolic space. The idea is that the vertices are accumulating at infinity, like the disk model of hyperbolic space. The beautiful thing is that many standard features of hyperbolic space are reflected in the geometry of this tree.

The double quotient

$$G(\mathcal{O}) \backslash G(K) / G(\mathcal{O}) \tag{9.127}$$

corresponds to the $G(\mathcal{O}) = \mathrm{PGL}_2(\mathbb{Z}_p)$-orbits on this *building*. Recall in actual hyperbolic space $\mathbb{H}$, the stabilizer of i was $G(\mathcal{O})$, and the other $G(\mathcal{O})$-orbits were labelled by their distance from some point i. Here we have a similar picture. $G(\mathcal{O})$ is the stabilizer of $\Lambda_0 = \mathbb{Z}_p^2$ in $\mathbb{Q}_p^2$ up to rescaling. I.e. this point in the middle of fig. 9.1 has stabilizer equal to $G(\mathcal{O})$. Then the other orbits are again these concentric "circles" indexed by distance from Λ_0, only now the distance is given by the tree metric, i.e. the number of edges. So the orbits correspond to $\mathbb{Z}_+$.

Then the Hecke operators are as follows. The "tree Laplacian" T_{p^n} sends a function f on $G(K)/G(\mathcal{O})$ to the function which takes the average over points of fixed distance n:

$$(T_{p^n} f)(v) = \sum_{\mathrm{dist}(w,v)=n} f(w) \ . \tag{9.128}$$

Again these all commute, and we find that the algebra of these operators is just the polynomial algebra $\mathbb{C}[T_p]$.

So just by writing Bun_G as:

$$\mathrm{Bun}_G = G(F) \backslash {\prod}' G(K_x)/G(\mathcal{O}_x) \ , \tag{9.129}$$

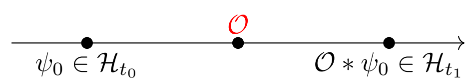
we get this action of all of these algebra which commute with one another. What is not clear yet, is that each of these are commutative themselves.

Chapter 10

Operators and defects in TFT

10.1 Space of states

One-dimensional quantum field theory is quantum mechanics. So we have $\mathcal{H}$ a Hilbert space of states, and it is acted on by an associative algebra of observables A. Typically A consists of all operators on $\mathcal{H}$. The schematic picture is that $\mathcal{H}$ consists of all states at a particular point in time. Then if we have a state $\psi \in \mathcal{H}_{t_0}$, we can place an operator $\mathcal{O}$ at some particular point in time after t_0, and then after passing the operator we have a new state $\mathcal{O} * \psi \in \mathcal{H}_{t_1}$. The picture to have in mind is:



. (10.1)

This all becomes much easier in topological quantum mechanics.

A 1-dimensional TFT (i.e. topological quantum mechanics) is a symmetric monoidal functor:

$$\mathcal{Z}\colon (\mathbf{Bord}_1^{\mathrm{or}}, \sqcup) \to (\mathbf{Vect}_{\mathbb{C}}, \otimes) \ . \tag{10.2}$$

The objects of $\mathbf{Bord}_1^{\mathrm{or}}$ are 0-manifolds, but they are 1-oriented, i.e. all the pictures we draw for an n-dimensional TFT are n-dimensional, because the bordisms have an n-dimensional tangent structure. One way of saying this is that you're not just a 0-manifold, but rather a germ of an *oriented* n-manifold. So here we can think that the objects are 0-manifolds with a small oriented interval attached. Therefore there are two 1-oriented 0-manifolds:

$$\longrightarrow\!\!\!\!\!\!\!\bullet\quad \qquad\qquad \longleftarrow\!\!\!\!\!\!\!\bullet\quad \ . \tag{10.3}$$

Therefore the functor $\mathcal{Z}$ sends these to two vector spaces:

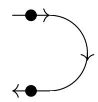
$$\mathcal{Z}\,(\text{-}\bullet\!\!\rightarrow) = V \qquad\qquad \mathcal{Z}\,(\leftarrow\!\!\bullet\text{-}) = V' \ . \tag{10.4}$$

Remark 49. V will eventually be our space of states.

Now what we would like to notice, is that the axioms of a TFT determine a close relationship between V and V'. There is a natural morphism in $\mathbf{Bord}_1^{\mathrm{or}}$ from the disjoint union of these two objects to the identity (the empty 0-manifold $\emptyset_0$):

$$(\text{—}\bullet\!\!\longrightarrow) \quad \sqcup \quad (\longleftarrow\!\!\bullet\text{—}) \qquad \to \qquad \emptyset_0 \ , \tag{10.5}$$

namely it is:

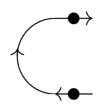


$$. \tag{10.6}$$

The functor $\mathcal{Z}$ is symmetric monoidal, so it sends this bordism to a functor which we will call the evaluation:

$$\mathrm{ev}\colon V \otimes V' \to \mathcal{Z}(\emptyset_0) = \mathbb{C}\ . \tag{10.7}$$

We also have the opposite bordism:

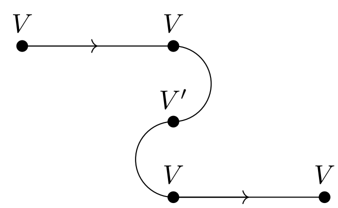


$$, \tag{10.8}$$

which gives the coevaluation:

$$\mathrm{coev}\colon \mathbb{C} \to V \otimes V'\ . \tag{10.9}$$

Then these maps satisfy a nice relation, known as the *Mark of Zorro*, since it comes from the following bordism:



$$. \tag{10.10}$$

Decomposing this bordism, we see that the functor $\mathcal{Z}$ sends it to the following composition:

$$\begin{tikzcd} V \arrow[r, "1\otimes\mathrm{coev}"] \arrow[rr, bend right, "\mathrm{id}"'] & V \otimes V' \otimes V \arrow[r, "\mathrm{ev}\otimes 1"] & V \end{tikzcd}\ , \tag{10.11}$$

where this diagram commutes, i.e. these maps compose to the identity, because we can pull that bordism straight, and it must induce the identity on V. One we unwind the definitions, this is a category theoretic way to say that V is a dualizable object with dual V'. This implies that V is in fact finite-dimensional and that $V' = V^*$.

We can summarize this by saying that we have a natural evaluation map: $V \otimes V^* \to \mathbb{C}$, and we have a natural identity map $\mathbb{C} \to \mathrm{End}(V)$. And for V finite-dimensional we have $V \otimes V^* \simeq \mathrm{End}(V)$ so we have:

$$\begin{tikzcd} & V \otimes V^* \arrow[r, "\mathrm{ev}"] \arrow[d, equal] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\sum e_i \otimes e^i"] \arrow[r, "\mathrm{id}"] \arrow[urr, bend right, "\mathrm{Tr}(\mathrm{id})=\dim(V)"'] & \mathrm{End}(V) \arrow[ur, "\mathrm{Tr}"] & \end{tikzcd}\ . \tag{10.12}$$

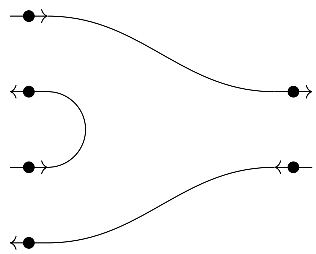
10.2 Operators

We have seen that $\mathcal{Z}$ assigns a finite-dimensional vector space V to a positively oriented point, and the dual is assigned to the negatively oriented point. This vector space V is the space of states. Now we would like to find the operators in this picture. First of all, since V is finite-dimensional, we have that $V \otimes V^* = \mathrm{End}\,(V)$, which consists of all operators on V. If you like, this is what $\mathcal{Z}$ attaches to the following object of $\mathbf{Bord}_1^{\mathrm{or}}$:

(10.13)

Remark 50. For now we will think of this bordism as the LHS of eq. (10.13). We will use the RHS interpretation below in eq. (10.20).

The algebra structure comes from the following bordism:

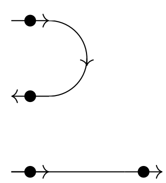


, (10.14)

since A sends it to a multiplication map:

$$\mathrm{End}\,(V) \otimes \mathrm{End}\,(V) \to \mathrm{End}\,(V)\ . \tag{10.15}$$

The action of $\mathrm{End}\,(V)$ on V comes from the following bordism:



, (10.16)

since $\mathcal{Z}$ sends it to a map:

$$\mathrm{End}\,(V) \otimes V \to V\ . \tag{10.17}$$

10.3 Interfaces

We can phrase both operators and states using the language of interfaces. Suppose V and W are two finite-dimensional vector spaces. Then we can form TFTs $\mathcal{Z}_V$ and $\mathcal{Z}_W$ where $\mathcal{Z}_V$ sends the positive point to V, and $\mathcal{Z}_W$ sends the positive point to W. Now we will define the notion of an *interface/morphism* between $\mathcal{Z}_V$ and $\mathcal{Z}_W$.

An interface is encoded in the notion of a *bipartite field theory*, which is a functor $\mathbf{Bord}_1^{\mathrm{or,\ bi}} \to \mathbf{Vect}$ satisfying some properties outlined below. The category $\mathbf{Bord}_1^{\mathrm{or,bi}}$ is the same as $\mathbf{Bord}_1^{\mathrm{or}}$, except the objects and morphisms are *bipartite* manifolds. This means the manifolds are decomposed into two parts, which we will call "blue" and "red" parts. For example:

(10.18)

is a bipartite 1-manifold. So some bipartite manifolds are all blue, some are all red, and some have some of each. Then one thing we require of a *bipartite field theory* $\mathcal{Z}$, is that on an entirely red manifold R, $\mathcal{Z}(R) = \mathcal{Z}_V(R)$, and on an entirely blue manifold B: $\mathcal{Z}(B) = \mathcal{Z}_W(B)$. Then we have the manifolds which are decomposed as a blue part and a red part. The example above gives a bordism from a blue interval to a red interval, so it gives a map $\mathcal{O}\colon W \to V$. So this bipartite manifold is a new kind of bordism we added to our category.

Now one can prove that such bipartite 1-dimensional field theories correspond to pairs of vector spaces V and W, equipped with a map:

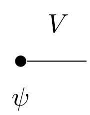
$$\mathcal{O}\colon W \to V \ . \tag{10.19}$$

The upshot is that we get a category of field theories (where the morphisms are interfaces) which is just be equivalent to the category of vector spaces. We want to think that V and W meet along an interface where quantum mechanics for V becomes quantum mechanics for W.

Example 50. If $V = W$ then the self-interfaces are labelled by operators $\mathcal{O} \in \mathrm{End}(V)$. The idea is that where these two lines meet, it looks like the RHS of (10.13):

$$\longrightarrow\!\!\!\!\bullet\!\!\!\longrightarrow \quad \longrightarrow\!\!\!\!\bullet\!\!\!\longrightarrow \ . \tag{10.20}$$

Likewise, a state $\psi \in V$ is the same as a map $\mathbb{C} \to V$ which sends $1 \mapsto \psi \in V$. So we can write this as an interface from the "trivial" theory $\mathcal{Z}_{\mathbb{C}}$ to $\mathcal{Z}_V$. We won't draw the trivial theory, so we can think of a state as a new kind of boundary condition:



(10.21)

where one can imagine that the trivial theory lives to the left of ψ.

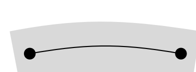
Lecture 20;
April 8, 2021

10.4 Two-dimensional TFT

We will discuss two-dimensional TFT, with an eye towards studying operators and defects. In particular we would like to see the inherent structure of the TFT and how it's reflected in the representation theory.

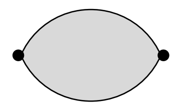
10.4.1 The basics

Let $\mathbf{Bord}_2^{\mathrm{or}}$ be the oriented bordism 2-category. The objects can be thought of as 0-manifolds with a 2-dimensional tangent space. The 1-morphisms are 1-manifold bordisms between 0-manifolds, where we're still thinking of this as having a 2-dimensional tangent space. This tangent space we're equipping the bordisms with is a way to keep track of the dimension, but also a way to prescribe orientations, framings, and other tangential structures. For example, we can picture a 1-morphism as:



. (10.22)

The 2-morphisms are 2-manifolds with corners. For example, here is a 2-morphism from the above 1-morphism to itself:



$$. \tag{10.23}$$

We saw this same bordism a while ago in fig. 6.3.

The important thing about this 2-category is that it's symmetric monoidal under the operation of disjoint union, and with unit given by the empty set $\emptyset$.

Remark 51. Bordisms come in families/classifying spaces. For example, a closed surface of genus g can be thought of as a bordism from the empty 1-manifold $\emptyset_1$ to itself. These form a family, given by $B\,\mathrm{Diff}\,(\Sigma^g)$, which is identified with the moduli space of curves of genus g, $\mathcal{M}_g$.

A *two-dimensional TFT* is a symmetric monoidal functor:

$$\mathcal{Z}\colon\ (\mathbf{Bord}_2^{\mathrm{or}}, \sqcup, \emptyset) \to (\mathcal{C}, \otimes, 1)\ . \tag{10.24}$$

This is the definition, but we already know that a 2-dimensional TFT should assign numbers to surfaces: $\mathcal{Z}\,(\Sigma) \in \mathbb{C}$, and a vector space to a 1-manifold $\mathcal{Z}\,(N) \in \mathbf{Vect}_{\mathbb{C}}$. Another way to say this is as follows. If we truncate to $\mathbf{Bord}^{\mathrm{or}}_{[1,2]}$, so we don't include corners, then we should really land in:

$$\mathbf{Bord}^{\mathrm{or}}_{[1,2]} \to \mathbf{Vect}_{\mathbb{C}}\ . \tag{10.25}$$

But another way to describe this truncated bordism category is endomorphisms of the empty 0-manifold:

$$\mathrm{End}\,(\emptyset_0)\ , \tag{10.26}$$

i.e. if we want numbers at the top level, and vector spaces one level down, then we should have that

$$\mathbf{Vect}_{\mathbb{C}} = \mathrm{End}\,(1_{\mathcal{C}})\ . \tag{10.27}$$

Now the additional data, beyond this truncated functor, that we would need to determine $\mathcal{Z}$ is just an object $\mathcal{Z}\,(\mathrm{pt}) \in \mathcal{C}$. But there is something canonical we can do to make this more concrete. Given any object $C \in \mathcal{C}$, we can attach a $\mathbb{C}$-linear category:

$$\mathrm{Hom}\,(1_{\mathcal{C}}, C)\ . \tag{10.28}$$

We're in a 2-category, so this does indeed form a category, and it is $\mathbb{C}$-linear because we can precompose with endomorphisms of the unit. So if you have a functor $\mathcal{Z}\colon \mathbf{Bord}_2 \to \mathcal{C}$, then this lifts to:

$$\begin{tikzcd} \mathbf{Bord}_2 \arrow[r] \arrow[dr] & \mathcal{C} \arrow[d] \\ & \mathbf{Cat}_{\mathbb{C}} \end{tikzcd}\ . \tag{10.29}$$

In practice $\mathcal{C} \to \mathbf{Cat}_{\mathbb{C}}$ will be either an equivalence or a full embedding.

Example 51. Let $\mathcal{C}$ be the Morita 2-category of associative algebras. I.e. the objects are associative algebras, the 1-morphisms between A and A' are bimodules ${}_AB_{A'}$, and the 2-morphisms are maps of bimodules. This sits as a full-subcategory of $\mathbb{C}$-linear categories given by sending an associate algebra to its category of modules.

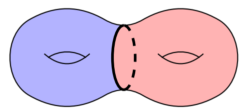


Figure 10.1: A closed bipartite 2-manifold.

10.4.2 Interfaces and higher categories

Now we want to interpret this operation in TFT language, rather than category theory language. Recall we have a notion of an *interface* between TFTs, which is a kind of "map" between topological field theories. To do this, we use bipartite 2-manifolds. Let $\mathcal{Z}$ and $\mathcal{W}$ be two field theories both valued in the same category:

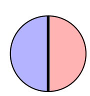
$$\mathcal{Z}, \mathcal{W} \colon \mathbf{Bord}_2 \to \mathcal{C} \ . \tag{10.30}$$

A morphism form $\mathcal{Z}$ to $\mathcal{W}$ is a functor on bipartite $\mathbf{Bord}_2^{\mathrm{bi}}$. We won't give a formal definition, but a closed bipartite 2-manifold is effectively a just a closed 2-manifold with a "red part" and "blue part" as in fig. 10.1. The idea is that this theory will be given by applying $\mathcal{Z}$ to the red manifolds, $\mathcal{W}$ to the blue manifolds, and then there is extra information coming from what these decomposed manifolds get assigned to. We can keep track of whether it is a morphism $\mathcal{Z} \to \mathcal{W}$ or $\mathcal{W} \to \mathcal{Z}$ by keeping track of orientations.

Using some version of the cobordism hypothesis with singularities, these interfaces correspond to maps:

$$\mathcal{Z}\,(\mathrm{pt}) \to \mathcal{W}\,(\mathrm{pt}) \ . \tag{10.31}$$

This morphism comes from the bipartite bordism:

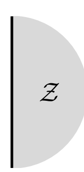


=

, (10.32)

so the whole structure is determined by $\mathcal{Z}$, $\mathcal{W}$, and what the theory assigns to this bordism.

Suppose $C = \mathcal{Z}\,(\mathrm{pt})$. Then $\mathrm{Hom}\,(1, C)$ corresponds to interfaces between the trivial theory associated to 1 and the theory $\mathcal{Z}$. We can picture these as interfaces between $\mathcal{Z}$ and nothing:



. (10.33)

This is what would be called a *boundary condition* or *boundary theory* for $\mathcal{Z}$.

By an "object of C" (where $C \in \mathcal{C}$) we mean, by definition, objects of the underlying category $\mathrm{Hom}\,(1, C)$. These are identified with morphisms $1 \to \mathcal{Z}_C$, which are identified with boundary theories for $\mathcal{Z}_C$. A boundary theory for $\mathcal{Z}_C$ really means we're extending the functor $\mathcal{Z}$ to 2-manifolds with a marked boundary component. I.e. I look at 2-manifolds, but we have a boundary component labelled by $\mathcal{B}$ and the interior of the manifold is labelled by $\mathcal{Z}$.

Remark 52. This is a specific case of a general construction which we will see in more detail. Namely, a *defect theory of codimension k* for a TFT $\mathcal{Z}$ means we extend $\mathcal{Z}$ from n-manifolds

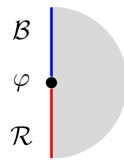


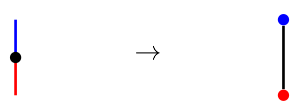
Figure 10.2: An interface between two interfaces. This is a bordism with corners which corresponds to a map $\varphi\colon \mathcal{B} \to \mathcal{R}$ of boundary conditions $\mathcal{B}, \mathcal{R} \in \operatorname{Hom}(1, \mathcal{Z}(\mathrm{pt}))$.

to n-manifolds with embedded marked codimension k submanifolds. Call such an extension $\mathcal{D}_k$, and then we can label these submanifolds with the letter $\mathcal{D}$.

If we're in codimension 1, then the submanifold divides the manifold in two parts, i.e. it's a bipartite manifold, and then we can label either part with a different theory. Or we could label them both with $\mathcal{Z}$, for which we get $\operatorname{End}(Z)$. If we label one side with the trivial theory, then we get boundary conditions. Really what is happening here is the 0-sphere is disconnected, so the link of a codimension 1 submanifold is disconnected so there is some choice of what to do on each component.

The point is that boundary conditions of a 2-dimensional TFT $\mathcal{Z}$ form a category, even though $\mathcal{Z}(\mathrm{pt})$ might not be a category itself. Any $\mathcal{B} \in \operatorname{Hom}(1, \mathcal{Z}(\mathrm{pt}))$ corresponds to an interface between $\mathcal{Z}$ and the trivial theory. Being an object of Hom in a 2-category means that they form a category by fiat. But the physical reason these are objects of a category, is that there are interfaces between interfaces as in fig. 10.2. This corresponds to a map $\mathcal{B} \to \mathcal{R}$ of boundary conditions $\mathcal{B}, \mathcal{R} \in \operatorname{Hom}(1, \mathcal{Z}(\mathrm{pt}))$.

Figure 10.2 is a new bordism with corners from:

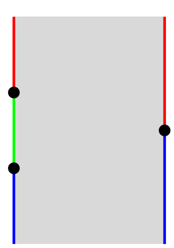


(10.34)

This is a perfectly good 1-manifolds with marked boundaries, so the theory assigns them to some vector space. So the collection of interfaces from $\mathcal{B}$ to $\mathcal{R}$ form a vector space. This vector space is already part of the theory. In particular, it is the assignment:

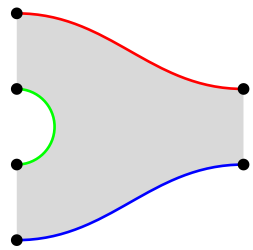
$\mathcal{Z}$ () . (10.35)

We can think of this interval as being the *link* of the singularity. These vector spaces are $\operatorname{Hom}(B, R)$. The bordism:



(10.36)

or equivalently the "pair of chaps" bordism:



$$(10.37)$$

both induce the composition map:

$$\mathrm{Hom}\left(\mathcal{B},\mathcal{G}\right)\otimes\mathrm{Hom}\left(\mathcal{G},\mathcal{R}\right)\rightarrow\mathrm{Hom}\left(\mathcal{B},\mathcal{R}\right)\ . \tag{10.38}$$

Therefore these vector spaces compose just like morphisms in a category.

The same idea works in any dimension. For an n-dimensional TFT, we can build an $(n-1)$-category of boundary theories, where the objects are boundary conditions, 1-morphisms are interfaces between the boundary conditions, 2-morphisms are interfaces between interfaces, and this continues up to level $(n-1)$ because the theory was n-dimensional.

10.4.3 Two-dimensional Yang-Mills for a finite group

This is also known as untwisted Dijkgraaf-Witten theory. Let G be a finite group. Then we will define a 2-dimensional TFT:

$$\mathcal{Z}_G\colon \mathbf{Bord}_2^{\mathrm{or}}\rightarrow\mathbf{Cat}_{\mathbb{C}}\ , \tag{10.39}$$

by linearizing spaces of G-local systems on manifolds.

This will be a Lagrangian field theory, by which we mean the following. We have this bordism category $\mathbf{Bord}_2^{\mathrm{or}}$, and then out of this we have a construction of fields, which lands in something geometric like groupoids, and then we linearize, e.g. by taking sheaves, to land in $\mathbf{Cat}_{\mathbb{C}}$:

$$\begin{tikzcd} \mathbf{Bord} \arrow[r, "\mathcal{F}"] \arrow[rr, bend right, "\mathcal{Z}=\mathbf{Shv}(\mathcal{F}(-))"'] & \mathbf{Grpd} \arrow[r, "\mathbf{Shv}"] & \mathbf{Cat}_{\mathbb{C}} \end{tikzcd}\ . \tag{10.40}$$

If Σ is a closed 2-manifold, then $\mathcal{Z}_G\left(\Sigma\right)$ is the number of G-local systems on Σ. Recall G-local systems comprise:

$$\mathrm{Loc}_G\,\Sigma=\left\{\pi_1\Sigma\rightarrow G\right\}/\sim\ , \tag{10.41}$$

but when we count these we weight them by automorphisms:

$$\mathcal{Z}_G\left(\Sigma\right)=\sum_{P\in\mathrm{Loc}_G\,\Sigma}\frac{1}{\left|\mathrm{Aut}\left(P\right)\right|}\ . \tag{10.42}$$

If N^1 is a 1-manifold we assign:

$$\mathcal{Z}_G\left(N^1\right)=\mathbb{C}\left[\mathrm{Loc}_G\,N\right]\ , \tag{10.43}$$

where we are just consider $\mathrm{Loc}_G\,N$ as a finite groupoid. If P^0 is a 0-manifold, then

$$\mathcal{Z}_G\left(P^0\right)=\mathbf{Vect}\left(\mathrm{Loc}_G\,P^0\right)\ . \tag{10.44}$$

For example,

$$\mathcal{Z}_G\left(S^1\right) = \mathbb{C}\left[\mathrm{Loc}_G S^1\right] = \mathbb{C}\left[\frac{G}{G}\right] , \tag{10.45}$$

which are *class functions*. For the zero-manifold pt we have:

$$\begin{aligned} \mathcal{Z}_G(\mathrm{pt}) &= \mathbf{Vect}\left(\mathrm{Loc}_G(\mathrm{pt})\right) && (10.46) \\ &= \mathbf{Vect}\left(\mathrm{pt}/G\right) && (10.47) \\ &= \mathbf{Rep}\left(G\right) . && (10.48) \end{aligned}$$

Let M be a bordism from N_{in} to N_{out}. Taking Loc_G gives a correspondence/span of groupoids:

$$\begin{tikzcd} & \mathrm{Loc}_G M \arrow[dl] \arrow[dr] & \\ \mathrm{Loc}_G N_{\mathrm{in}} & & \mathrm{Loc}_G N_{\mathrm{out}} \end{tikzcd} . \tag{10.49}$$

Taking functions on these groupoids gives:

$$\pi_{\mathrm{out}*}\pi_{\mathrm{in}}^* \colon \mathbb{C}\left[\mathrm{Loc}_G N_{\mathrm{in}}\right] \to \mathbb{C}\left[\mathrm{Loc}_G N_{\mathrm{out}}\right] . \tag{10.50}$$

So this is how bordisms between manifolds become maps between the associated spaces of functions.

Question 4. How do you see *representations* of G?

Abstractly we have $\mathcal{Z}_G(\mathrm{pt}) = \mathbf{Rep}(G)$. So a representation $V \in \mathbf{Rep}(G)$ is equivalent to an object of $\mathcal{Z}_G(\mathrm{pt})$, which is equivalent to a functor $\mathbf{Vect} \to \mathbf{Rep}(G)$ which sends $\mathbb{C}$ to V. I.e. this is an interface from the trivial theory to $\mathcal{Z}_G$, i.e. a boundary condition for $\mathcal{Z}_G$.

But what does this have to do with colored boundaries? Given V a representation of G, which we're thinking of as "red", we can extend $\mathcal{Z}_G$ to manifolds with red boundary components. For example, we have a new kind of 1-bordism:

(10.51)

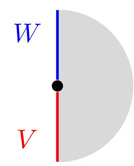
which corresponds to the object $V \in \mathcal{Z}_G(\mathrm{pt})$.

Similarly a cylinder with one boundary component marked red gives an element of $\mathcal{Z}_G\left(S^1\right)$. This element turns out to be the *character* of V: $\chi_V \in \mathbb{C}\left[\frac{G}{G}\right]$.

The physics interpretation is as follows. $V \in \mathbf{Rep}(G)$ is certainly a vector space, so we can think of it as the space of states defining a theory of topological quantum mechanics. But making it into a representation means we have *coupled* the 1-dimensional theory to the *bulk gauge theory* $\mathcal{Z}_G$. So we're considering a theory of quantum mechanics which lives on 1-manifolds which are the boundary of some bulk, where we have the theory $\mathcal{Z}_G$.

Example 52. Recall the regular representation $Reg = \mathbb{C}[G]$. This corresponds to the Dirichlet boundary condition for $\mathcal{Z}_G$. The trivial representation $\mathbb{C}$ corresponds to the Neumann boundary condition.

The main thing we can do with these boundary conditions is talk about interfaces between them. So if I have a representation V, and a blue representation W, then an interface:



(10.52)

is an element of the vector space that $\mathcal{Z}_G$ assigns to the link of this singularity:

$$\mathcal{Z}\left(\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\right) = \mathrm{Hom}_{\mathbf{Rep}(G)}\left(V, W\right) \ . \tag{10.53}$$

And the composition is realized by this collision of defects, i.e. the pair of chaps bordism.

Now we will write the category of boundary conditions in terms of the fields. Sets with an action of G comprise an easy source of representations of G, since $G \circlearrowright X$ implies $\mathbb{C}[X]$ is a G-representation. So we can think of a boundary condition as being specified by $G \circlearrowright X$, where we're secretly associating it with the representation $\mathbb{C}[X]$. In the bulk we have Loc_G, so our total space of fields consists of G-local systems equipped with a section of the associated X-bundle on the boundary. So we're thinking of mapping our whole surface to pt/G, which is why we get Loc_G, and then on the boundary labelled by X, we're lifting this map to X/G:

$$\begin{tikzcd} & X/G \arrow[d] \\ \partial\Sigma \arrow[r] \arrow[ur] & \mathrm{pt}/G \end{tikzcd} \ . \tag{10.54}$$

Example 53. If $X = G$ (so we're considering the regular representation) then $X/G = \mathrm{pt}$ and we're consider G-local systems plus a trivialization on the boundary, which is the Dirichlet boundary condition.

transition

Lecture 21;
April 13, 2021

10.5 Boundary conditions in Lagrangian theories

The basic picture is that we're thinking of a theory of maps from some spacetime Σ to some target Y. I.e. the fields are maps, and we're linearizing them. We might allow Σ to have a boundary, and then we can put conditions on this boundary. E.g. if we choose some submanifold $Z \subset Y$ then we might insist that the maps $\psi \colon \Sigma \to Y$ send $\partial\Sigma$ to Z:

$$\varphi|_{\partial\sigma} \subset Z \ . \tag{10.55}$$

Remark 53. In order to formulate this problem we don't need Z to be a submanifold. We can just take some map $Z \to Y$, and insist that the map to Y lifts to Z on the boundary:

$$\begin{tikzcd} \partial\Sigma \arrow[r] \arrow[d, hook] & Z \arrow[d] \\ \Sigma \arrow[r, "\varphi"] & Y \end{tikzcd} \ . \tag{10.56}$$

So now we have $\varphi \colon \Sigma \to Y$ as well as the data of a lift of $\varphi|_{\partial Sigma}$ to

$$\widetilde{\varphi_\partial} \colon \partial\Sigma \to Z \ . \tag{10.57}$$

The point is that we aren't really imposing a condition, but rather adding the extra structure of a map. As such, boundary *theory* might be a better name than boundary *condition*.

Recall we were studying Yang-Mills for a finite group. This is the theory with target $Y = \mathrm{pt}/G$. Then a map $\varphi\colon \Sigma \to \mathrm{pt}/G$ gives a G-bundle on Σ. For G finite this means we just get $\mathrm{Loc}_G\,\Sigma$.

Giving a map $Z \to Y = \mathrm{pt}/G$ is equivalent to space $G \circlearrowright X$. Explicitly, for a map:

$$\begin{array}{c} Z \\ \downarrow \\ Y = \mathrm{pt}/G \end{array}\ , \tag{10.58}$$

we can form a G-space $G \circlearrowright X$ as the fiber product:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathrm{pt} \arrow[r] & Y = \mathrm{pt}/G \end{tikzcd} . \tag{10.59}$$

Conversely, starting with a G-space $G \circlearrowright X$ we can form:

$$\begin{array}{c} Z = X/G \\ \downarrow \\ Y = \mathrm{pt}/G \end{array} . \tag{10.60}$$

Therefore a G-space $G \circlearrowright X$ gives rise to a boundary theory. Before we had

$$\mathrm{Loc}_G\left(\Sigma\right) = \left\{\Sigma \to \mathrm{pt}/G\right\}\ . \tag{10.61}$$

When Σ has boundary, then we can ask to lift to:

$$\begin{tikzcd} \partial\Sigma \arrow[r, "\widetilde{\varphi_\partial}"] \arrow[d, hook] & X/G \arrow[d] \\ \Sigma \arrow[r, "\varphi"] & \mathrm{pt}/G \end{tikzcd} . \tag{10.62}$$

These comprise:

$$\mathrm{Loc}_G^X\left(\Sigma, \partial\Sigma\right)\ . \tag{10.63}$$

Concretely φ corresponds to a G-bundle:

$$\begin{tikzcd} \mathcal{P} \arrow[d, "G"] \\ \Sigma \end{tikzcd} , \tag{10.64}$$

and the data of a lift $\widetilde{\varphi_\partial}$ is equivalent to a section of the associated X-bundle:

$$\begin{tikzcd} P|_{\partial\Sigma} \times^G X \arrow[d] \\ \partial\Sigma \end{tikzcd} . \tag{10.65}$$

The fibers look like X and the transition data is given by the action $G \circlearrowright X$.

Exercise 1. Show that the data of a lift to X/G is equivalent to a section of this associated. X-bundle.

This data is also equivalent to a G-equivariant map $\mathcal{P}|_{\partial\Sigma} \to X$.

Remark 54. In physics words we are "coupling a Σ-model to X on the boundary with bulk G-gauge theory".

I.e. on the boundary we have a theory of one dimension lower of maps into X, and in the interior we have a theory of G-bundles. Then we can put them together to make fields for this coupled theory.

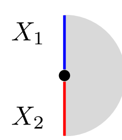
Example 54. If $X = G/H$, then a section of the associated X-bundle $\mathcal{P} \times^G X = \mathcal{P} \times^G G/H$ is the same as a reduction of $\mathcal{P}$ from G to H. I.e. this is the same as writing $\mathcal{P}$ as being induced from an H-bundle.

For example if $X = G = G/1$ then a section of the associated X-bundle is the same thing as a trivialization of $\mathcal{P}|_{\partial\Sigma}$. This is the gauge theory version of the Dirichlet boundary condition. Recall in our two-dimensional TFT $\mathcal{Z}_G$ (the Yang-Mills theory) the boundary conditions correspond to $\mathbf{Rep}\,(G)$. Explicitly a G-space gives rise to the representation $\mathbb{C}\,[X]$. The boundary condition associated to $X = G/1$ corresponds to the regular representation $\mathbb{C}\,[G] \in \mathbf{Rep}\,(G)$.

The other extreme is when $X = \mathrm{pt} = G/G$. This is what is called the Neumann condition. This corresponds to the trivial representation $\mathbb{C} \in \mathbf{Rep}\,(G)$.

10.5.1 Interfaces between boundary conditions

Now we will consider interfaces between boundary conditions, which will correspond to maps between representations once we have linearized. So consider two boundary conditions given by representations V and W. Assume V came from a G-space X_1, and W as coming from a G-space X_2. Then consider an interface between them:

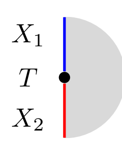


. (10.66)

What kind of thing does the interface get labelled by? We have seen that this corresponds to a map of representations, but in terms of the G-spaces X_i we form the fiber product:

$$\begin{array}{ccc} T & \longrightarrow & X_2/G \\ \big\downarrow & & \big\downarrow \\ X_1/G & \longrightarrow & \mathrm{pt}/G \end{array} , \tag{10.67}$$

and the interface is labelled by this:



. (10.68)

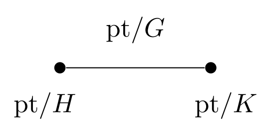
This is the natural thing to put here: if we have a lift to X_1 on one part of the boundary, and a lift to X_2 on another part of the boundary, then on the overlap we should lift to something which maps to T.

Example 55. If $X_1 = G/H$ and $X_2 = G/K$, then $X/G = \mathrm{pt}/H$ and $X_2/G = \mathrm{pt}/K$, so the fiber product is:

$$\begin{tikzcd} H\backslash G/K \arrow[r] \arrow[d] & \mathrm{pt}/K \arrow[d] \\ \mathrm{pt}/H \arrow[r] & \mathrm{pt}/G \end{tikzcd} . \tag{10.69}$$

I.e. if we have a G-bundle on Σ with a reduction from the structure group from G to H on one part of the boundary, and a reduction from G to K on another part of the boundary, then on the overlap we're reducing to both H and K, and these are given by maps to $H\backslash G/K$. I.e. geometric./Lagrangian interfaces between pt/H and pt/K are given by $H\backslash G/K$. Self interfaces are given by $K\backslash G/K$.

Pictorially we have that:

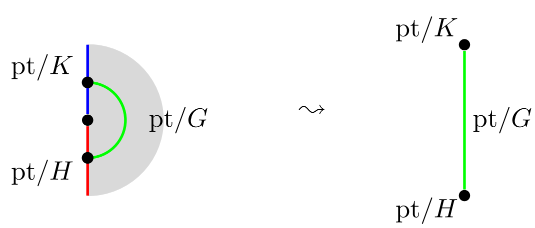
$H\backslash G/K =$ 

. (10.70)

I.e. we have a G-bundle on the interior which is reduced to an H-bundle on one endpoint and to a K-bundle on the other endpoint. The space of all such bundles is $H\backslash G/K$.

Question 5. How does this picture relate to the pictures we had before?

We saw this a bit earlier, but the basic idea is that they're related by passing from a singularity to its *link*. This is the boundary of a small tubular neighborhood of the singularity. So in this example we're studying the interface between pt/H and pt/K and the link is the green interval:

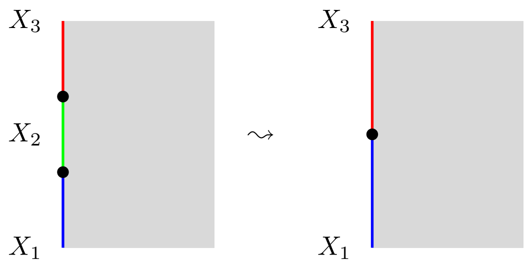


. (10.71)

10.5.2 Composition of interfaces between boundary conditions

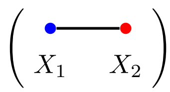
Now we want to understand the composition of interfaces. This is how the Hecke algebra acquires an algebra structure from the physics.

The physical picture (known as the operator product expansion (OPE)) is that we have two interfaces, but looking far away it looks like two interfaces:



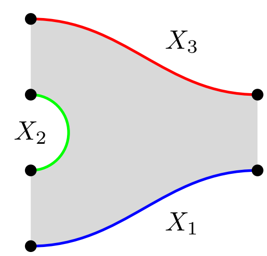
(10.72)

Formally we can take the link of the singularity as in the discussion surrounding eq. (10.71) to get:

$\mathcal{Z}$ 

$= \mathrm{Hom}\left(X_1, X_2\right)$. (10.73)

Then the "pair of chaps" bordism:

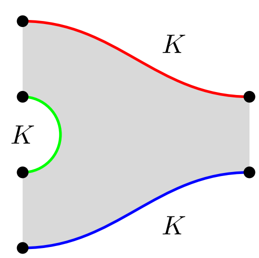


(10.74)

induces the composition map:

$$\mathrm{Hom}\left(X_1, X_2\right) \otimes \mathrm{Hom}\left(X_2, X_3\right) \to \mathrm{Hom}\left(X_1, X_3\right) \ . \tag{10.75}$$

The *Hecke algebra* occurs when we're looking at G-bundles with a reduction to K on all three boundaries:



. (10.76)

This picture gives the multiplication map:

$$\mathbb{C}\left[K \backslash G / K\right] \otimes \mathbb{C}\left[K \backslash G / K\right] \to \mathbb{C}\left[K \backslash G / K\right] \ . \tag{10.77}$$

So that is the physics reinterpretation of Hecke algebras.

Example 56. If $K = 1$ (Dirichlet boundary condition), then $K \backslash G / K = G$, and this picture is recovering the group algebra $\mathbb{C}G$.

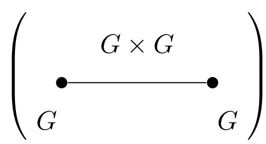
Example 57. If $K = G$ (Neumann boundary condition) then the space of fields is pt/G. But $\mathbb{C}\left[\mathrm{pt}/G\right] = \mathbb{C}$, and the Hecke algebra is $\mathrm{End}_{\mathbf{Rep}}\left(\mathbb{C}\right) = \mathbb{C}$.

For V a vector space, $\mathrm{End}(V) \circlearrowright V$. If $V \in \mathbf{Rep}(G)$ then we get a possible more interesting structure $\mathrm{End}_G(V) \circlearrowright V$. For example if $V = \mathbb{C}[G/K]$ then there is an action of $\mathcal{H}_{G,K} = \mathrm{End}_G V$. These algebras are not commutative, so they don't fit into this paradigm of spectral decomposition, i.e. there is no Fourier theory for us to do. So we were looking for commutative symmetries. Then we observed that, even though Hecke algebras are not commutative in general, sometimes they are.

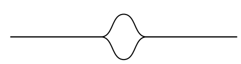
The one such example of this that we have seen is $G \subset G \times G$. Here we have the Hecke algebra

$$\mathcal{H}_{G\times G,G} = \mathrm{End}_{G\times G}\,\mathbb{C}[G] \tag{10.78}$$
$$= \mathbb{C}[G\backslash (G\times G)/G] \tag{10.79}$$
$$= \mathbb{C}\left[\frac{G}{G}\right] . \tag{10.80}$$

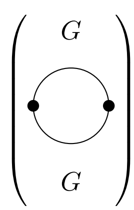
We can write this as:

$\mathcal{H}_{G\times G,G} = \mathcal{Z}_{G\times G}$  , (10.81)

i.e. we're looking at $G \times G$-bundles on the interval with reductions to G on the endpoints. This is the same as two G-bundles on the interval along with an identification at the boundary. More poetically we're studying G-bundles on the *ravioli*:

$I \sqcup_{I\backslash \mathrm{pt}} I \quad =$  . (10.82)

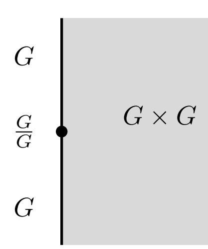
Since we're just doing topology, we can give it a little more air and consider this as G-local systems on the circle:

Loc_G  . (10.83)

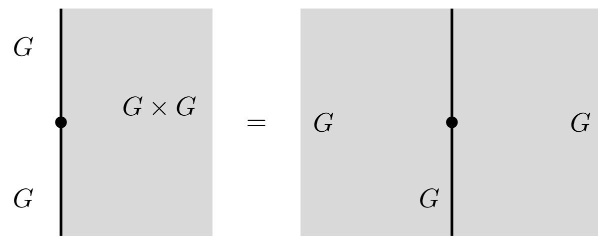
In other words, we're looking at the G-theory on the circle, rather than the $G \times G$-theory on the interval with these particular boundary conditions.

Remark 55. This trick doesn't have an analogue for a general Hecke algebra.

Another way to write this is as follows. We're consider the Yang-Mills theory $\mathcal{Z}_{G\times G}$, and we considering self-interfaces of the diagonal boundary condition:

 . (10.84)

Now we can imagine splitting the bulk $G\times G$ theory into two copies, each of which is labelled by the G theory:



(10.85)

This is saying that a boundary condition for the $G\times G$-theory is the same as a self-interface for the G-theory. We can upgrade this to apply to products of groups: a boundary condition for $G \times H$ is the same as an interface between the G-theory and the H-theory.

Algebraically (with enough duality) what we're doing here is:

$$\mathrm{Hom}\,(V, W) = V^* \otimes W\ , \tag{10.86}$$

where the interfaces are in the LHS, and the boundary condition for the product sits on the RHS. The duality we're using is the self-duality of $\mathbb{C}\,[G]$:

$$\begin{aligned} \mathbb{C}\,[G \times G]\,\textbf{-mod} &= \mathbb{C}\,[G] \otimes \mathbb{C}\,[G]\,\textbf{-mod} && (10.87)\\ &= \mathbb{C}\,[G]^{\mathrm{op}} \otimes \mathbb{C}\,[G]\,\textbf{-mod} && (10.88)\\ &= \mathbb{C}\,[G]\,\textbf{-mod-}\ . && (10.89) \end{aligned}$$

Remark 56. This is a general fact about field theory. Once you understand boundary conditions you understand interfaces and vice versa: interfaces between field theories are the same as a boundary condition for the product.

Question 6. Why is $\mathcal{H} = \mathcal{H}_{G\times G,G} = \mathbb{C}\left[\frac{G}{G}\right]$ commutative?

The reason is that $\mathcal{H}$ consists of trivial self-interfaces of $\mathcal{Z}_G$, which comprise $\mathrm{End}\left(\mathrm{id}_{\mathbf{Rep}(G)}\right)$. This is the *Bernstein center* of $\mathbf{Rep}\,(G)$. For $\mathcal{C}$ any category, the *Bernstein center* is

$$\mathcal{Z}\,(\mathcal{C}) \coloneqq \mathrm{End}\,(\mathrm{id}_{\mathcal{C}})\ . \tag{10.90}$$

The claim is that this is always commutative because there are two "directions" in which I can compose these operators. I.e. $\mathcal{Z}\,(\mathcal{C}) = \mathrm{End}\,(\mathrm{id})$ has two compatible monoid structures. On one hand, we can collide two interfaces along the identity interface. On the other hand, we can collide two copies of the identity interface itself to get:

$$\mathrm{id} \circ \mathrm{id} = \mathrm{id}\ . \tag{10.91}$$

This second composition does not have an analogue for general Hecke algebras.

Exercise 2. A set with two compatible monoid structures is a commutative monoid.

This is a toy version of the Eckmann-Hilton argument in topology, which shows that $\pi_2\,(X)$ is commutative.

Example 58. If $\mathcal{C} = A\text{-}\mathbf{mod}$, then

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &= \operatorname{End}(\mathrm{id}_{A\text{-}\mathbf{mod}}) && (10.92)\\ &= \operatorname{End}({}_A A_A)\ , && (10.93) \end{aligned}$$

since the following functors are equivalent:

$$(-) \otimes {}_A A_A = \mathrm{id}_{A\text{-}\mathbf{mod}}\ . \tag{10.94}$$

Again we are "unfolding" the bulk $A \otimes A^{\mathrm{op}}$ theory with A on the boundary to get a bulk A-theory with a trivial interface. But this is just the center of the algebra:

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &= \operatorname{End}({}_A A_A) && (10.95)\\ &= \mathcal{Z}(A)\ . && (10.96) \end{aligned}$$

So for any category $\mathcal{C}$, we have a commutative algebra $\mathcal{Z}(\mathcal{C})$. Then we can consider a space:

$$\mathcal{M}_{\mathcal{C}} \coloneqq \operatorname{Spec} \mathcal{Z}(\mathcal{C})\ . \tag{10.97}$$

Then we should try to understand $\mathcal{C}$ using spectral decomposition over $\mathcal{M}_{\mathcal{C}}$.

Example 59. If $\mathcal{C} = A\text{-}\mathbf{mod}$, then we're doing spectral decomposition over $\operatorname{Spec} \mathcal{Z}(A)$.

This may or not be interesting, since the center might be trivial. For example, in quantum mechanics algebras of observables are things like the Weyl algebra, which has trivial center. So this gives us nothing. But the claim is that higher-dimensional QFT has inherent commutativity. The idea is as follows. We were considering the theory $\mathcal{Z}_G$ on the bulk, with a trivial interface. This is equivalent to having a 0-dimensional singularity for the theory $\mathcal{Z}_G$. This is a *local operator*. So the punchline is that this commutativity really came from local operators rather than boundary conditions.

10.5.3 Local operators in physics

A local operator is some kind of observable localized near a point in spacetime. Formally this is something like the following. The idea is that, around a point in spacetime, we consider a sphere S^{n-1}_ϵ of radius ϵ. Then we consider the states on this sphere as $\epsilon \to 0$, and this consists of local operators. So this is something like functionals on fields in a small punctured neighborhood of the point. Note we are allowing for singularities at this point.

A physicist would tell you that, given a local operator $\mathcal{O}_x$ at a point x, we can take the expectation value:

$$\langle \mathcal{O}_x \rangle = \int_{\text{Fields}} \mathcal{O}_c(\varphi)\, e^{-S(\varphi)} D\varphi \tag{10.98}$$

where we are integrating over fields φ on $M \setminus x$. More generally we have a correlation function between operators at different points:

$$\left\langle \mathcal{O}_x \mathcal{O}'_y \mathcal{O}''_z \right\rangle = \int \mathcal{O}_x(\varphi)\, \mathcal{O}_y(\varphi)\, \mathcal{O}''_z(\varphi)\, e^{-S(\varphi)} D\varphi\ . \tag{10.99}$$

The other thing that physicists will tell you, is that there is an operator product expansion (OPE), where we try to take a limit where these points collide. So if we look from far away, the points look like they are close to one another and sit inside the same small sphere, so they look like a new composite operator.

In (conformal and) TFT, we have a state-operator correspondence:

$$\mathcal{Z}\left(S^{n-1}\right) = \text{space of local operators. ,} \tag{10.100}$$

where we don't have to worry about the radius of the sphere anymore. Then $M \setminus B_\epsilon(x)$, i.e. the complement of a neighborhood of the point, is a bordism from S^{n-1} to $\emptyset$, which gives rise to the expectation value:

$$\langle - \rangle : \mathcal{Z}\left(S^{n-1}\right) \to \mathcal{Z}\left(\emptyset_{n-1}\right) = \mathbb{C}\ . \tag{10.101}$$

So we can just define the space of local operators as the assignment to the codimension 1 sphere.

Example 60. Consider two-dimensional Yang-Mills $\mathcal{Z}_G$ for finite G. Then local operators are:

$$\mathcal{Z}_G\left(S^1\right) = \mathbb{C}\left[\frac{G}{G}\right]\ . \tag{10.102}$$

For $\gamma \in \frac{G}{G}$ we have the delta function $\delta_\gamma \in \mathbb{C}\left[\frac{G}{G}\right]$. Let Σ be a closed topological surface. Then $\Sigma \setminus D_x$, the complement of a disk around a point, is a bordism $S^1 \to \emptyset$. This gives rise to

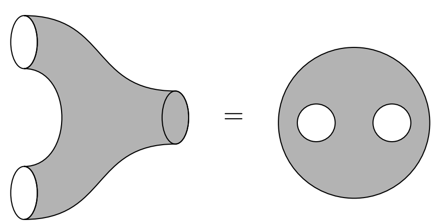
$$\langle - \rangle : \mathbb{C}\left[\frac{G}{G}\right] \to \mathbb{C}\ . \tag{10.103}$$

Then

$$\langle \delta_\gamma \rangle = \#\,\mathrm{Loc}_G\left(\Sigma \setminus D, \gamma\right) \tag{10.104}$$

is the count of G-local systems with monodromy γ around x. Note that this depends on the ambient Riemann surface.

The key point is that $\mathcal{H}_{G\times G,G}$ is just the algebra of local operators in $\mathcal{Z}_G$, and by drawing pictures such as:



(10.105)

we get an algebra structure which is commutative, since we can move x and y around one another. So local operators in TFT (in dimension ≥ 2) are commutative algebras. Lecture 22; April 15, 2021

10.6 Local operators

transition

In *topological* field theory, local operators comprise $\mathcal{Z}\left(S^{n-1}\right)$.

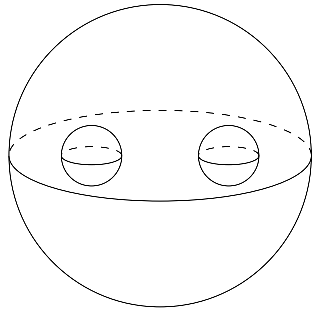


Figure 10.3: A 3-ball after excising two small 3-balls. This is a bordism from the disjoint union $S^2 \sqcup S^2$ to S^2.

Figure 10.4: This is a 3-ball with three 3-balls excised. We can think of two of them as sitting inside of an intermediate sphere, which shows the associativity of the product defined by fig. 10.3.

10.6.1 Algebra structure

Consider two points in the n-dimensional disk D^n. After excising small neighborhoods of the two points we have fig. 10.3, which is a bordism between:

$$S^{n-1} \sqcup S^{n-1} \to S^{n-1} \ . \tag{10.106}$$

The TFT $\mathcal{Z}$ sends this to a map:

$$\mathcal{Z}\left(S^{n-1}\right) \otimes \mathcal{Z}\left(S^{n-1}\right) \to \mathcal{Z}\left(S^{n-1}\right) \ . \tag{10.107}$$

So we get a family of products of local operators labelled by configurations of two balls in D^n. Because $\mathcal{Z}$ is a topological field theory, this is locally constant over the configuration space of these two points in and n-disk, so only depends on π_0 of this. So what is π_0 of this configuration space?

If $n = 1$, then we're considering two (small intervals containing) points inside of an interval. So there are two cases: x is less than y vs. y is less than x. So this space is disconnected, i.e. we have two products which are a priori different: $\mathcal{O}_x * \mathcal{O}'_y$ versus $\mathcal{O}'_y * \mathcal{O}_x$. So we get an associative, but not necessarily commutative algebra. fix fig. 10.4

We actually get associativity in any dimension. To see this, consider fig. 10.4. This bordism is the composition of two copies of fig. 10.3. In 2-dimensions we're excising two

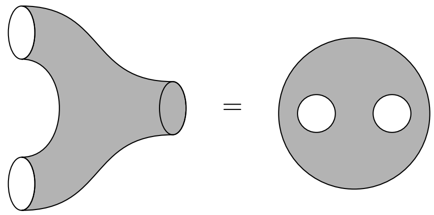
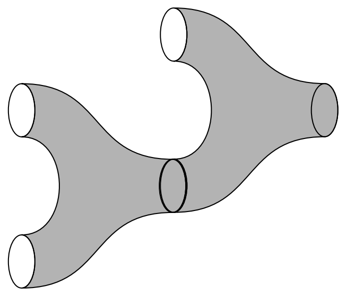


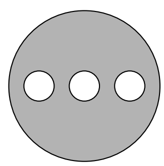
Figure 10.5: The pair of pants bordism. This is a bordism from the disjoint union $S^1 \sqcup S^1$ to S^1.

small disks from a larger disk as in fig. 10.5. Then composition corresponds to:



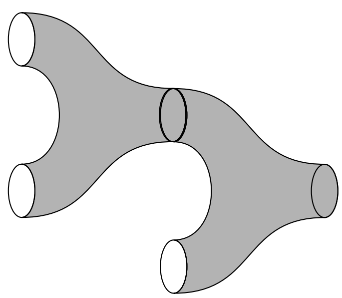
$$. \tag{10.108}$$

This is homotopic to:



$$, \tag{10.109}$$

which is also homotopic to:



$$, \tag{10.110}$$

which shows associativity.

If $n > 1$, then this configuration space of pairs of disks (or homotopically pairs of points) in a disk is connected. This tells us that:

$$\left(\mathcal{Z}\left(S^{n-1}\right), *\right) \tag{10.111}$$

is a commutative algebra.

Remark 57. This is a crucial feature of topological quantum field theory. So quantum mechanics is fundamentally noncommutative, but once we get into higher dimensions we do have this commutativity around.

10.6.2 Spectral decomposition

As we have seen, the general principle is that if someone hands you a commutative algebra, you should take Spec of it. So given an n-dimensional TFT ($n > 1$) we can define an affine variety:

$$\mathcal{M}^0_{\mathcal{Z}} := \operatorname{Spec}\left(\mathcal{Z}\left(S^{n-1}\right), *\right) \ . \tag{10.112}$$

This is called the *moduli space of vacua*, i.e. a point of this space is a *vacuum* of the field theory. A *vacuum* is part of the data needed to define the partition functions, expectation values, etc. in a quantum field theory. E.g. for a quantum field theory defined on $\mathbb{R}^n$, then we might try to define a partition function on $\mathbb{R}^n$, but this will not be well-defined because of the non-compactness of $\mathbb{R}^n$. So we need to fix some data at infinity. I.e. we need to compactify at infinity, and we're seeing here an element of $\mathcal{Z}\left(S^{n-1}\right)$.

Remark 58. We first met $\mathcal{Z}\left(S^{n-1}\right)$ by looking at operators very close to a point, e.g. in the *ultraviolet*. Now we're seeing $\mathcal{Z}\left(S^{n-1}\right)$ appear by looking very far away, i.e. at infinity. These agree because a *topological* quantum field theory doesn't known the difference between a large and small sphere. In other words, the ultraviolet and infrared become the same for a topological field theory. In ordinary quantum field theory the connection between vacua and local operators is not so direct.

Given a local operator $\mathcal{O}$ and vacuum $\langle 0\rangle$, we can then define a vacuum expectation value of $\mathcal{O}$, which is some number written $\langle 0|\mathcal{O}|0\rangle$. So we can just think of the vacuum as something we need to choose to make these expectation values well-defined. I.e. operators give *functions* on the space of vacua. In the TFT setting we will use this as the *definition* of a vacuum: a vacuum is a point in $\operatorname{Spec}\mathcal{Z}\left(S^{n-1}\right)$.

The main point is that a TFT tautologically spectrally decomposes over its moduli space $\mathcal{M}^0_{\mathcal{Z}}$. I.e. everything in TFT is linear over the algebra $\mathcal{Z}\left(S^{n-1}\right)$.

This is an algebra, so it certainly acts on itself, but there are many other natural modules around. Let M be an $(n-1)$-manifold. Then the claim is that $\mathcal{Z}(M)$ is a module over $\mathcal{Z}\left(S^{n-1}\right)$. Consider the bordism $M \times I$, which induces the identity map on $\mathcal{Z}(M)$. Then at some point of the interval, we can think of inserting an operator, i.e. we can excise some small ball around this point to get fig. 10.6. This resulting manifold $(M \times I) \setminus D_x$ is a bordism: redo figure

$$M \sqcup S^{n-1} \to M \ , \tag{10.113}$$

which induces a map:

$$\underset{x}{act}\colon \mathcal{Z}(M) \otimes \mathcal{Z}\left(S^{n-1}\right) \to \mathcal{Z}(M) \ . \tag{10.114}$$

This map realizes $\mathcal{Z}(M)$ as a module by drawing similar pictures to fig. 10.4. Therefore we can think of $\mathcal{Z}(M)$ as a sheaf on the moduli space:

$$\mathcal{Z}(M) \in \mathbf{QC}\left(\mathcal{M}^0_{\mathcal{Z}}\right) = \mathcal{Z}\left(S^{n-1}\right)\textbf{-mod} \ . \tag{10.115}$$

Remark 59. We have an action of $\mathcal{Z}\left(S^{n-1}\right)$ on $\mathcal{Z}(M)$ for every point $x \in M$. So this huge tensor product $\otimes_{x\in M}\mathcal{M}\left(S^{n-1}\right)$ acts on $\mathcal{Z}(M)$, but actually factors through something called factorization homology:

$$\int_M \mathcal{Z}\left(S^{n-1}\right) \ , \tag{10.116}$$

which consists of functions on locally constant maps from M to the moduli space of vacua:

$$\mathcal{O}\left(\operatorname{Map}_{\text{l.c.}}\left(M, \mathcal{M}_{\mathcal{Z}}\right)\right) \ . \tag{10.117}$$

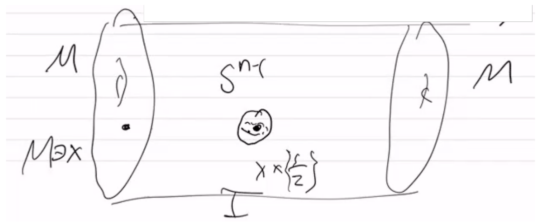


Figure 10.6: Excising a small ball from $M^{n-1} \times I$ results in a bordism from $M^{n-1} \sqcup S^{n-1}$ to M^{n-1}. This induces a module structure on $\mathcal{Z}\left(M^{n-1}\right)$.

10.6.3 Two-dimensional TFT

Local operators comprise the space of states $\mathcal{Z}\left(S^1\right)$. We studied this last time when we considered the identity interface between $\mathcal{Z}$ and itself, and then took $\mathrm{End}\left(\mathrm{id}_{\mathcal{C}}\right)$ (where $\mathcal{C}$ is the category of boundary conditions). Recall this is the Bernstein center

$$\mathrm{End}\left(\mathrm{id}_{\mathcal{C}}\right) = \mathcal{Z}\left(\mathcal{C}\right) \ . \tag{10.118}$$

So this is the same as $\mathcal{Z}\left(S^1\right)$.

If $\mathcal{Z} = \mathcal{Z}_G$ is two-dimensional Yang-Mills for a finite group G, then

$$\mathcal{Z}_G\left(S^1\right) = \mathbb{C}\left[\frac{G}{G}\right] \tag{10.119}$$

(with convolution) is the center of the group algebra $\mathbb{C}G$.

The moduli space of vacua is:

$$\mathcal{M}^0_{\mathcal{Z}_G} = \mathrm{Spec}\, \mathcal{Z}_G\left(S^1\right) \tag{10.120}$$

$$= \mathrm{Spec}\, \mathbb{C}\left[\frac{G}{G}\right] \tag{10.121}$$

$$=: \widehat{G} \ , \tag{10.122}$$

which we will call the *unitary dual* of a group G. The claim is that $\widehat{G}$ is the set of isomorphism classes of irreducible representations of G. Representations of a finite group decompose into irreducible representations by Maschke's theorem:

$$\mathbf{Rep}\left(G\right) \simeq \bigoplus_{\text{iso classes of irreps } V_\lambda} \left(\mathbf{Vect}\right) \otimes V_\lambda \ . \tag{10.123}$$

I.e. any representation can be written:

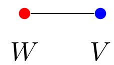
$$V \simeq \bigoplus_{\lambda \in \widehat{G}} V_\lambda^{\oplus m_\lambda} \ . \tag{10.124}$$

In other words:

$$\mathbf{Rep}\left(G\right) \simeq \mathbf{Vect}\left(\widehat{G}\right) = \mathbf{QC}\left(\widehat{G}\right) \ . \tag{10.125}$$

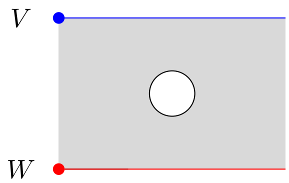
This is a version of spectral decomposition.

We said that $\mathcal{Z}\left(M^{n-1}\right)$ was a module for $\mathcal{Z}\left(S^{n-1}\right)$. But in 2-dimensions we don't get much because there aren't many compact 1-manifolds. But we do have interesting 1-manifolds with boundary, e.g.



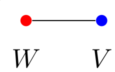
, (10.126)

for $V, W \in \mathbf{Rep}(G)$. Then we have a bordism from this to itself with a small disk excised:



. (10.127)

Therefore $\mathcal{Z}\left(S^1\right)$ acts on:



$$\mathcal{Z}\left(S^1\right) \circlearrowright \mathcal{Z}\left(\quad \right) = \mathrm{Hom}\,(V, W) \tag{10.128}$$

I.e. the category $\mathbf{Rep}(G)$ is enriched in $\mathcal{Z}\left(S^1\right)$-modules. Equivalently this is a $\mathcal{Z}\left(S^1\right)$-linear category.

This isn't something special about finite group theory. In any 2-dimensional TFT, the category of boundary conditions $\mathcal{C}$ is $\mathcal{Z}\left(S^1\right) = \mathcal{Z}(\mathcal{C})$-linear. In fact any category $\mathcal{C}$ is linear over $\mathcal{Z}(\mathcal{C}) = \mathrm{End}\,(\mathrm{id})$. This is the universal algebra over which $\mathcal{C}$ is linear. I.e. Hom spaces in $\mathcal{C}$ are naturally in $\mathbf{QC}\,(\mathrm{Spec}\,\mathcal{Z}(\mathcal{C}))$. The idea is that $\mathcal{C}$ forms a sheaf of categories over $\mathcal{M} = \mathrm{Spec}\,\mathcal{Z}(\mathcal{C})$. This means that the Hom-spaces are upgraded to a sheaf on this space, i.e. they are modules over $\mathcal{Z}(\mathcal{C})$.

In general we can say $\mathcal{C}$ is R-linear if the Hom-spaces are R-modules, or equivalently if we have a map $R \to \mathcal{Z}(\mathcal{C})$. If we are in a "big setting", i.e. $\mathcal{C}$ has enough colimits (cocomplete), then this is equivalent to saying that $\mathcal{C}$ is a module category for R-**mod**.

So we have three different ideas that we would like to internalize as being the same:

1. $\mathcal{C}$ forms a (quasicoherent) sheaf of categories.

2. $\mathcal{C}$ is R-linear, i.e. there is a map $R \to \mathcal{Z}(\mathcal{C})$.

3. $\mathcal{C}$ is a module category for R-**mod**.

I.e. $\mathcal{C}$ being a (quasicoherent) sheaf of categories over $\mathrm{Spec}\,R$ is equivalent to there being a map $R \to \mathcal{Z}(\mathcal{C})$, which is equivalent to $\mathcal{C}$ being enriched in R-modules, which says that $\mathcal{C}$ has an action of $(R\text{-}\mathbf{mod}, \otimes_R)$. This last characterization looks a bit different. To define something like $M \otimes C$ for $M \in R$-**mod** and $C \in \mathcal{C}$ we first resolve M as a complex of free modules:

$$\cdots R^{\oplus n_2} \to R^{\oplus n_1} \to M\ , \tag{10.129}$$

where the maps are matrices with entries in R. We know that the tensor product of C with a free module $R^{\oplus n_1}$ is just $C^{\oplus n_1}$. Since the category is R-linear, we can write these same matrices as maps between:

$$\cdots \to C^{\oplus n_2} \to C^{\oplus n_1}\ , \tag{10.130}$$

and this is a resolution of $M \otimes C$. This is where we use the hypotheses about colimits: we needed to write down these resolutions and take a colimit of it at the end. E.g. modules for a group/associative algebra satisfy this.

Localization

If $\mathcal{C}$ is a module over R-**mod**, then we can define its localization over $\operatorname{Spec} R$. Recall for ordinary modules we define the localization of $M \circlearrowleft R$ to be:

$$\underline{M}(U) = M \otimes_R \mathcal{O}(U) \ , \tag{10.131}$$

for $U \subset \operatorname{Spec} R$ open. The point is that $\mathcal{O}(U) = R\left[S^{-1}\right]$, for some multiplicative set S, and so $\underline{M}(U) = M\left[S^{-1}\right]$. This gives an equivalence of categories between R-**mod** and quasicoherent sheaves on $\operatorname{Spec} R$.

In the categorical setting the same picture works. Let $\mathcal{C} \circlearrowleft R$-**mod**, and then we can form:

$$\underline{\mathcal{C}}(U) = \mathcal{C} \otimes_{R\text{-}\mathbf{mod}} R\left[S^{-1}\right]\text{-}\mathbf{mod} \tag{10.132}$$

where $U = \operatorname{Spec} R\left[S^{-1}\right]$. This is $\mathcal{C}$ where we invert all morphisms coming from $S \subset R$.

For A an associative algebra (e.g. $\mathbb{C}G$) A-**mod** sheafifies over $\operatorname{Spec} \mathcal{Z}(A)$. Similarly $\mathbf{Rep}(G)$ sheafifies over $\widehat{G}$, and the fact that we can decompose representations as irreducibles gives us:

$$\mathbf{Rep}(G) \simeq \mathbf{QC}\left(\widehat{G}\right) \ . \tag{10.133}$$

This sounds like some version of a Fourier transform, but it is not canonical. Something is suspicious about this picture. What is tautologically true is that $\mathbf{Rep}(G)$ sheafifies over the center, Given any subset S of $\widehat{G}$, we can form a summand of $\mathbf{Rep}(G)$ consisting of all representations whose characters lie in S. But in order to get an identification between $\mathbf{Rep}(G)$ and $\mathbf{Vect}G^*$, we need to pick representatives of the isomorphism classes of representations. Alternatively, Wedderburn's theorem (finite group Peter-Weyl theorem) tells us that:

$$\mathbb{C}G \simeq \bigoplus_{V_\lambda \text{irreps}} \operatorname{End}(V_\lambda) \ , \tag{10.134}$$

where $\operatorname{End}(V_\lambda) = M_{n_\lambda \times n_\lambda}(\mathbb{C})$ where $n_\lambda = \dim V_\lambda$. I.e. we have that $\mathbb{C}G$ decomposes as a direct sum over $\chi \in \widehat{G}$ of associative algebras, which turn out to be matrix algebras over division algebras.

This isn't such a small issue. If we keep track of *reality*, i.e. we're considering real representations, this isomorphism is no longer true. So we're thinking of real representations of G, and we look at the set of irreducible characters $\widehat{G}$, which has a natural real structure. There are three kinds of irreducible representations V. Schur's lemma tells us that we get a division algebra over an extension of $\mathbb{R}$:

$$\operatorname{End}(V) = \begin{cases} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{cases} . \tag{10.135}$$

$\mathbb{R}$ and $\mathbb{H}$ correspond to real points of the dual: $\widehat{G}_{\mathbb{R}}$. The complex case means we have complex conjugate pairs of points of the dual, i.e. the characters are not real valued. But even if the characters are real valued, $\operatorname{End}(V)$ is not $\mathbb{R}$, but just some division algebra over $\mathbb{R}$. This tells us that the Wedderburn decomposition doesn't give us endomorphisms of V, we get this quaternionic version. So $\mathbf{QC}\left(\widehat{G}\right)$ acts on $\mathbf{Rep}_{\mathbb{R}}(G)$ as an invertible module category, but nontrivially. I.e. $\mathbf{Rep}_{\mathbb{R}}(G)$ is a categorified line bundle (invertible module for $\mathbf{QC}\left(\widehat{G}\right)$), i.e. a Gerbe. These correspond to elements of the Brauer group (e.g. the quaternions $\mathbb{H}$ are in the Brauer group of $\mathbb{R}$). This gerbe is related to what ate called *anomalies* in physics.

What is tautologically true, is that $\mathcal{C}$ sheafifies over $\mathcal{M}$. The simplest answer would just be that it is $\mathbf{QC}(\mathcal{M})$ itself. The next most complicated would be a gerbe (an anomalous or twisted version of $\mathbf{QC}$), or it might be even more complicated.

For $\mathcal{Z}$ a two-dimensional TFT, we attach the commutative ring of local operators $\mathcal{Z}\left(S^1\right)$, which is a version of the algebra of observables in $\mathcal{Z}$, and we make observations by acting with these operators. So we find that $\mathcal{Z}$ sheafifies over $\mathcal{Z}_{\mathcal{Z}}$.

Remark 60. Mednykh's formula tells us that:

$$\mathcal{Z}_G\left(\Sigma^G\right) = \#\operatorname{Loc}_G \Sigma^g \tag{10.136}$$

$$= \sum_{\chi\in\widehat{G}} \left(\frac{|G|}{\dim\left(V_\chi\right)}\right)^{2g-2} . \tag{10.137}$$

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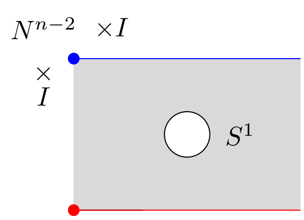
So far we have seen that:

- $\mathcal{Z}\left(S^{n-1}\right)$ (the space of local operators) forms a commutative ring for $n \geq 2$,
- $\mathcal{Z}\left(M^{n-1}\right)$ is naturally a module over $\mathcal{Z}\left(S^{n-1}\right)$, and
- $\mathcal{Z}\left(N^{n-2}\right)$ is naturally linear over $\mathcal{Z}\left(S^{n-1}\right)$.

We only saw this last point for 2-dimensional TFT, but in it is true in general. To see this, consider the identity bordism from $N^{n-2}\times I$ to itself, i.e. $\left(N^{n-2}\times I\right)\times I$. Then we can excise a small disk to form the following bordism:



, (10.138)

where we're thinking of the point as N^{n-2}, i.e. we're implicitly crossing this whole picture with N^{n-2}. So we have that

$$\mathcal{Z}\left(\bullet\!\!-\!\!-\!\!-\!\!\bullet\right) , \tag{10.139}$$

which is a Hom-space in $\mathcal{Z}\left(N^{n-2}\right)$, is actually a module over the commutative ring $\mathcal{Z}\left(S^{n-1}\right)$.

Then we found that we could organize this structure using the variety:

$$\mathcal{M}^{\text{aff}}_{\mathcal{Z}} = \mathcal{M}^0_{\mathcal{Z}} \coloneqq \operatorname{Spec}\mathcal{Z}\left(S^{n-1}\right) , \tag{10.140}$$

called the moduli space of vacua. The idea is that we can relate the arbitrary theory $\mathcal{Z}$ to a Lagrangian field theory where the spaces of fields are mapping spaces into $\mathcal{M}^0_{\mathcal{Z}}$. These are the $\mathcal{B}$-models we will define in the next section.

Example 61. If $\mathcal{Z}_G$ is 2-dimensional Yang-Mills for a finite group G, then $\mathcal{M}^0_{\mathcal{Z}} = \widehat{G}$, and this gave us a decomposition of $\mathcal{Z}$ into irreducibles.

10.7 $\mathcal{B}$-models

Let R be a commutative ring. (Or equivalently we can think of the variety $X = \operatorname{Spec} R$.) From this we can construct a (truncated) TFT of any dimension. Here *truncated* means that we don't concern ourselves with attaching numbers to top-dimensional bordisms. This is a Lagrangian field theory, so we have a notion of *fields* on a manifold M, which are given by:

$$\operatorname{Map}\left(M, \operatorname{Spec} R = X\right) \ . \tag{10.141}$$

How do we make an interesting interaction between smooth manifolds and an affine variety? We will consider the "stupidest" interaction we can consider between them: locally constant maps. I.e. we're considering:

$$\operatorname{Map}_{\mathrm{lc}}\left(M, X\right) = \operatorname{Map}\left(\pi_0\left(M\right), X\right) \ . \tag{10.142}$$

In terms of rings this is:

$$\operatorname{Map}_{\mathrm{lc}}\left(M, X\right) = \operatorname{Spec}\left(R \otimes \pi_0\left(M\right)\right) \ . \tag{10.143}$$

Then we can define the (truncated) n-dimensional $\mathcal{B}$-model on $X = \operatorname{Spec} R$ as follows. To an $(n-1)$-manifold we assign:

$$\mathcal{B}^n_X\left(M^{n-1}\right) = \mathcal{O}\left(\operatorname{Map}\left(M, X\right)\right) = R \otimes \pi_0\left(M\right) \ . \tag{10.144}$$

To an $(n-2)$-manifold we assign:

$$\mathcal{B}^n_X\left(N^{n-2}\right) = \mathbf{QC}\left(\operatorname{Map}\left(N, X\right)\right) = \left(R \otimes \pi_0\left(N\right)\right)\text{-}\mathbf{mod} \ . \tag{10.145}$$

Then we can define all of the push-pull operations you want from the span:

$$\begin{tikzcd} & \operatorname{Map}\left(M, X\right) \arrow[dl] \arrow[dr] & \\ \operatorname{Map}\left(\partial_{\mathrm{in}}, X\right) & & \operatorname{Map}\left(\partial_{\mathrm{out}}, X\right) \end{tikzcd} \ . \tag{10.146}$$

What does this have to do with spectral decomposition? We started with an n-dimensional TFT $\mathcal{Z}$, then we constructed this ring $\left(\mathcal{Z}\left(S^{n-1}\right), *\right)$, and from this we can define the $\mathcal{B}$-model

$$\mathcal{B}^n_{\mathcal{M}^0_{\mathcal{Z}}} \ . \tag{10.147}$$

The naive idea is that $\mathcal{Z}$ looks like the theory $\mathcal{B}^n_{\mathcal{M}^0_{\mathcal{Z}}}$.

Example 62. Consider 2-dimensional Yang-Mills for a finite group G, $\mathcal{Z}_G$. Then $\mathcal{M}^0_{\mathcal{Z}} = \widehat{G}$,

$$\mathcal{Z}_G\left(S^1\right) = \mathbb{C}\frac{G}{G} \simeq \mathbb{C}\left[\widehat{G}\right] \ , \tag{10.148}$$

is the ring of local operators, and

$$\mathcal{Z}_G\left(\mathrm{pt}\right) = \mathbf{Rep}\left(G\right) \simeq \mathbf{QC}\left(\widehat{G}\right) \ , \tag{10.149}$$

which is also $\mathcal{B}^2_{\mathcal{M}^0_{\mathcal{Z}}}\left(\mathrm{pt}\right)$. I.e. we can recreate the theory just by looking at this ring of class functions.

Remark 61. Mirror symmetry is exactly this kind of idea. In fact the "intrinsic mirror symmetry" of Gross-Siebert is closely modelled on this idea. Start with an interesting theory, cite the $\mathcal{A}$-model in symplectic topology, and construct a variety (the mirror) as a spectrum of a ring of local operators. Then mirror symmetry says that the $\mathcal{A}$-model you started with is equivalent to the $\mathcal{B}$-model that you built.

Even for G a finite group, $\mathbf{Rep}_{\mathbb{R}} G$ won't exactly match the naive answer $\mathbf{QC}\left(\widehat{G}\right)$. Instead, by abstract nonsense, we get the following. We know that $\mathcal{Z}\left(M^{n-1}\right)$ is a module for $\mathcal{Z}\left(S^{n-1}\right)$. We got this structure by crossing with an interval and inserting a local operator, i.e. excising a small ball as in fig. 10.6. So really we got a family of module structures for every point in M such that whenever we path between two points the associated actions are identified. I.e. this depends locally constantly on $x \in M$. So we naturally get that $\mathcal{Z}(M)$ is a module for $\pi_0(M)$-many copies of$\mathcal{Z}\left(S^{n-1}\right)$, i.e. an action of:

$$\pi_0(M) \otimes \mathcal{Z}\left(S^{n-1}\right) = \mathcal{O}\left(\mathrm{Map}\left(M, \mathcal{M}^0_{\mathcal{Z}}\right)\right) \ . \tag{10.150}$$

So we don't get an isomorphism between this vector space and this ring, but rather that $\mathcal{Z}(M)$ is a module for $\mathcal{O}\left(\mathrm{Map}\left(M, \mathcal{M}^0_{\mathcal{Z}}\right)\right)$, i.e. it is an object:

$$\mathcal{Z}(M) \in \mathbf{QC}\left(\mathcal{M}^0_{\mathcal{Z}}\right) \ , \tag{10.151}$$

i.e.

$$\mathcal{Z}(M) \in \mathcal{B}^{n+1}_{\mathcal{M}^0_{\mathcal{Z}}}(M) \ . \tag{10.152}$$

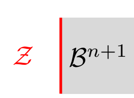
Likewise $\mathcal{Z}(N)$ is an R-linear category, but in fact it can be refined to a category which is linear over $\pi_0(N) \otimes R$, i.e.

$$\mathcal{Z}(N) \in \mathcal{B}^{n+1}_{\mathcal{M}^0_{\mathcal{Z}}}(N) = \text{2-category of sheaves of categories}/\mathcal{M}^0_{\mathcal{Z}} \ . \tag{10.153}$$

To summarize, spectral decomposition for field theories says the following. We start with $\mathcal{Z}$ an n-dimensional TFT (i.e. our states states). Then we have the observables (local operators), which act on the states. We can reformulate this as saying that

$$\mathcal{Z}(-) \in \mathcal{B}^{n+1}_{\mathcal{M}^0_{\mathcal{Z}}}(-) \ . \tag{10.154}$$

Equivalently this is saying that the theory $\mathcal{Z}$ is a boundary condition for this $\mathcal{B}$-module of one dimension higher: $\mathcal{B}^{n+1}_{\mathcal{M}^0_{\mathcal{Z}}}$. I.e. we're thinking of $M \times I$ with a red ($\mathcal{Z}$-labelled) boundary:



$M \times I$. (10.155)

We originally thought of boundary conditions as representations of a group, or in general objects of a category. What we are doing here is just a fancier version of the same idea.

Recall we were headed towards these 4-dimensional $\mathcal{A}$-models. We only talked about them for the group $G = \mathrm{GL}_1$, where we got the topological Maxwell theory $\mathcal{A}_{\mathrm{GL}\,1}$. The vector space $\mathcal{A}_{\mathrm{GL}_1}\left(M^3\right)$, which this theory attached to a 3-manifold, was locally constant functions on $\mathrm{Pic}\left(M^3\right)$. Recall $\mathrm{Pic}\left(M^3\right)$ is the space of $\mathrm{U}(1)$-connections modulo gauge equivalence. Now we can ask for the local operators and try to play the spectral decomposition game as above. Recall that $\mathrm{Pic}(M)$ looks like a lattice $H^2(M,\mathbb{Z})$ crossed with a torus which has $\pi_1 = H^1(M,\mathbb{Z})$. The local operators will come from the assignment to S^3, but this has neither H^2 nor H^1, so we get

$$\mathrm{Pic}\left(S^3\right) = \mathrm{pt}\ . \tag{10.156}$$

If we're being careful about automorphisms we might get something like $B\,\mathrm{U}(1)$ here instead. In any case, locally constant functions on this are trivial, so we don't have any local operators. In the derived version we would get:

$$H^*\left(B\,\mathrm{U}(1)\right) = \mathbb{C}[u]\ , \tag{10.157}$$

but this is still not very interesting/useful.

10.8 Defects

Now we want to generalize our story for local operators to line operators. To do this we will introduce general defect operators, and to do this we will need to discuss the cobordism hypothesis.

10.8.1 Cobordism hypothesis

The cobordism hypothesis was introduced by Baez-Dolan, and proved by fix and cite

Hopkins-Lurie, Lurie, Ayala-Francis, Schommer-Pries

The cobordism hypothesis is an equivalence between certain notions in topological field theory and certain notions in category theory. We will think of it as a way to build a topological field theory from categorical data. This can also be thought of the other way, i.e. as giving an explanation of why various forms of *pictorial calculus* describe category theory, i.e. it tells us we can draw pictures instead of thinking about categories.

An *extended topological field theory* is a representation of a bordism category, i.e. a symmetric monoidal functor:

$$\mathcal{Z}\colon \left(\mathbf{Bord}_n^{\square}, \sqcup\right) \to (\mathcal{C}, \otimes)\ , \tag{10.158}$$

where $\square$ can consist of any choice of *tangential structure*. For example, we might ask for the bordisms to be oriented, framed, or in general to have a G-structure, whenever we have a map $G \to \mathrm{O}(n)$. If $G = 1$, then a G-structure is an n-framing. If $G = \mathrm{SO}(n)$, then a G-structure is equivalent to choosing an orientation on M. If $G = \mathrm{O}(n)$, then a G-structure on a manifold M does not consist of any extra structure. In any case, the basic picture to draw is that our manifolds in this category always have an n-dimensional tangent bundle. I.e. instead of a point, we can picture a small "chunk" of $\mathbb{R}^n$.

The *cobordism hypothesis* says that the structure of a field theory $\mathcal{Z}$ is completely determined by the data of $\mathcal{Z}(\mathrm{pt}) \in \mathcal{C}$, subject to:

- dualizability (finiteness condition),
- self-duality structure (this depends on $\square$).

For a one-dimensional theory dualizability was asking for the vector space assigned to a point to have a dual. The self-duality constraint has to do with the choice of tangential structure.

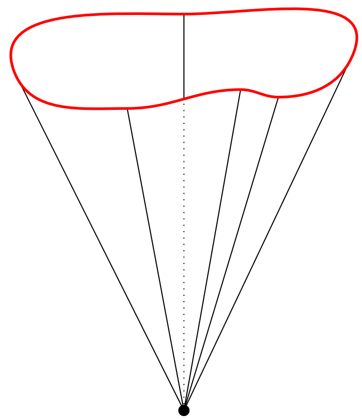


Figure 10.7: The cone of a manifold M is formed by crossing with an interval, and then quotienting out by the entire copy of M on one end. The original manifold is highlighted in red.

We can think of this as saying that $\mathbf{Bord}_n$ is freely generated by the single object pt, so we only need to determine where this is sent in $\mathcal{C}$. With the caveat that we need some finiteness and self-duality structure in order to evaluate the theory on bordisms up to the top dimension.

10.8.2 Cobordism hypothesis with singularities

This appears in Lurie, and is a generalization of a conjecture of Baez-Dolan called the *tangle hypothesis.* Cite

We described the cobordism hypothesis as some kind of equivalence between topological field theory and category theory. So far, the statement of the cobordism hypothesis tells us an equivalence at the level of objects, but doesn't reflect some of the diagrammatic structure we see in category theory, e.g. wave morphisms, higher morphisms, endomorphisms etc. which will become more visible with the help of the cobordism hypothesis with singularities. The general idea is that we are describing field theories that are now defined on bigger sources, i.e. on bordisms which are manifolds with certain prescribed singularities, which will be matched with certain categorical structures.

Given a manifold M, we can build a singularity by forming the cone on M, defined by:

$$C(M) := (M \times I) / (M \times \{0\}) \ , \tag{10.159}$$

as in fig. 10.7. The link around the singular point is M. We want to add singularities of this same flavor. E.g. we might allow for singularities with link given by some manifold, or we might do some stratified version where we build inductively out of such cones.

The *cobordism hypothesis with singularities* roughly says that: symmetric monoidal functors from $\mathbf{Bord}_n$ with possible $C(M)$-singularities to $(\mathcal{C}, \otimes)$ are equivalent to:

1. $\mathcal{Z} \colon \mathbf{Bord}_n \to \mathcal{C}$ (or equivalently $C \in \mathcal{C}$) along with
2. an object of $\mathcal{Z}(M)$, i.e. a morphism:

$$1 \to \mathcal{Z}(M) \ . \tag{10.160}$$

The idea is that the cone on M is a new bordism from $\emptyset \to M$, and $\mathcal{Z}$ sends this to a map $1 \to \mathcal{Z}(M)$, i.e. an element of $\mathcal{Z}(M)$.

The only case we will need is when M is a sphere. When we take a cone modelled on a sphere, this isn't really a singularity at all, but we can still imagine that we're specifying a marked point when we form a cone modelled on the sphere. For example, if we look at $\mathbf{Bord}_n$ with marked points, then the cobordism hypothesis with singularities says this marked point is equivalent to specifying an object $\mathcal{O} \in \mathcal{Z}\left(S^{n-1}\right)$, which is a local operator. So this is what we've already seen. If you like, this is the state-field correspondence for local operators.

This tells us that if we look at n-manifolds together with an embedded $(k-1)$-manifold, then we need to look at the link of the singularity (the embedded submanifold) which is a sphere S^{n-k}. So we need to give an element of $\mathcal{Z}\left(S^{n-k}\right)$. These are called *defect operators*.

For example, line defect/operators are embedded 1-manifolds, which correspond to a choice of a state

$$L \in \mathcal{Z}\left(S^{n-2}\right) \ . \tag{10.161}$$

So before local operators were elements of the vector space $\mathcal{Z}\left(S^{n-1}\right)$, and now the assignment $\mathcal{Z}\left(S^{n-2}\right)$ is the *category of line operators*. Likewise $\mathcal{Z}\left(S^{n-3}\right)$ is the 2-category of surface defects.

The upshot is that, given an n-dimensional TFT $\mathcal{Z}$, once we have specified an element $L \in \mathcal{Z}\left(S^{n-2}\right)$ (satisfying the appropriate dualizability conditions) we can extend the theory to be defined on manifolds with arbitrary embedded 1-manifolds. I.e. for any knot $K \subset M$, we get a new value:

$$\mathcal{Z}_L\left(M \supset K\right) \ . \tag{10.162}$$

Example 63. If $n=3$ this is saying that we need to specify an element $L \in \mathcal{Z}\left(S^{n-2}\right)$ to determine a knot invariant.

We saw that $\mathcal{Z}\left(S^{n-1}\right)$ formed a commutative ring (for $n \geq 2$) and then we took Spec:

$$\mathcal{M}^0_{\mathcal{Z}} = \operatorname{Spec} \mathcal{Z}\left(S^{m-1}\right) \ , \tag{10.163}$$

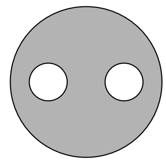
and spectrally decomposed over this, i.e. we studied states as a module for local operators.

Now we want to play the same game with line operators. So consider the category of line operators $\mathcal{Z}\left(S^{n-2}\right)$. In fact, this forms a tensor category just as the vector space of local operators formed an algebra. This product still comes from what is called the OPE, but now of line operators instead of local operators. The product comes from the bordism in fig. 10.3, where we are now regarding it as living in 4-dimensional space, i.e. the spheres are S^{n-2}-spheres in n-dimensional space arising as the links around two embedded lines (rather than S^{n-1}-spheres arising in n-dimensional space as the links around two points).

In other words, the data we need to get this structure is a configuration of points (or disks) inside of a (bigger) disk. This is the same picture we had for local operators, but we have more dimensions we're ignoring, so the categorical level has jumped. I.e. $\mathcal{Z}\left(S^{n-2}\right)$ has multiplications labelled by (locally constantly) pairs of disks inside of a larger disk. (This means this is defined over the *little* $(n-1)$*-discs operad*, which is also called an $\mathbb{E}_{n-1}$-structure.)

Example 64. For example line operators in two-dimensions are codimension 1, and so we have multiplications labelled by intervals inside of an interval. So the multiplication is associative but not commutative. This is an $\mathbb{E}_1$-structure.

Example 65. In three-dimensions we have the pair of pants:



$$, \tag{10.164}$$

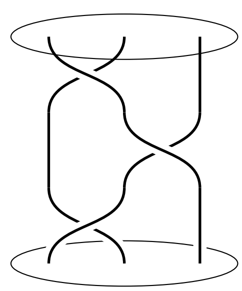
which is a bordism from $S^1 \sqcup S^1$ to S^1, and therefore tells us that $\mathcal{Z}\left(S^1\right)$ has an operation $*$ which depends locally constantly on the configuration space of two points inside of a disk. More precisely, we mean that given a path between configurations we get an isomorphism between the corresponding multiplications. I.e. we get a map from the fundamental groupoid:

$$\pi_{\leq 1}\left(\mathrm{Conf}_2\, D^2\right) \otimes \mathcal{Z}\left(S^1\right)^{\otimes 2} \to \mathcal{Z}\left(S^1\right) \ . \tag{10.165}$$

This space of pairs of points in the disk is connected, so at the level of local operators this multiplicative was commutative. Now this just says that we have an isomorphism $L_1 * L_2 \simeq L_2 * L_1$, but when we square it, we aren't guaranteed to get the identity:

$$\begin{tikzcd} L_1 * L_2 \arrow[r, "\sim"] \arrow[rr, bend right, "\text{id?}"'] & L_2 * L_1 \arrow[r, "\sim"] & L_1 * L_2 \end{tikzcd} \ . \tag{10.166}$$

So what we find is that $\mathcal{Z}\left(S^1\right)$ is still monoidal, but now it is even *braided* monoidal, i.e. this multiplication operation satisfies the braid relations. We can imagine that a configuration of multiple points in the disk becomes another configuration via a braid:



$$. \tag{10.167}$$

This gives us an isomorphism:

$$L_1 * L_2 * L_3 \to L_3 * L_2 * L_1 \ , \tag{10.168}$$

and the braid relations say that these isomorphisms satisfy the relations of the braid group.

So in 2-dimensions line operators are $\mathbb{E}_1$, i.e. a monoidal category. In 3-dimensions line operators are $\mathbb{E}_2$, i.e. a braided monoidal category. But in 4-dimensions and above, line operators are $\mathbb{E}_\infty$, i.e. symmetric monoidal. So the fundamental groupoid is:

$$\pi_{\leq 1}\left(\mathrm{Conf}_2\, D^{\geq 3}\right) = 0 \ , \tag{10.169}$$

so we square to the identity:

$$\begin{tikzcd} L_1 * L_2 \arrow[r, "\sim"] \arrow[rr, bend right, "\text{id}"'] & L_2 * L_1 \arrow[r, "\sim"] & L_1 * L_2 \end{tikzcd} \ . \tag{10.170}$$

I.e. these spaces aren't just connected, but in fact simply connected.

Now we can state the punchline. In 4-dimensions and above, line operators:

$$\left(\mathcal{Z}\left(S^2\right), *\right) \tag{10.171}$$

form a tensor category, and so we can define:

$$\mathcal{M}^1_{\mathcal{Z}} \coloneqq \operatorname{Spec}\left(\mathcal{Z}\left(S^2\right), *\right) \ . \tag{10.172}$$

This is a refinement of $\mathcal{M}^0_{\mathcal{Z}}$. Explicitly it is given by the affinization:

$$\mathcal{M}^1_{\mathcal{Z}} \twoheadrightarrow \mathcal{M}^0_{\mathcal{Z}} = \operatorname{Aff}\left(\mathcal{M}^1_{\mathcal{Z}}\right) \ . \tag{10.173}$$

transition

April 22, 2021

10.8.3 Local operators versus line operators

Recall defect operators in TFT are the values of our TFT on sphere of various dimensions:

- $\mathcal{Z}\left(S^{n-1}\right)$ consists of local operators,
- $\mathcal{Z}\left(S^{n-2}\right)$ consists of line operators,
- $\mathcal{Z}\left(S^{n-3}\right)$ consists of surface operators, etc.

For n (the dimension of the theory) large enough and the dimension of the defect small enough, the defects form a commutative algebra. I.e. once there is a enough room for the spheres to move around, the product we get from "colliding" them becomes commutative. For example we saw that local operators are merely associative for $n = 1$. But for $n > 1$ they were commutative. Similarly, line operators are associative for $n = 2$, braided for $n = 3$ and symmetric for $n \geq 4$.

We're mostly interested in line operators for $n = 4$-dimensional TFTs. These form the symmetric monoidal category $\left(\mathcal{Z}\left(S^2\right), *\right)$, so we take the spectrum:

$$\mathcal{M}^1_{\mathcal{Z}} \coloneqq \operatorname{Spec}\left(\mathcal{Z}\left(S^2\right), *\right) \ . \tag{10.174}$$

Recall that local operators $\mathcal{Z}\left(S^{n-1=3}\right)$ are a commutative algebra so we could take the spectrum of this too to get:

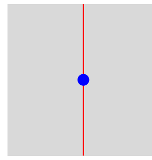
$$\mathcal{M}^0_{\mathcal{Z}} = \operatorname{Spec} \mathcal{Z}\left(S^{n-1}\right) \ . \tag{10.175}$$

Question 7. What is the relationship between $\mathcal{M}^1_{\mathcal{Z}}$ and $\mathcal{M}^0_{\mathcal{Z}}$? I.e. what is the relationship between line operators and local operators?

One way we can see the relationship is with the following cartoon. Consider a local operator, i.e. a little blue dot in the middle of much nothingness:

. (10.176)

We can interpret it as actually sitting on a transparent line shown in red:

 , (10.177)

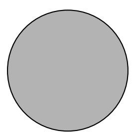
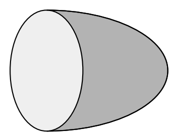
i.e. we're thinking of a local operator as sitting on a 1-dimensional defect which is trivial. This trivial defect is the unit with respect to the convolution structure on line operators. I.e. **local operators are the same as self-interfaces of the trivial line operator.**

Suspension

As it turns out, this picture can be rephrased in terms of forming suspensions of spheres. Suppose we are studying S^{n-2}, coming to us as the link of a line. Then line operators, $\mathcal{Z}\left(S^{n-2}\right)$, has a distinguished object:

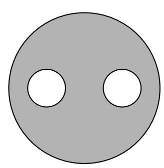
$$\mathcal{Z}\left(D^{n-1}\right) \in \mathcal{Z}\left(S^{n-2}\right) \ , \tag{10.178}$$

since we can think of the disk as a bordism:

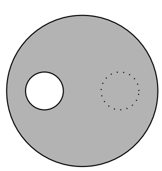
 $\simeq$  , (10.179)

from $S^{n-2} \to \emptyset$.

The claim is that this distinguished object is the unit. Recall our multiplication came from the pair of pants bordism:

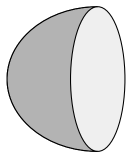
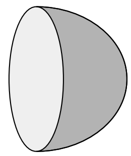
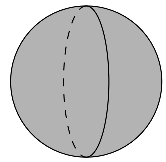
 , (10.180)

so multiplication by the disk is the identity because once we have composed the pair-of-pants with the disk we get:

 $\simeq$  $\simeq$ $S^{n-2} \times I$, (10.181)

which is the identity bordism.

We can also compose the disk with itself to form a sphere:

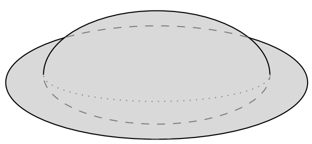
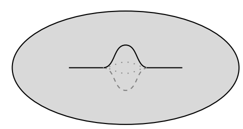
 $\sqcup_{S^{n-2}}$  $=$  $= \quad S^{n-1}\ .$ (10.182)

This is exactly the process of suspending S^{n-2} to get S^{n-1}. The point is that this picture gives us:

$$\mathcal{Z}\left(S^{n-1}\right) = \mathrm{Hom}_{\mathcal{Z}(S^{n-2})}\left(\mathcal{Z}\left(D^{n-1}\right), \mathcal{Z}\left(D^{n-1}\right)\right) = \mathrm{End}_{\mathcal{Z}(S^{n-2})}\left(1\right)\ . \tag{10.183}$$

I.e. $\mathcal{Z}\left(S^{n-1}\right)$ is endomorphisms of the unit in $\mathcal{Z}\left(S^{n-2}\right)$.

Remark 62. If we're being more careful about framings, etc. the two copies of the disk compose to give the *ravioli/UFO*:

 $= \quad D^{n-1} \sqcup_{D^{n-1}\setminus \mathrm{pt}} D^{n-1} \quad =$  .

(10.184)

The punchline is that local operators $\mathcal{Z}\left(S^{n-1}\right)$ are identified with:

$$\mathcal{Z}\left(S^{n-1}\right) = \mathrm{End}\left(1_{\mathcal{Z}(S^{n-2})}\right) = \mathrm{End}\left(\text{trivial line operator}\right)\ . \tag{10.185}$$

Remark 63. This is the same maneuver as in section 10.4.3, example 58, section 10.6.3, section 10.7 (example 62) where we said that:

$$\mathcal{Z}_G\left(S^1\right) = \mathbb{C}\left[\frac{G}{G}\right] = \mathrm{End}\left({}_G\mathbb{C}G_G\right)\ , \tag{10.186}$$

where we interpreted ${}_G\mathbb{C}G_G$ as the trivial line defect of $\mathcal{Z}_G$.

This is a general construction for any sphere. For example line operators are given by the endomorphisms of the trivial surface operator.

Affinization

So we have seen the relationship between local and line operators via the TFT, but now we want to see it spectrally. Recall we have $\mathcal{M}^0_{\mathcal{Z}} = \mathrm{Spec}\,\mathcal{Z}\left(S^{n-1}\right)$ and $\mathcal{M}^1_{\mathcal{Z}} = \mathrm{Spec}\,\mathcal{Z}\left(S^{n-2}\right)$. What we just saw is that $\mathcal{Z}\left(S^{n-1}\right)$ is given as endomorphisms of the unit/trivial object of $\mathcal{Z}\left(S^{n-2}\right)$. We know that the unit in $\mathbf{QC}\left(\mathcal{M}^1_{\mathcal{Z}}, \otimes\right)$ is the structure sheaf:

$$\mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}} \in \mathbf{QC}\left(\mathcal{M}^1_{\mathcal{Z}}, \otimes\right)\ , \tag{10.187}$$

so this is saying that:

$$\begin{aligned} \mathcal{Z}\left(S^{n-1}\right) &= \mathcal{O}\left(\mathcal{M}^0_{\mathcal{Z}}\right) && (10.188)\\ &= \mathrm{End}_{\mathbf{QC}(\mathcal{M}^1_{\mathcal{Z}})}\left(\mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}}\right) && (10.189)\\ &= \mathrm{Hom}\left(\mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}}, \mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}}\right) && (10.190)\\ &= \Gamma\left(\mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}}\right) \ , && (10.191) \end{aligned}$$

since Hom from $\mathcal{O}$ to anything, is global sections of that thing. I.e.

$$\mathcal{M}^0_{\mathcal{Z}} = \mathrm{Spec}\,\Gamma\left(\mathcal{O}_{\mathcal{M}^1_{\mathcal{Z}}}\right) =: \mathbf{Aff}\left(\mathcal{M}^1_{\mathcal{Z}}\right) \ , \tag{10.192}$$

since this is the definition of the affinization. This is a general construction in algebraic geometry:

$$X \to \mathbf{Aff}\,(X) \coloneqq \mathrm{Spec}\,(\mathcal{O}\,(X)) \ , \tag{10.193}$$

which is universal in the sense that any map from X to an affine variety factors through a map to $\mathbf{Aff}\,(X)$. I.e. given a scheme (or stack) we can consider functions on it, and this doesn't know everything about it (i.e. most objects aren't affine), but we can still take Spec of it, to form the affinization: an affine object which is an "approximation" of the original scheme.

So we have

$$\mathcal{M}^1_{\mathcal{Z}} \twoheadrightarrow \mathcal{M}^0_{\mathcal{Z}} \tag{10.194}$$

is affinization, i.e. $\mathcal{M}^1_{\mathcal{Z}}$ is a new and improved version of $\mathcal{M}^0_{\mathcal{Z}}$, i.e. in general, line operators know strictly more than local operators.

Remark 64. Affinization can lose a lot of information. For example, the affinization of projective space $\mathbb{P}^n$ is a point, and the affinization of a stack like BG is also a point. The point is that the world of algebraic stacks has a lot more to it than just rings of functions.

So we have seen that the categorical notion of taking endomorphisms of the unit corresponds to the topological notion of suspension, which corresponds to the algebro-geometric notion of affinization of the associated moduli space.

Remark 65. We can do even better by forming some object $\mathcal{M}^2_{\mathcal{Z}}$ associated to surface operators:

$$\ldots \to \mathcal{M}^2_{\mathcal{Z}} \to \mathcal{M}^1_{\mathcal{Z}} \to \mathcal{M}^0_{\mathcal{Z}} \ . \tag{10.195}$$

We won't need any of these higher versions, but the construction of $\mathcal{M}^1_{\mathcal{Z}}$ will be very useful for us.

From a 4-dimensional TFT we constructed a stack $\mathcal{M}^1_{\mathcal{Z}}$, the *spectrum of the TFT* $\mathcal{Z}$. This is good for the following. The idea is that this gives us approximations by/relations to the simple Lagrangian TFTs we are calling $\mathcal{B}$-models, introduced in section 10.7. I.e. we can ask: how close is $\mathcal{Z}$ to a theory of maps into its moduli space/spectrum $\mathcal{M}^1_{\mathcal{Z}}$?

Let X be a variety or stack. Recall for any n, we can build an n-dimensional truncated TFT, called the $\mathcal{B}$-model, $\mathcal{B}^n_X$ which sends an $(n-1)$-manifold to:

$$\mathcal{B}^n_X\left(M^{n-1}\right) = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}\left(M, X\right)\right) \ , \tag{10.196}$$

where the maps (fields) are locally constant. For X a stack, we can do something more interesting than simply forming:

$$\mathrm{Map}_{\mathrm{lc}}(M, X) = \mathrm{Map}(\pi_0 M, X) \ , \tag{10.197}$$

as we did then X was simply a variety. Recall that a stack is a functor $\mathbf{Ring} \to \mathbf{Grpd}$. For M a manifold, we can construct the fundamental groupoid:

$$\pi_{\leq 1} M \ , \tag{10.198}$$

whose objects are points and morphisms are paths up to homotopy. So now we can now interpret locally constant maps as:

$$\mathrm{Map}_{\mathrm{lc}}(M, X) = \mathrm{Map}(\pi_{\leq 1} M, X) \ . \tag{10.199}$$

The point is that any manifold M gives rise to a functor $\mathbf{Ring} \to \mathbf{Grpd}$ which is just the constant functor:

$$R \mapsto \pi_{\leq 1}(M) \ . \tag{10.200}$$

Therefore we can think of manifolds and stacks as living in:

$$\mathbf{Functors}(\mathbf{Ring}, \mathbf{Grpd}) \ . \tag{10.201}$$

Now we can calculate $\mathrm{Map}_{\mathrm{lc}}(M, X)$ in this common category of functors. The upshot of this is that these maps remember the fundamental groupoid of M, as opposed to just π_0.

Example 66. Let $X = \mathrm{pt}/G$ for G finite. Then:

$$\mathrm{Maps}_{\mathrm{lc}}(M, \mathrm{pt}/G) = \mathrm{Loc}_G\, M \ . \tag{10.202}$$

Then our $\mathcal{B}$-model for X is given by linearizing this space $\mathrm{Map}_{\mathrm{lc}}(M, X)$:

$$\mathcal{B}^n_X\left(M^{n-1}\right) = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}(M, X)\right) \tag{10.203}$$
$$\mathcal{B}^n_X\left(N^{n-2}\right) = \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}(N, X)\right) \tag{10.204}$$
$$\mathcal{B}^n_X\left(P^{n-3}\right) = \mathbf{ShvCat}\left(\mathrm{Map}_{\mathrm{lc}}(P, X)\right) \ . \tag{10.205}$$

Moral of the story

So we started with an n-dimensional TFT $\mathcal{Z}$, and built the spectrum $\mathcal{M}^1_{\mathcal{Z}}$. Out of this we can build two things: $\mathcal{B}^n_{\mathcal{M}_{\mathcal{Z}^1}}$ or $\mathcal{B}^{n+1}_{\mathcal{M}_{\mathcal{Z}^1}}$. Then the best approximation to $\mathcal{Z}$ in algebraic geometry is $\mathcal{B}^n_{\mathcal{M}^1_{\mathcal{Z}}}$. A better thing to realize is that $\mathcal{Z}$ is linear over $\mathcal{M}^1_{\mathcal{Z}}$, i.e. $\mathcal{Z}$ is a boundary for $\mathcal{B}^{n+1}_{\mathcal{M}^1_{\mathcal{Z}}}$.

Recall $\mathcal{Z}\left(N^{n-2}\right)$ was acted on by $\mathcal{Z}\left(S^{n-2}\right)$, so we find that $\mathcal{Z}\left(N^{n-2}\right)$ has a module structure over $\mathcal{Z}\left(S^{n-2}\right)$ for every point of N^{n-2}. I.e. without making any choices, we get an action of $\pi_0(N)$-many copies of $\mathcal{Z}\left(S^{n-2}\right)$. But really we can replace π_0 by $\pi_{\geq 1}$. So before we had that if there was a path between two points, the associated actions were isomorphic. Now we just have an isomorphism between the actions for every path between them. I.e. the module structure is locally constant with respect to $x \in N$, so we actually have an action of $\mathbf{QC}$ of locally constant maps from N to the moduli space:

$$\mathcal{Z}\left(N^{n-2}\right) \circlearrowleft \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)\right) \ . \tag{10.206}$$

I.e. we have a category $\mathcal{Z}\left(N^{n-2}\right)$, we have another category

$$\mathcal{B}^n_{\mathcal{M}^1_{\mathcal{Z}}}(N) = \mathbf{QC}\left(\mathrm{Maps}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)\right) \ , \tag{10.207}$$

and the former is a module over the latter. I.e. $\mathcal{Z}\left(N^{n-2}\right)$ is an object of $\mathcal{B}_{\mathcal{M}^1_{\mathcal{Z}}}(N)$, which is sheaves of categories on $\mathcal{M}^1_{\mathcal{Z}}$.

Remark 66. This is general way of assembling all of these line operators together into something that acts on $\mathcal{Z}\left(N^{n-2}\right)$. This procedure is known as *factorization homology.* The following notation is often used:

$$\int_M R = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}\left(M, \mathrm{Spec}\left(R\right)\right)\right) \ , \tag{10.208}$$

or in the case of categories:

$$\int_M (\mathcal{C}, \otimes) = \mathbf{QC}\left(\mathrm{Maps}_{\mathrm{lc}}\left(M, \mathrm{Spec}\left(\mathcal{C}\right)\right)\right) \ . \tag{10.209}$$

Now we will see that $\mathrm{Maps}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)$ has the right kind of structure. For a point $x \in N$ we get an evaluation map:

$$\begin{array}{c} \mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right) \\ \Big\downarrow {\scriptstyle \mathrm{ev}_x} \\ \mathcal{M}^1_{\mathcal{Z}} \ . \end{array} \tag{10.210}$$

Therefore we get a pullback map:

$$\mathrm{ev}_x^* \colon \mathbf{QC}\left(\mathcal{M}^1_{\mathcal{Z}}\right) \to \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)\right) \ . \tag{10.211}$$

So if we have an action of $\mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)\right)$ on $\mathcal{Z}\left(N^{n-2}\right)$, we get an action of $\mathbf{QC}\left(\mathcal{M}^1_{\mathcal{Z}}\right)$ too:

$$\mathrm{ev}_x^* \colon \mathbf{QC}\left(\mathcal{M}^2_{\mathcal{Z}}\right) = \mathcal{Z}\left(S^{n-2}\right) \to \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^2_{\mathcal{Z}}\right)\right) \circlearrowright \mathcal{Z}\left(N^{n-2}\right) \ . \tag{10.212}$$

Remark 67. This diagram is telling us the local to global compatibility of this operation. I.e. that we start with these actions associated to points $x \in N$, they assemble together to given an action of this larger space $\mathbf{QC}\left(\mathrm{Maps}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)\right)$, and then these evaluation maps retrieve the local actions.

The general theorem to prove for the local-to-global part is that categories with an action of $(\mathcal{C}, \otimes)$ for each $x \in N$ (depending locally constantly on x) are equivalent to categories with an action of

$$\mathcal{C} \otimes N := \int_N \mathcal{C} := \mathbf{QC}\left(\mathrm{Maps}_{\mathrm{lc}}\left(N, \mathrm{Spec}\,\mathcal{C}\right)\right) \ , \tag{10.213}$$

for $\mathcal{C}$ Tannakian ($\mathcal{C} = \mathbf{QC}\left(\mathrm{Spec}\,\mathcal{C}\right)$).

So the punchline is that any n-dimensional TFT $\mathcal{Z}$ is a boundary condition for $\mathcal{B}^{n+1}_{\mathcal{M}^1_{\mathcal{Z}}}$, i.e. $\mathcal{Z}\left(N\right)$ sheafifies over $\mathrm{Map}_{\mathrm{lc}}\left(N, \mathcal{M}^1_{\mathcal{Z}}\right)$.

10.8.4 Back to topological Maxwell theory

Recall (from sections 4.5, 4.8.1, 6.1.1 and 6.4) we had a TFT $\mathcal{A}_{\mathrm{GL}_1}$ which sends 3-manifolds to the space of locally constant functions on $\mathrm{Pic}\,(M)$. To a Riemann surface we had a few different versions. The Betti version was:

$$\mathcal{A}_{\mathrm{GL}_1}\left(\Sigma^2\right) = \mathbf{Loc}\,(\mathrm{Pic}\,\Sigma)\ , \tag{10.214}$$

and in the de Rham version we had:

$$\mathcal{A}_{\mathrm{GL}_1}\left(\Sigma^2\right) = \mathbf{D\text{-}Mod}\,(\mathrm{Pic}\,\Sigma)\ . \tag{10.215}$$

Recall these categories are not the same, but they are very similar. For what we are doing now, it won't matter which one we choose.

So, by the above abstract nonsense, for every $x \in \Sigma^2$ we get an action of $\mathcal{A}\left(S^2\right)$ (line operators) on $\mathcal{A}\left(\Sigma^2\right)$. For $S^2 = \Sigma$ we choose a complex structure so we can think of it as $\mathbb{P}^1$, and we have

$$\mathcal{A}\left(S^1\right) = \mathrm{Loc}\left(\mathrm{Pic}\left(\mathbb{P}^1\right)\right) = \mathrm{Loc}\,(\mathbb{Z}) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}}\ , \tag{10.216}$$

where the composition product of operators is just the tensor product of graded vector spaces, i.e. the product induced by the group structure on $\mathbb{Z}$. We can think of this as saying that when we have

Remark 68. The fact that the composition product is induced by addition of integers is a version of Gauss' law. This composition product is induced by the bordism $\mathbb{P}^1 \sqcup \mathbb{P}^1 \to \mathbb{P}^1$ given by excising two points from the interior of a 3-ball. We can think of having a magnetic monopole at both of these points, where the integer value is interpreted as their magnetic fluxes, and then when we look further away, i.e. inside the large/target $\mathbb{P}^1$, the magnetic fluxes add.

I.e. our line operators ('t Hooft/Dirac line operators) correspond to $\mathbb{Z}$-graded vector spaces. This tells us that our moduli space is given by Tannakian reconstruction on $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$. Recall from example 37 that we have:

$$\mathcal{M}_{\mathcal{A}_{\mathrm{GL}_1}} = \mathrm{Spec}\left(\mathbf{Vect}^{\mathbb{Z}\text{-gr}}\right) = \mathrm{pt}/\mathbb{G}_m\ . \tag{10.217}$$

Therefore:

$$\mathrm{Map}_{\mathrm{lc}}\left(\Sigma, \mathcal{M}^1_{\mathcal{A}}\right) = \mathrm{Map}_{\mathrm{lc}}\left(\Sigma, \mathrm{pt}/\mathbb{G}_m\right) \tag{10.218}$$
$$= \mathrm{Loc}_1\left(\Sigma\right)\ . \tag{10.219}$$

So we find that $\mathcal{A}\left(\Sigma^2\right)$ is acted on by: $\mathbf{QC}\,(\mathrm{Loc}_1\,\Sigma)$, which we recall from section 6.4.2 is what the $\mathcal{B}$-side attaches to the surface Σ:

$$\mathbf{QC}\,(\mathrm{Loc}_1\,\Sigma) = \mathcal{B}\,(\Sigma)\ . \tag{10.220}$$

So we find that the magnetic theory $\mathcal{A}\,(\Sigma)$ is acted on by the electric theory $\mathcal{B}\,(\Sigma)$. But this isn't what we said. We said they were equivalent. So what we then show is that $\mathcal{A}\,(\Sigma)$ is a rank 1 free module, so isomorphic to $\mathcal{B}\,(\Sigma)$ once we have chosen a generator, i.e. we just need to keep track of the unit.

Remark 69. We talked about E-M duality for a while as this Fourier transform involving abelian group theory. What we have discovered here is that you don't technically need all of that abelian group theory. Just starting from the field theory $\mathcal{A}$, the spectral/electric side is something we can just build intrinsically from $\mathcal{A}$.

So the punchline is that the automorphic side is:

$$\mathcal{A}_G\left(\Sigma^2\right) = \mathbf{Shv}\left(\mathrm{Bun}_G\left(\Sigma\right)\right) \tag{10.221}$$

(where $\mathcal{A}_G$ is the GL-twisted $\mathcal{N} = 4$ super Yang-Mills theory with parameter $\Psi = 0$) and then out of this we build some kind of spectral side $\mathbf{QC}\left(-\right)$ which acts on $\mathcal{A}_G\left(\Sigma\right)$. The Geometric Langlands Conjecture says that these are the same/closely related.

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10.8.5 Topological Maxwell theory

transition

Recall we had a 4-dimensional TFT $\mathcal{Z}$, and for a topological surface Σ, we looked at $\mathcal{Z}\left(\Sigma\right)$. This had an algebra of symmetries, i.e. the algebra of observables $\mathcal{Z}\left(S^2\right)$, which acts on $\mathcal{A}\left(\Sigma\right)$. In particular, for every $x \in \Sigma$ we can remove a ball around

$$x \in \Sigma \times \{i\} \subset \Sigma \times I \tag{10.222}$$

to get a bordism $\Sigma \sqcup S^2 \to \Sigma$ which $\mathcal{Z}$ sends to an action $\mathcal{Z}\left(S^2\right) \otimes \mathcal{Z}\left(\Sigma\right) \to \mathcal{Z}\left(\Sigma\right)$. I.e. we get an action for every point $x \in \Sigma$. Then given a path from x to y, we get an identification of the corresponding actions, and similarly given a path between paths (and so on) we get an identification of the identifications (and so on). So, from the TFT, we got an action of a commutative algebra (in categories) labelled by $x \in \Sigma$, relations given by paths, relations between relations from paths between paths, and so on.

This is the following general construction. Given R a commutative algebra (in any symmetric monoidal (higher) category $(\mathcal{C}, \otimes)$ with certain colimits) and Σ a topological space. Then, thinking of Σ just as a simplicial set, we can construct $R \otimes \Sigma$ which is again a commutative algebra in $\mathcal{C}$. In short this is $R^{\otimes \#\mathrm{vertices}(\Sigma)}$ with relations given by the 1-simplices, and so on. This is called the *factorization homology over* Σ *of* R, and it is written

$$\int_\Sigma R = R \otimes \Sigma\ . \tag{10.223}$$

This has a concrete geometric description when the commutative ring R is nice (Tannakian):

$$R \otimes \Sigma = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}\left(\Sigma, \mathrm{Spec}\left(R\right)\right)\right)\ . \tag{10.224}$$

For R a nice tensor category (rather than commutative ring) we have:

$$R \otimes \Sigma = \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(\Sigma, \mathrm{Spec}\left(R\right)\right)\right)\ . \tag{10.225}$$

Now we will use this to study gauge theories. Recall we were in the setting of topological Maxwell theory $\mathcal{A}_{\mathrm{GL}_1}$. This had no local operators, but has interesting line operators, i.e. codimension 3 defects. We saw these defects in section 4.5.1. Given a 3-manifold M^3, we can create a Dirac monopole at a point, and we get a new version of the vector space $\mathcal{A}\left(M\right)$ by insisting on a certain magnetic flux through the 2-sphere around this point. Another variant of the same idea is as follows. Consider M^3 and allow time to pass, i.e. consider $M^3 \times I$, and a codimension 3 singularity will be a loop/knot in $M^3 \times I$. The link of this singularity will be S^2, and now we can study bundles on this cylinder $M^3 \times I$, which have a certain integral/Chern class/flux through this 2-sphere. The former manifestation of line operators modified the vector space attached to M^3, and the latter picture gave us operators between $\mathcal{Z}(M)$ and itself.

We know that line operators comprise $\mathcal{Z}\left(S^2\right)$, which in this case is given by:

$$\mathcal{A}_{\mathrm{GL}_1}\left(S^2\right) = \text{sheaves on } \operatorname{Pic}\left(S^2\right) = \mathbf{Vect}^{\mathbb{Z}\text{-gr}} . \tag{10.226}$$

I.e. given a 3-manifold M, a point $p \in M$, and a $\mathbb{Z}$-graded vector space, then we get a new version of the Hilbert space attached to M, which is a weighted sum where we look at different charges/monopole numbers. We also have the graded dimension:

$$\dim^{\mathrm{gr}} \colon \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \to \mathbb{C}\mathbb{Z} , \tag{10.227}$$

i.e. given a $\mathbb{Z}$-graded vector space we get a complex-linear combination of integers. Recall that we labelled knots in $M \times I$ with complex-linear combinations of monopole numbers. So these graded vector spaces give us these line operators, which we can wrap around an honest knot/loop in a 4-manifold.

The point is that in Gauge theory, there is lots of room for codimension 3 defects. In more detail: in dimension 3 and above we can look at codimension 3 defects, the link is a 2-sphere, and the solutions to the Yang-Mills equations on S^2 are interesting. Here we're talking about the abelian case, where it's just given by the monopole number, and now we will consider the nonabelian case. Operators of this general type are called *'t Hooft operators*.

So we have that $\mathcal{A}_{\mathrm{GL}_1}(\Sigma)$ is acted on by $\mathbf{Vect}^{\mathbb{Z}\text{-gr}}$ for every point $x \in \Sigma$. We can assemble these actions into an action of $\Sigma \otimes \mathbf{Vect}^{\mathbb{Z}\text{-gr}}$, which can be written as:

$$\begin{aligned} \Sigma \otimes \mathbf{Vect}^{\mathbb{Z}\text{-gr}} &= \Sigma \otimes \mathbf{Rep}\left(\mathbb{G}_m\right) && (10.228)\\ &= \Sigma \otimes \mathbf{QC}\left(\mathrm{pt}/\mathbb{G}_m\right) && (10.229)\\ &= \mathbf{QC}\left(\mathrm{Maps}_{\mathrm{lc}}\left(\Sigma, \mathrm{pt}/\mathbb{G}_m\right)\right) && (10.230)\\ &= \mathbf{QC}\left(\mathrm{Loc}_{\mathbb{G}_m}(\Sigma)\right) . && (10.231) \end{aligned}$$

This category was what the $\mathcal{B}$-side/electric/spectral side attached to Σ:

$$\mathcal{B}_{\mathrm{GL}_1}(\Sigma) \coloneqq \Sigma \otimes \mathcal{A}_{\mathrm{GL}_1}\left(S^2\right) = \mathbf{QC}\left(\mathrm{Loc}_{\mathbb{G}_m}(\Sigma)\right) . \tag{10.232}$$

So the $\mathcal{B}$/electric/spectral side acts on the $\mathcal{A}$/magnetic/automorphic side. This reduces to the statement that these theories are equivalent (under a Fourier transform) once we realize that $\mathcal{A}$ is a free rank 1 module over $\mathcal{B}$.

We can see how this works at the level of the Fourier transform. We're used to the Fourier transform giving us an isomorphism:

$$(\mathrm{Fun}(G), *) \simeq (\mathrm{Fun}(G^\vee), \cdot) , \tag{10.233}$$

but we can rephrase it as saying that $\mathrm{Fun}(G)$ is a rank 1-module over $\mathrm{Fun}(G^\vee, \cdot)$ with a distinguished unit. I.e. we have an action:

$$(\mathrm{Fun}(G^\vee, \cdot)) \circlearrowright \mathrm{Fun}(G) \tag{10.234}$$

given by convolution with characters. Then we claim that $\mathrm{Fun}(G^\vee)$ is in fact a rank 1 free module. For G a group, $\delta_e \in \mathrm{Fun}(G)$ is the unit, so we can act on δ_e to get:

$$\mathrm{Fun}(G^\vee) \xrightarrow{\sim} \mathrm{Fun}(G) \ . \tag{10.235}$$

Then the Fourier-transform says that this is an isomorphism. So we want to shift our point of view to think of the Fourier transform in this way.

In the Betti version, for Σ a Riemann surface, we attached a category:

$$\mathcal{A}_{\mathrm{GL}_1}(\Sigma) = \mathbf{Loc}(\mathrm{Pic}(\Sigma)) \ . \tag{10.236}$$

Now we need to pick a unit in order to get an identification with the thing it is a module over. We want to say the skyscraper at the unit of Pic: $\mathcal{O} \in \mathrm{Pic}$. In the de Rham version we would have $\mathcal{A}_{\mathrm{GL}_1}(\Sigma) = \textbf{D-Mod}(\mathrm{Pic})$, so we can literally just take this. But in the Betti version, this skyscraper isn't actually in $\mathbf{Loc}(\mathrm{Pic}(\Sigma))$ so we need to find a replacement as follows. We have the inclusion:

$$i\colon \{\mathcal{O}\} \hookrightarrow \mathrm{Pic} \ , \tag{10.237}$$

and then we can take the universal cover of Pic (with respect to the basepoint $\mathcal{O}$) which gives a contractible space $\widetilde{\mathrm{Pic}}$ mapping to Pic:

$$\begin{tikzcd} \{\mathcal{O}\} \arrow[r, hook, "i"] & \mathrm{Pic} \\ \widetilde{\mathrm{Pic}} \arrow[ur, "\widetilde{\pi}"] & \end{tikzcd} \tag{10.238}$$

and we push the constant sheaf forward to get the unit:

$$\widetilde{\pi}(\underline{k}) \in \mathbf{Loc}(\mathrm{Pic}(\Sigma)) \ . \tag{10.239}$$

Example 67. Recall $\mathrm{Loc}(S^1) \simeq \mathbf{QC}(\mathbb{G}_m)$. The unit $\mathcal{O}_1 \in \mathbf{QC}(\mathbb{G}_m)$ corresponds to the following local system on the circle. Consider the universal cover:

$$p\colon \mathbb{R} \to S^1 \ . \tag{10.240}$$

Then we can push the constant sheaf forward to get the local system corresponding to the unit:

$$p_*\underline{k} \ . \tag{10.241}$$

This is kind of a weird object, for example it is not finite rank (the fiber over any point is $\mathbb{Z}$), but the claim is that acting on this unit object gives the Fourier transform.

The theory $\mathcal{A}_G$ will linearize Bun_G, where G is a reductive group over $\mathbb{C}$ (e.g. $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$, $\mathbb{E}_8$, etc.). The slogan was that the theory $\mathcal{A}_G$ studies the *topology* of Bun_G.

Example 68. For F a number field, recall from section 5.2.2 that

$$\mathcal{A}_G(\mathrm{Spec}\,\mathcal{O}_F) = H^*(\mathrm{Bun}_G(\mathrm{Spec}\,\mathcal{O}_F)) \ , \tag{10.242}$$

where we were writing:

$$\mathrm{Bun}_G(\mathrm{Spec}\,\mathcal{O}_F) = G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_{\mathbb{A}}) \ . \tag{10.243}$$

Example 69. For a function field $C/\mathbb{F}_q$, we can study compactly-supported complex-valued functions on Bun_G:

$$\mathbb{C}_c\left[\mathrm{Bun}_G\left(C\right)\left(\mathbb{F}_q\right)\right] \ . \tag{10.244}$$

So we're thinking of $\mathcal{A}_G$ as a theory which linearizes spaces of G-bundles. There is a well-defined 4-dimensional TFT $\mathcal{A}_G$ in the sense of physics, called the 4-dimensional $\mathcal{N}=4$ super Yang-Mills with gauge group G_c (compact form of G) in the "A-twist". This was studied in the seminal paper [KW07]. There is not a fully-developed mathematical definition of $\mathcal{A}_G$, but we do know how to evaluate the theory on surfaces. If C is a Riemann surface then we study the moduli stack $\mathrm{Bun}_G\left(C\right)$, and attach to it a category of sheaves:

$$\mathbf{Shv}\left(\mathrm{Bun}_G\left(C\right)\right) \ , \tag{10.245}$$

which will have various flavors (Betti (constructible sheaves), de Rham ($\mathcal{D}$-modules), restricted, etc.) but they will always be *topological* sheaves. So we are studying the topology of $\mathrm{Bun}_G\left(C\right)$, rather than its algebraic geometry.

Now the key point is that we can study the line operators in $\mathcal{A}_G$.

Remark 70. There are "no" local operators (except in some derived sense, and even then they are not very interesting), so the line operators are where everything interesting is happening.

The main principle is to study the 't Hooft monopoles. I.e. that we have interesting codimension 3 singularities in Yang-Mills theory which correspond to studying the link of a line in 4-space, i.e. a 2-sphere. The line operators are labelled by G-bundles on this sphere, which after choosing a complex structure we can think of as $\mathbb{P}^1$. The Grothendieck-Birkhoff factorization tells us the following about bundles on $\mathbb{P}^1$. Take $G=\mathrm{GL}_n$ for simplicity. Then the factorization says that if V is a GL_n-bundle on $\mathbb{P}^1$ (rank n vector bundle) then V is a direct sum of line bundles:

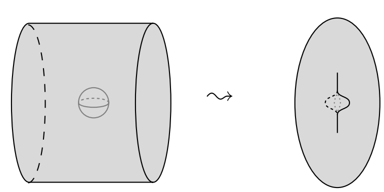
$$V \simeq L_1 \oplus L_2 \oplus \cdots \oplus L_n \ , \tag{10.246}$$

uniquely up to permutation. In other words the points of this are in bijection with the co weight lattice associated to GL_n:

$$\mathrm{Bun}_n\,\mathbb{P}^1 \simeq \mathbb{Z}^n/S_n \ , \tag{10.247}$$

where S_n is the symmetric group. So these vectors bundles on $\mathbb{P}^1$ all give me interesting line operators, and are labelled by something combinatorial.

We want to write this in terms of algebraic geometry, but the problem is that we are crossing our surface C with a time interval I which doesn't really make sense from the point of view of algebraic geometry. So we will reinterpret this picture by collapsing the time dimension:



. (10.248)

Formally, when we collapse time, we are passing from:

$$(C\times I)\setminus x \qquad \rightsquigarrow \qquad C\sqcup_{C\setminus x} C \ . \tag{10.249}$$

Recall we can think of local operators as endomorphisms of the trivial line defect. After we have collapsed time, this picture is telling us a higher dimensional version of the same statement: **line operators are endomorphisms of the trivial surface defect.** I.e. consider the trivial surface defect:

$$\{x\} \times I \times \mathbb{R} \subset C \times I \times \mathbb{R} \ . \tag{10.250}$$

Then the line defect $\{x\} \times \{i\}$ can be thought of as an endomorphism of the trivial surface defect. The point is that when we instead think of a line operator as coming from $C \sqcup_{C\setminus x} C$, this is still an endomorphism of the trivial surface defect.

We can describe G-bundles on $C \sqcup_{C\setminus x} C$:

$$\begin{aligned}
\mathrm{Bun}_G\left(C \sqcup_{C\setminus x} C\right) &= \mathrm{Bun}_G\left(C\right) \times_{\mathrm{Bun}_G(C\setminus x)} \mathrm{Bun}_G\left(C\right) &(10.251)\\
&= \left\{P_1, P_2 \in \mathrm{Bun}_G\left(C\right), P_1|_{C\setminus x} \xrightarrow{\sim} P_2|_{C\setminus x}\right\} \ , &(10.252)
\end{aligned}$$

i.e. such a bundle consists of a pair of bundles on C, and an isomorphism away from x. There is a correspondence of spaces of G-bundles, i.e. fields for our theory:

$$\begin{tikzcd}
& \mathcal{H}\mathrm{ecke}_x = \mathrm{Bun}_G\left(C \sqcup_{C\setminus x} C\right) \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G\left(C\right) & & \mathrm{Bun}_G\left(C\right) \ .
\end{tikzcd} \tag{10.253}$$

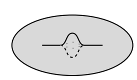
Then we can think of this correspondence as defining an operator on any linearization of Bun_G to itself, by pulling and pushing. So this is how we will interpret the actions of line operators. Explicitly

$$\mathcal{A}_G\left(C\right) = \mathbf{Shv}\left(\mathrm{Bun}_G\left(C\right)\right) \tag{10.254}$$

for some topological notion of sheaves, and then we will construct operators on this using pull pull for this Hecke correspondence.

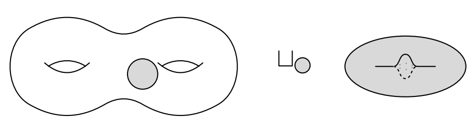
We can think of passing from C to $C \sqcup_{C\setminus x} C$ as gluing a bubble:

$$D \sqcup_{D\setminus x} D \quad = \tag{10.255}$$



into C along some disk D around $x \in C$:

$$C \sqcup_{C\setminus x} C = C \sqcup_D \left(D \sqcup_{D\setminus x} D\right) \quad = \tag{10.256}$$



In terms of G-bundles this means we can write:

$$\begin{aligned}
\mathcal{H}\mathrm{ecke}_x &= \mathrm{Bun}_G\left(C \sqcup_{C\setminus x} C\right) &(10.257)\\
&= \mathrm{Bun}_G\left(C\right) \times_{\mathrm{Bun}_G(C\setminus x)} \mathrm{Bun}_G\left(C\right) &(10.258)\\
&= \mathrm{Bun}_G\left(C\right) \times_{\mathrm{Bun}_G(D)} \left(\mathrm{Bun}_G\left(D\right) \times_{\mathrm{Bun}_G(D^*)} \mathrm{Bun}_G\left(D\right)\right) \ . &(10.259)
\end{aligned}$$

I.e. instead of describing the elements of $\mathcal{H}\mathrm{ecke}_x$ as pairs of G-bundles with an isomorphism away from x, we can think of them as a single bundle with a modification along the disk D.

Recall $\mathrm{Bun}_G(C,x)$ is the space of G-bundles equipped with a trivialization on a disk around $x \in C$. Then we can rewrite $\mathcal{H}\mathrm{ecke}_x$ as: the product of $\mathrm{Bun}_G(C,x)$ with the space of all modifications of the trivial G-bundle on the disk, where we quotient out by the group of changes of trivialization $G(\mathcal{O}_x)$:

$$\mathcal{H}\mathrm{ecke}_x = \left(\mathrm{Bun}_G(C,x) \times \text{modifications}\right)/G(\mathcal{O}_x) \ . \tag{10.260}$$

This is really a version of something we already saw, e.g. in Theorem 5:

$$\mathrm{Bun}_G = G(F_x)\backslash {\prod_{x\in C}}' G(K_x) / \prod_{x\in C} G(\mathcal{O}_x) \ . \tag{10.261}$$

When we trivialize around $x \in C$ we play the same game except we don't mod out by one of the factors on the right:

$$\mathrm{Bun}_G(C,x) = G(F)\backslash {\prod_{y}}' G(K_y) / \prod_{y\neq x} G(\mathcal{O}_x) \ . \tag{10.262}$$

Then this space carried an action of the whole loop group $G(K_x)$. So we have $\mathrm{Bun}_G(C)$, which has a principal bundle for the group $G(\mathcal{O}_x)$:

$$\begin{tikzcd} \mathrm{Bun}_G(C,x) \arrow[d, "G(\mathcal{O}_x)"] \\ \mathrm{Bun}_G(C) \end{tikzcd} \tag{10.263}$$

and the total space has an action of the larger group $G(K_x)$. In other words, $\mathrm{Bun}_G(C)$ is the quotient of a $G(K_x)$-space by the subgroup $G(\mathcal{O}_x) \subset G(K_x)$, which is exactly where Hecke modifications come from in general. So we have that:

$$\mathcal{H}\mathrm{ecke}_x \simeq \left(\mathrm{Bun}_G(C,x) \times G(K_x)/G(\mathcal{O}_x)\right)/G(\mathcal{O}_x) \ , \tag{10.264}$$

i.e. this is the associated $G(K)/G(\mathcal{O})$-bundle over Bun_G to the principal $G(\mathcal{O})$-bundle $\mathrm{Bun}_G(C,x)$.

So we have:

$$\begin{tikzcd} \mathrm{Bun}_G(C,x) \arrow[d, "G(\mathcal{O}_x)"] \arrow[loop right, "G(K_x)"] \\ \mathrm{Bun}_G(C) \end{tikzcd} \ , \tag{10.265}$$

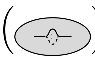
and this gives an action of the Hecke groupoid:

$$\begin{aligned} G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x) &= \mathrm{Bun}_G\left(D \sqcup_{D\backslash x} D\right) && (10.266)\\ &= \text{modifications of bundles on } D && (10.267)\\ &= \left\{P_1, P_2 \in \mathrm{Bun}_G(D),\, P_1|_{D^*} \xrightarrow{\simeq} P_2|_{D^*}\right\} \ . && (10.268) \end{aligned}$$

The punchline of all of this is the following. In the physics picture

$$\mathcal{A}_G\left(S^2\right) = \mathbf{Shv}\left(\mathrm{Bun}_G\left(S^2\right)\right) \tag{10.269}$$

acted by line operators. In the group theory picture, we have an action of this Hecke guy, i.e. sheaves on G-bundles on the ravioli:

$\mathbf{Shv}\left(G\left(\mathcal{O}\right)\backslash G\left(K\right)/G\left(\mathcal{O}\right)\right) = \mathbf{Shv}\Big(\mathrm{Bun}_G\Big($  $\Big)\Big) \ .$ (10.270)

But the idea is that these two aren't far from each other once we put enough filling into the ravioli.

This is the definition of the *spherical Hecke category*:

$$\mathbf{Sph} \coloneqq \mathbf{Shv}\left(G\left(\mathcal{O}\right)\backslash G\left(K\right)/G\left(\mathcal{O}\right)\right) \ . \tag{10.271}$$

This is a central object in the geometric Langlands program. This acts on $\mathbf{Shv}\left(\mathrm{Bun}_G C\right)$ by Hecke operators. So we saw this adélic description from the number theory, which said that Bun_G had lots of symmetries realized by correspondences: it isn't a group, but it's a quotient of a space with a group action. This is exactly when we have see we get this leftover symmetry realized by Hecke operators. Then we can view these Hecke operators as line operators for the theory, but line operators are commutative. So $\mathbf{Sph}$ is a commutative tensor-category even though Hecke algebras/categories are generally not commutative.

move example?

Example 70 (Classical Hecke correspondences)**.** Let V be a rank n vector bundle on C, and let $x \in C$. Now we want to "modify V at x". The easiest thing to do is to look at:

$$V\left(-x\right) = V \otimes \mathcal{O}\left(-x\right) \hookrightarrow V \ , \tag{10.272}$$

which is defined to have sections given by the sections of V which vanish at x. This is what we did in the abelian case, i.e. given a line bundle this is just modifying the corresponding divisor.

The following is a more interesting modification. Let W be a k-dimensional subspace of $V|_x$. Then we can modify W to form $V\left(-W\right)$ which has sections given by sections of V which are in W at x, i.e.:

$$\Gamma\left(V\left(-W\right)\right)\left\{s \in \Gamma\left(V\right) \mid s|_x \in W \subset V|_x\right\} \ . \tag{10.273}$$

This is in fact a vector bundle, and is a sub bundle of V.

This gives us a way of modifying a vector bundle at a point for any choice of subspace W, i.e. a choice of a point in the Grassmannian of k-dimensional subspaces of the fiber:

$$W \in \mathrm{Gr}_k\left(V|_x\right) \ . \tag{10.274}$$

Therefore we have a correspondence:

$$\begin{tikzcd} & \mathcal{H}\mathrm{ecke}_x^k \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n\left(C\right) & & \mathrm{Bun}_n\left(C\right) \end{tikzcd} \tag{10.275}$$

where

$$\mathcal{H}\mathrm{ecke}_x^k = \text{“}\,\mathrm{Gr}_k\left(E|_x\right)\text{”} = \left\{\left(V, W \in \mathrm{Gr}_k\left(V|_x\right)\right)\right\} \tag{10.276}$$

where E is the universal bundle. I.e. this is a Grassmannian bundle over $\mathrm{Bun}_n(C)$ in two ways, defining the above correspondence:

$$\begin{tikzcd} & \mathcal{H}\mathrm{ecke}_x^k & \\ & (V, W \in \mathrm{Gr}_k(V|_x)) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ V & & V(-W) \end{tikzcd} . \tag{10.277}$$

So $\mathrm{Bun}_n(C)$ doesn't have that many maps from it back to itself, as we might have expected if it was a group, but it is the quotient of a G-space, so it carries lots of interesting correspondence from it to itself, for example here we got one for any non-negative integer $k \leq n$. These are the basic Hecke modifications, and they will generate the lattice of all Hecke operators.

Chapter 11

Geometric Satake

transition

11.1 General theme

Recall the general theme that when given $X \circlearrowleft X \supset K$, we have an action of a groupoid:

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$$\begin{tikzcd} X/K \arrow[loop right] \arrow[dr] & & K\backslash G/K \arrow[dl] \\ & \mathrm{pt}/K & \end{tikzcd} \tag{11.1}$$

or equivalently we have a correspondence given by $\mathcal{H}$ecke:

$$\begin{tikzcd} & \mathcal{H}\text{ecke} \arrow[dl] \arrow[dr] & \\ X/K & & X/K \end{tikzcd} , \tag{11.2}$$

where $\mathcal{H}$ecke is the G/K-bundle over X/K associated to the principal K-bundle $X \to X/K$, i.e. we think of G/K as a space with a K-action, and then we have that:

$$\mathcal{H}\text{ecke} = X \times^K G/K \ , \tag{11.3}$$

where $\times^K$ denotes the balanced product, i.e. we have quotiented out by the right action of K on X and the left action of K on G/K.

11.2 Affine Grassmannian

For $x \in C$ we will construct a version of $\mathcal{H}$ecke, called $\mathcal{H}\text{ecke}_x$:

$$\begin{tikzcd} & \mathcal{H}\text{ecke}_x \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(C) & & \mathrm{Bun}_G(C) \end{tikzcd} . \tag{11.4}$$

This is a fibration with fibers given by the *affine Grassmannian*:

$$\begin{aligned}\mathcal{G}\mathrm{r}_G &\coloneqq G\left(K_x\right)/G\left(\mathcal{O}_x\right) &(11.5)\\ &= LG/LG_+ \ . &(11.6)\end{aligned}$$

We can think that:

$$G\left(K_x\right) = LG = \mathrm{Map}\left(D^*, G\right) \tag{11.7}$$

and inside of this we have:

$$G\left(K_x\right) \supset G\left(\mathcal{O}_x\right) = LG_+ = \mathrm{Map}\left(D, G\right) \ . \tag{11.8}$$

For any curve C, we have that $\mathcal{G}\mathrm{r}$ can be written:

$$\begin{aligned}\mathcal{G}\mathrm{r}_G &= LG/LG_+ &(11.9)\\ &= G\text{-bundles on } C + \text{ triv. on } C\setminus x &(11.10)\\ &= G\text{-bundles on } D + \text{ triv. on } D\setminus x \ , &(11.11)\end{aligned}$$

i.e. this consists of modifications of the trivial bundle on D at the point x.

We can write $\mathcal{H}\mathrm{ecke}_x$ explicitly as the collection of triples:

$$\begin{aligned}\mathcal{H}\mathrm{ecke}_x &= \left\{P_1, P_2 \in \mathrm{Bun}_G\left(X\right), P_1|_{C\setminus x} \xrightarrow{\sim} P_2|_{C\setminus x}\right\} &(11.12)\\ &= \left\{P_1,\, P_2|_D\,, P_1|_{D^*} \xrightarrow{\sim} P_2|_{D^*}\right\} \ . &(11.13)\end{aligned}$$

The top description makes it clear we have two maps to $\mathrm{Bun}_G\left(C\right)$ (one sends the triple to P_1, the other to P_2) but the bottom description tells us that $\mathcal{H}\mathrm{ecke}$ is associated to the principal LG_+-bundle over Bun_G given by $\mathrm{Bun}_G\left(C, x\right)$ (the space of G-bundles with a trivialization on D_x). I.e. we have that:

$$\mathcal{H}\mathrm{ecke}_X = \mathrm{Bun}_G\left(C, x\right) \times^{LG_+} \mathcal{G}\mathrm{r} \ . \tag{11.14}$$

I.e. we start with $P \in \mathrm{Bun}_G$ and trivialize on D, so we have the pair P and a trivialization in $\mathrm{Bun}_G\left(C, x\right)$. Then we modify the trivial bundle on D to get P and P' in $\mathrm{Bun}_G\left(C, x\right)\times\mathcal{G}\mathrm{r}$. Then we mod out by $G\left(\mathcal{O}\right) = LG_+$.

So $\mathcal{H}\mathrm{ecke}_x$ is a twisted $\mathcal{G}\mathrm{r}$-bundle over Bun_G in two ways:

$$\begin{tikzcd} & \mathcal{H}\mathrm{ecke}_x \arrow[dl, "\mathrm{Gr}"'] \arrow[dr, "\mathrm{Gr}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd} , \tag{11.15}$$

e.g. by mapping to P versus P'.

What is this good for? $\mathcal{H}\mathrm{ecke}$ is a groupoid, so we might try to use it to act on Bun_G. It's hard to identify points in $\mathcal{H}\mathrm{ecke}$, but we can pick out orbits. Locally it looks like $\mathcal{G}\mathrm{r}\times\mathrm{Bun}_G$ once we choose a trivialization. If we change the trivialization then the identification changes by the left action $LG_+ \circlearrowright \mathcal{G}\mathrm{r}$. This is just a general fact about associated bundles: any LG_+-invariant object on $\mathcal{G}\mathrm{r}$ makes sense over the entire space $\mathcal{H}\mathrm{ecke}_x$. For example if you take any LG_+-orbit $\mathbb{O} \subset \mathcal{G}\mathrm{r}$, i.e. $\mathbb{O}$ corresponds to a coset $LG_+ \backslash LG / LG$, then this gives a correspondence between Bun_G and itself:

$$\begin{tikzcd} & \mathbb{O} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd} . \tag{11.16}$$

So given this single object $\mathcal{H}\mathrm{ecke}_x$, we should think of it as giving many different correspondences labelled by orbits in $\mathcal{G}\mathrm{r}$.

Example 71. Recall in example 70 we took $G = \mathrm{GL}_n$ and for any integer k such that $0 \leq k \leq n$ we got a correspondence:

$$\begin{tikzcd} & \mathcal{H}\mathrm{ecke}_x^k \arrow[dl, "\mathrm{Gr}_k^n"'] \arrow[dr, "\mathrm{Gr}_k^n"] & \\ \mathrm{Bun}_n(C) & & \mathrm{Bun}_n(C) \end{tikzcd} . \tag{11.17}$$

This $\mathcal{H}\mathrm{ecke}_x^k$ is an example of the correspondences coming from the orbits in $\mathcal{G}\mathrm{r}$. In fact the closed orbits of the action of $\mathrm{GL}_n(\mathcal{O}) = L(\mathrm{GL}_n)_+$ on

$$\mathcal{G}\mathrm{r}_{\mathrm{GL}_n} = \mathrm{GL}_n(K) / \mathrm{GL}_n(\mathcal{O}) \tag{11.18}$$

are in bijection with integers $0 \leq k \leq n$, and are isomorphic to ordinary Grassmannians Gr_k^n. Concretely, $\mathcal{G}\mathrm{r}_{\mathrm{GL}_n}$ consists of rank n vector bundles V on the disk D, equipped with a trivialization on D^*. We can write this very concretely as follows. Consider the regular sections of V on the disk, $V(D)$. This sits inside of sections on the punctured disk, which is trivialized:

$$V(D) \subset V(D^*) \cong K^n , \tag{11.19}$$

so this is a subspace of K^n, i.e. n copies of Laurent series. In fact, this is not just a subspace but an $\mathcal{O}$-submodule of K^n. If we invert the parameter we get:

$$V(D) \otimes_{\mathcal{O}} K \simeq K^n . \tag{11.20}$$

In other words **the affine Grassmannian is the collection of $\mathcal{O}$-lattices inside of** K^n.

We can bound the order of poles by some N, so we find that this subspace $V(D)$ is sandwiched between:

$$t^{-N}\mathcal{O}^n \subset V(D) \subset t^N \mathcal{O}^n \subset K^n . \tag{11.21}$$

The quotient of the ends of the sandwich is a finite-dimensional vector space, so $V(D)$ gives a subspace of this vector space. It is also closed under multiplication by the coordinate. In any case, it is a closed subvariety of the finite-dimensional Grassmannian. This gives us a description of $\mathcal{G}\mathrm{r}_n$ as

$$\mathcal{G}\mathrm{r}_n = \cup_N \text{projective variety} . \tag{11.22}$$

I.e. $\mathcal{G}\mathrm{r}_n$ is an ind-proper ind-scheme. For other groups we can embed the associated $\mathcal{G}\mathrm{r}_G$ into $\mathcal{G}\mathrm{r}_n$ by choosing a faithful representation into GL_n.

For $N = 1$ we can see the finite-dimensional Gr_k^n inside as orbits.

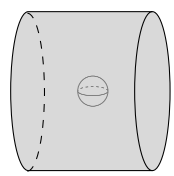
So the idea is that we're studying $\mathcal{G}\mathrm{r}$, but we only want to study the geometry which is invariant under $G(\mathcal{O})$. Any such geometry makes sense over Bun_G. For example we can consider invariant sheaves, $\mathbf{Shv}^{G(\mathcal{O})}(\mathcal{G}\mathrm{r})$, and then this maps down to $\mathbf{Shv}(\mathcal{H}\mathrm{ecke}_x)$. Therefore we get an integral kernel on Bun_G by the groupoid action of $\mathcal{H}\mathrm{ecke}_x$, i.e. we have maps:

$$\begin{array}{c} \mathbf{Shv}^{G(\mathcal{O})}(\mathcal{G}\mathrm{r}) \\ \downarrow \\ \mathbf{Shv}(\mathcal{H}\mathrm{ecke}_x) \\ \downarrow \\ \mathrm{End}(\mathbf{Shv}(\mathrm{Bun}_G)) \end{array} . \tag{11.23}$$

Again:

$$\mathbf{Shv}^{G(\mathcal{O})}(\mathcal{G}\mathrm{r}) = \mathbf{Shv}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})) =: \mathbf{Sph} . \tag{11.24}$$

So an orbit $\mathcal{O}$ gives me an operator by looking at the image of the constant sheaf on $\mathcal{O}$ under these two maps. In the physics picture we're inserting $\mathcal{O}$ at some point $x \in C$ and some time $t \in I$:



(11.25)

Associativity follows from the fact that configurations of three points on the line $\{x\} \times I$ are associative. We have commutativity because given two such points where we are placing operators, we have enough dimensions to commute the points. Now we want to distil this idea from the physics in order to see that these Hecke operators are commutative. The point will be that $\mathcal{G}\mathrm{r}$ has some extra algebraic structure, which is not of a very familiar form, coming from moving the point $x \in C$.

11.2.1 Homotopy type of $\mathcal{G}\mathrm{r}$

Recall for $G = \mathrm{GL}_1$ we had $\mathcal{G}\mathrm{r}_1 = \mathbb{Z}$. Then this is where out Fourier series description came from. So $\mathcal{G}\mathrm{r}$ is some kind of generalization of $\mathbb{Z}$. By definition, $\mathcal{G}\mathrm{r}$ consists of G-bundles on $\mathbb{A}^1$ with a trivialization on $\mathbb{A}^1 \setminus 0$. If we are only considering $\mathcal{G}\mathrm{r}$ up to homotopy, we can think of it as:

$$\mathcal{G}\mathrm{r} = G\text{-bundle on } \mathbb{R}^2 + \text{trivialization on } \mathbb{R}^d \setminus D_0 . \tag{11.26}$$

We can go a bit further and take:

$$\begin{aligned} \mathcal{G}\mathrm{r} &\simeq G_c\text{-bundles on } \mathbb{R}^2 + \text{trivialization on } \mathbb{R}^2 \setminus D_0 && (11.27) \\ &\simeq \mathrm{Map}\left(\left(\mathbb{R}^2, \mathbb{R}^2 \setminus D\right), (BG, *)\right) && (11.28) \\ &\simeq \Omega^2 BG && (11.29) \\ &\simeq \Omega G && (11.30) \\ &\simeq \Omega G_c , && (11.31) \end{aligned}$$

where D_0 is a disk around the origin, and $G_c \subset G$ is a maximal compact subgroup.

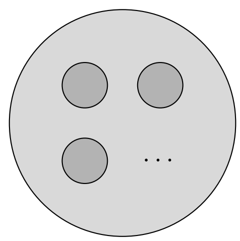
Remark 71. This is even a homeomorphism (i.e. more than just the same homotopy type) between ΩG_c and $\mathcal{G}\mathrm{r}$.

Therefore we have:

$$\pi_0\,\mathcal{G}\mathrm{r} = \pi_2\left(BG\right) = \pi_1\left(G\right) \ . \tag{11.32}$$

So $\mathcal{G}\mathrm{r}$ will be connected if G is simply-connected, e.g. $G = \mathrm{SL}_n\left(\mathbb{C}\right)$ is simply-connected so has connected $\mathcal{G}\mathrm{r}$. The more interesting thing, is that two-fold based loops on anything, $\Omega^2\left(-\right)$, is a weakly commutative group up to homotopy. This is the reason π_2 of any space is commutative.

To formalize the "commutative" multiplication on $\Omega^2 X$, we introduce the little 2-disk ($\mathbb{E}_2$) operad. In these terms $\Omega^2 X$ is an $\mathbb{E}_2$-space. Consider a configuration of k disks inside of a disk D:



$$\in \mathrm{Conf}_k\left(D\right) \ . \tag{11.33}$$

We can think of labelling each of the small disks $\{D_i\}_{i=1}^k$ with a map $\{f_i\}_{i=1}^k$ where

$$\left(f_1, \ldots, f_k\right) \in \left(\Omega^2 X\right)^k \ . \tag{11.34}$$

Then we can define a new map, which sends the complement of the little disks:

$$D \setminus \left(\bigcup_{i=1}^k D_i\right) \tag{11.35}$$

to the basepoint, and evaluates the f_i on the D_i. This defines a k-fold multiplication map:

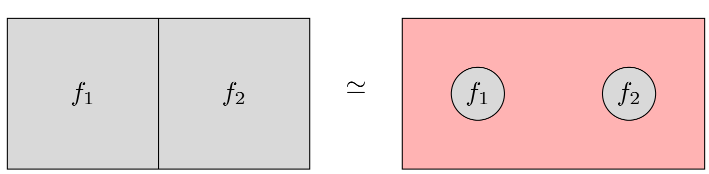
$$\begin{array}{c} \mathrm{Conf}_k\left(D\right) \times \left(\Omega^2 X\right)^k \\ \big\downarrow \\ \Omega^2 X \end{array} \ . \tag{11.36}$$

We can compose the multiplication maps as in fig. 11.1.

Digression 5. One should think of this as being analogous to the following argument from topology. For

$$f_1, f_2 \in \pi_2\left(X\right) = \pi_0\left(\Omega^2 X\right) \ , \tag{11.37}$$

we can imagine that the red portion gets mapped to the basepoint, and:



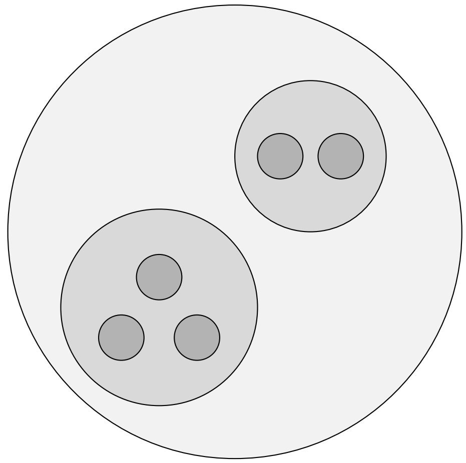
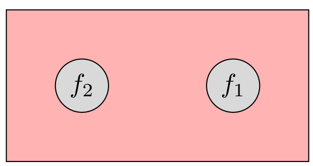
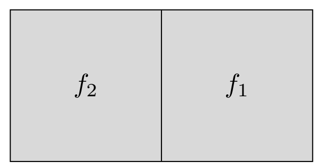


Figure 11.1: Composition of a 2-fold multiplication with a 3-fold multiplication yields a 5-fold multiplication.

$\simeq$  $\simeq$  .

So $\mathcal{G}\mathrm{r}$ is an $\mathbb{E}_2$-space. Therefore when we linearize it, we will get an $\mathbb{E}_2$-object. For example **Shv** $(\mathcal{G}\mathrm{r})$ is an $\mathbb{E}_2$-category: it has a "weakly commutative" multiplication.

Remark 72. This corresponds to how line operators for a 3-dimensional field theory form a braided $\otimes$-category. The category of braided $\otimes$-categories is equivalent to the category of $\mathbb{E}_2$-categories.

I.e. $\mathcal{G}\mathrm{r}$ is something like a commutative group in the world of homotopy theory. So we could look at some kind of topological sheaves, like local systems, but we need something more refined, e.g. the types of sheaves one might take in algebraic geometry. The claim is that $\mathcal{G}\mathrm{r}$ is some kind of commutative group in the world of algebraic geometry as well. This is the notion of a factorization space.

Remark 73. Right now we are moving points in my curve C. So we get this $\mathbb{E}_2$-structure of 2-disks in a larger 2-disk. But then we said that line operators are actually better: they're $\mathbb{E}_3$ because we have this extra time dimension. Then when we came to $\mathcal{G}\mathrm{r}$, we looked at it up to the $G(\mathcal{O})$-action. I.e. we are considering:

$$G(\mathcal{O}) \backslash \mathcal{G}\mathrm{r} = G(\mathcal{O}) \backslash \mathcal{G}\mathrm{r} / G(\mathcal{O}) \ , \tag{11.38}$$

which is actually an $\mathbb{E}_3$-space, i.e. better than $\mathbb{E}_2$.

Remark 74 (Caveat). We are ignoring the non-reduced structure of $\mathcal{G}\mathrm{r}$. If G is semisimple then $\mathcal{G}\mathrm{r}$ is reduced. Even $\mathcal{G}\mathrm{r}_{\mathrm{GL}_1}$ is actually $\mathbb{Z}\times$ some infinite-dimensional formal group.

We know that:

$$\begin{aligned} \mathcal{G}\mathrm{r}_{\mathrm{GL}_1} &= \mathrm{GL}_1(K) / \mathrm{GL}_1(\mathcal{O}) && (11.39) \\ &= K^* / \mathcal{O}^* \ , && (11.40) \end{aligned}$$

where we're thinking of both of these as infinite-dimensional schemes. From this we should have that the tangent space is:

$$T\,\mathcal{G}\mathrm{r}_{\mathrm{GL}_1} = \mathbf{Lie}\,(K^*)\,/\mathbf{Lie}\,(\mathcal{O}^*) = K/\mathcal{O} = \mathbb{C}\,((t))\,/\mathbb{C}\,[[t]] \;, \tag{11.41}$$

which is an infinite-dimensional vector space. Everything is abelian so $K/\mathcal{O}$ is an infinite-dimensional Lie algebra, so we can exponentiate $\exp(K/\mathcal{O})$, which is an infinite-dimensional formal group. Then

$$K^*/\mathcal{O}^* \simeq \mathbb{Z} \times \exp(K/\mathcal{O}) \;, \tag{11.42}$$

which is something like $\mathbb{Z} \times \widehat{\mathbb{A}^\infty}$.

Even though this doesn't show up for semisimple G, we can't ignore it. Recall $\mathrm{Pic}\,(C) = H^1\,(C, \mathcal{O}^*)$ and

$$T\,\mathrm{Pic}\,(C) = H^1\,(C, \mathcal{O}) \simeq \mathcal{O}\,(C \setminus x)\,\backslash K_x/\mathcal{O}_x \tag{11.43}$$

and

$$K/\mathcal{O} \twoheadrightarrow \mathcal{O}\,(C \setminus x)\,\backslash K_x/\mathcal{O}_x \;. \tag{11.44}$$

So we have a map from divisors at x to Pic:

$$K^*/\mathcal{O}^* \to \mathrm{Pic} \;, \tag{11.45}$$

so the extra stuff in $K_x/\mathcal{O}_x$ that we're ignoring is what makes Pic not discrete.

11.2.2 Geometric structure of $\mathcal{G}\mathrm{r}$

See [PS86] for a good reference.

The first thing we will consider is Morse-theory/Białynicki-Birula decomposition/cell decomposition of $\mathcal{G}\mathrm{r}$. The cells of the decomposition are labelled by homomorphisms to the maximal torus, which comprise the coweight lattice:

$$\left\{\mathbb{G}_m \xrightarrow{\varphi} T \subset G\right\} = \Lambda \;. \tag{11.46}$$

We get such a decomposition when we have an action of $\mathbb{G}_m$. First we get an action $\mathbb{G}_m \circlearrowright \mathcal{G}\mathrm{r}$ from rotating Laurent series. We can also pick a generic copy of $\mathbb{G}_m$ inside of T:

$$\mathbb{G}_m \xrightarrow{\rho^\vee} T \subset G \;. \tag{11.47}$$

Generic means that the centralizer $Z\,(\rho^\vee) = T$. Then

$$\mathbb{G}_m \to T \subset G \subset LG_+ \tag{11.48}$$

acts on Gr too. So we can take the diagonal action where we both rotate the disk and act by this copy of $\mathbb{G}_m$.

What are the fixed points for this combined action? We're asking for a map $D^* \to G \in G\,(K)$ which is fixed under both of the actions. So the fixed points are given by cocharacters:

$$\{\varphi \colon \mathbb{G}_m \to T\} \;. \tag{11.49}$$

These can be restricted to $D^* \subset \mathbb{G}_m$, and then we can remember that $T \subset G$. I.e. we have a map:

$$\begin{array}{c} \{\varphi\colon \mathbb{G}_m \to T\} \\ \big\downarrow \\ \{\varphi\colon D^* \to G\} \end{array} \quad , \tag{11.50}$$

i.e. a cocharacter maps to $\varphi \in G(K)$ and we can ask for the associated coset.

Example 72. For GL_n we are considering matrices:

$$\begin{pmatrix} t^{i_1} & & & \\ & t^{i_2} & & \\ & & \ddots & \\ & & & t^{i_n} \end{pmatrix} \in \mathrm{GL}_n(K) \ . \tag{11.51}$$

The upshot is that we get a cell decomposition of $\mathcal{G}\mathrm{r}$ with one fixed point at the center of each cell.

We have actually already seen this cocharacter lattice Λ. It is the Grassmannian for T:

$$\Lambda \simeq T(K)/T(\mathcal{O}) = \mathcal{G}\mathrm{r}_T \ . \tag{11.52}$$

I.e. we have:

$$\Lambda \cong \mathcal{G}\mathrm{r}_T \hookrightarrow \mathcal{G}\mathrm{r}_G \ . \tag{11.53}$$

This is the first step of reducing the story for nonabelian G to the story for abelian T.

The next step is finding the Cartan decomposition. These cells are (not quite the same, but) contained in the $G(\mathcal{O})$-orbits. The $G(\mathcal{O})$-orbits in $\mathcal{G}\mathrm{r}$ correspond to the points of Λ modulo the Weyl group:

$$G(\mathcal{O}) \backslash \mathcal{G}\mathrm{r} = LG_+ \backslash LG/LG_+ \leftrightarrow \Lambda/W \leftrightarrow \Lambda_+ \ . \tag{11.54}$$

Secretly there is an affine Weyl group W_{Aff} and:

$$\Lambda/W = W \backslash W_{\mathrm{Aff}}/W \ . \tag{11.55}$$

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11.3 Orbits in $\mathcal{G}\mathrm{r}$

transition

The Geometric Satake correspondence is the nonabelian counterpart to the theory of Fourier series. Recall the theory of Fourier series identified

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Rep}(\mathbb{G}_m) \ . \tag{11.56}$$

Then we identified the LHS as:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \simeq \mathbf{Shv}(\mathrm{Gr}_{\mathrm{GL}_1}) \ , \tag{11.57}$$

and we identified the RHS as:

$$\mathbf{Rep}(\mathbb{G}_m) \simeq \mathbf{QC}(\mathrm{pt}/\mathbb{G}_m) \simeq \mathbf{QC}\left(\mathrm{Loc}_{\mathbb{G}_m} \mathbb{P}^1\right) \ . \tag{11.58}$$

We are ignoring derived structure for now, since everything is abelian. So the reinterpretation of Fourier series is:

$$\mathbf{Shv}\left(\mathcal{G}\mathrm{r}_{\mathrm{GL}_1}\right) \simeq \mathbf{QC}\left(\mathrm{Loc}_{\mathbb{G}_m}\mathbb{P}^1\right) \ , \tag{11.59}$$

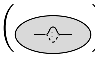
and we want to replace the LHS with something in terms of the affine Grassmannian for G:

$$\mathcal{G}\mathrm{r}_G = G\left(K\right)/G\left(\mathcal{O}\right) = LG/LG_+ \ . \tag{11.60}$$

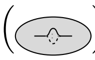
But we are only interested in (sheaves on) this up to the $G\left(\mathcal{O}\right)$-action, i.e. we're interested in the orbits:

$$\begin{aligned} G\left(\mathcal{O}\right)\backslash\,\mathcal{G}\mathrm{r}_G & \qquad (11.61)\\ = G\left(\mathcal{O}\right)\backslash G\left(K\right)/G\left(\mathcal{O}\right) & \qquad (11.62)\\ = LG_+\backslash LG/LG_+ \ . & \qquad (11.63)\end{aligned}$$

Recall we came about this in the physics as the space of G-bundles on the ravioli:

$$LG_+\backslash\,\mathcal{G}\mathrm{r}_G = LG_+\backslash LG/LG_+ = \mathrm{Bun}_G\left(\right.$$  $$\left.\right) \ . \tag{11.64}$$

We have that:

$$\mathrm{Bun}_G\left(\right.$$  $$\left.\right) = \mathrm{Bun}_G\left(\mathbb{P}^1\right) \ , \tag{11.65}$$

and we can think of $\mathbb{P}^1$ as a 2-sphere made out of a disk D glued to a copy of $\mathbb{A}^1$ away from the origin. Then LG_+ can be thought of as automorphisms of bundles on D, but then we need to mod out by changes of trivializations on $\mathbb{A}^1$, i.e. maps $\mathbb{A}^1 \to G$:

$$LG_- = \left\{\mathbb{A}^1 \to G\right\} \ . \tag{11.66}$$

Therefore:

$$LG_-\backslash LG/LG_+ = \mathrm{Bun}_G\left(\mathbb{P}^1\right) \ . \tag{11.67}$$

So the ravioli and $\mathbb{P}^1$ look topologically very similar, and the isomorphism classes of bundles on the ravioli are in bijection with isomorphism classes of bundles on $\mathbb{P}^1$. I.e. we have bijections:

$$LG_+\backslash LG/LG_+ \leftrightarrow \Lambda/W \leftrightarrow \mathrm{Bun}_G\,\mathbb{P}^1 \tag{11.68}$$

where $\Lambda = \mathrm{Hom}\left(\mathbb{G}_m, T\right)$ is the coweight lattice. So these are the same as a set, but the poset structure is very different. The orbits of $LG_+ \circlearrowright \mathcal{G}\mathrm{r}$ form a poset under the closure relation. The orbits are all finite dimensional, and each has finitely many orbits in its closure. On the other hand, $\mathrm{Bun}_G\,\mathbb{P}^1$ is the same as a set, but the poset is reversed.

Example 73. Let us consider $G = \mathrm{GL}_2$. The first Chern class is a map:

$$\mathrm{Bun}_{\mathrm{GL}_2}\left(\mathbb{P}^1\right) \to \mathbb{Z} \ . \tag{11.69}$$

For example $\deg\left(\mathcal{O}\oplus\mathcal{O}\right) = 0$, and $\deg\left(\mathcal{O}\oplus\mathcal{O}\left(1\right)\right) = 1$. $\mathrm{Bun}_{\mathrm{GL}_2}\left(\mathbb{P}^1\right)$ is disconnected, and it has connected components consisting of bundles with the same degree, i.e. there is one for every $n \in \mathbb{Z}$.

Consider the bundles of degree 0. This is identified with SL_2-bundles:

$$\mathrm{Bun}_{\mathrm{SL}_2}\left(\mathbb{P}^1\right) = \deg^{-1}\left(0\right) \ . \tag{11.70}$$

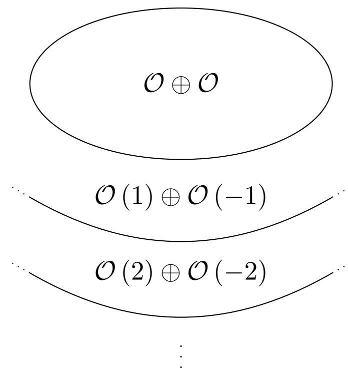


Figure 11.2: The connected component of $\mathrm{Bun}_{\mathrm{GL}_2}(\mathbb{P}^1)$ containing the trivial bundle has a single open dense orbit. The layers in this picture correspond to other strata, which are necessarily contained in the closure of the open stratum in the center. We can also think of this as a picture of the LG_- orbits on $\mathcal{G}\mathrm{r}$.

In here we have the trivial bundle $\mathcal{O} \oplus \mathcal{O}$, and other bundles such as $\mathcal{O}(1) \oplus \mathcal{O}(-1)$. This is a connected space, with an open, dense stratum containing $\mathcal{O} \oplus \mathcal{O}$. So all other strata are contained in the closure of the one containing the trivial bundle. So the poset looks like $\mathbb{N}$. The picture is as in fig. 11.2. The poset structure on $\mathcal{G}\mathrm{r}$ is the opposite: we have one closed orbit in the closure of everything else.

We can also think of fig. 11.2 as a picture of the LG_- orbits in $\mathcal{G}\mathrm{r}$: there is this open orbit, and all of the orbits are finite codimension. As opposed to the LG_+ orbits in $\mathcal{G}\mathrm{r}$, where the orbits are finite-dimensional, and the closure relations go the other way.

11.4 Equivariant sheaves on $\mathcal{G}\mathrm{r}$

Inside $\mathcal{G}\mathrm{r}_G$ we have this coweight lattice:

$$\Lambda \simeq \mathcal{G}\mathrm{r}_T \subset \mathcal{G}\mathrm{r}_G \tag{11.71}$$

which sits as "centers of cells", i.e. it sits as the fixed points for the $\mathbb{G}_m$-action. Again, $LG_+ \backslash \mathcal{G}\mathrm{r}_G$ is in bijection with Λ/W. Λ/W looks like irreducible representations of a group $G^\vee$ with maximal torus $T^\vee$ dual to T, i.e. we have that

$$\Lambda = \mathrm{Hom}(T^\vee, \mathbb{G}_m) \ . \tag{11.72}$$

We will (Tannakian re) construct this dual group $G^\vee$ from LG_+-equivariant sheaves on $\mathcal{G}\mathrm{r}_G$:

$$\mathbf{Sph} = \mathbf{Shv}_{LG_+}(\mathcal{G}\mathrm{r}_G) \ . \tag{11.73}$$

By **Shv** we mean the abelian category of either $\mathcal{D}$-modules, or what are known as *perverse sheaves*. The idea is that, on every orbit, these sheaves will have a vector bundle with flat connection, or equivalently a local system shifted by the dimension of the orbit. These categories both have simple objects which are in bijection with irreducible local systems on the orbits. The LG_+-orbits are not quite the cells, but they are paved by them. In particular all of the LG_+-orbits are simple connected. In fact we can describe them explicitly as affine bundles over partial flag manifolds for G. This implies that the simple objects in **Sph** are in bijection with the orbits Λ/W.

As it turns out, **Sph** is a *semisimple category*, i.e. every object is a direct sum of simples. This has to do with *parity* of dimensions. I.e. the parity of the $\mathbb{C}$-dimension of the orbits is constant on each component. The point is that this limits the possible interaction between a sheaf on one orbit, and a sheaf on an orbit in its closure: an orbit in the closure is complex codimension 2 away. This means there are no possible extensions, i.e. Ext^1 vanishes for sheaves on different orbits.

The upshot is that this category is very simple: every object is a direct sum of objects labelled by orbits, i.e. we have:

$$\mathbf{Sph} \simeq \mathbf{Vect}\left(\Lambda/W\right) \ , \tag{11.74}$$

as abstract categories.[1] This is good since this is what the category of representations of a reductive group looks like. Recall from that compact groups are dual to discrete groups under Pontryagin duality. When we discussed the Tannakian version, we saw that a reductive group is dual to a semisimple category. So this duality is suggesting that when we have some kind of discreteness/semi-simplicity, then the dual side will involve something like a complex reductive group.

ref

The main piece of structure is that **Sph** is a symmetric monoidal category, i.e. it has a commutative tensor product. We explained this from the point of view of line operators. This has two different structures which are compatible in some precise way. The first is the structure coming from the fact that this is sheaves on the double coset:

ref

$$\mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \ . \tag{11.75}$$

Sheaves on a space of double cosets always has an associative multiplication. There is also another structure, which is something like a braided multiplication, with a completely different origin. In particular, we saw that $\mathcal{G}\mathrm{r}_G$ has a multiplicative structure even before quotienting. I.e. up to homotopy:

$$\mathcal{G}\mathrm{r}_G = \Omega^2\left(BG\right) \ , \tag{11.76}$$

which gave it a weakly commutative multiplication. In algebraic geometry we have to say this a bit differently, using the notion of *factorization* of Beilinson-Drinfeld.

cite

11.4.1 Factorization structure

$\mathcal{G}\mathrm{r}_G$ has some adelic version, which is an algebro-geometric version of the $\mathbb{E}_2$-structure on $\Omega^2(BG)$. Then Beilinson-Drinfeld introduced the BD Grassmannian. The idea is that there is some kind of multiplicative structure encoded by the way Grassmannians fit together.

Fix a curve C (e.g. $C = \mathbb{A}^1$, $\mathbb{P}^1$, or whatever you like[2]). Then we will define something called $\mathcal{G}\mathrm{r}_{G,C^n}$ living over C^n for all n:

$$\begin{array}{c} \mathcal{G}\mathrm{r}_{G,C^n} \\ \big\downarrow \\ C^n \end{array} \quad , \tag{11.77}$$

[1]The identification $\mathbf{Sph} \simeq \mathbf{Vect}(\Lambda/W)$ is not a natural identification. In particular, **Sph** is monoidal, and $\mathbf{Vect}(\Lambda/W)$ is a bad model in terms of the monoidal structure.

[2]The curve will not affect the final answer.

as follows.

For $n = 1$ this is:

$$\begin{tikzcd} \mathcal{G}\mathrm{r}_{G,C} \arrow[d] \\ C \end{tikzcd} \quad , \tag{11.78}$$

where the fiber over $x \in C$ consists of G-bundles on C equipped with a trivialization on $C \setminus x$. So we have this interpretation of $\mathcal{G}\mathrm{r}$ as bundles with a trivialized away from the origin. Then Beilinson-Drinfeld let the origin be any point x, and end up with this bundle over C. $\mathcal{G}\mathrm{r}_{G,C}$ is the associated bundle for the action of changes of trivialization on the disk on $\mathcal{G}\mathrm{r}_G$: $\mathrm{Aut}\,(D) \circlearrowright \mathcal{G}\mathrm{r}_G$.

For $n = 2$,

$$\begin{tikzcd} \mathcal{G}\mathrm{r}_{G,C^2} \arrow[d] \\ C^2 \end{tikzcd} \quad , \tag{11.79}$$

the fiber over $(x, y) \in C^2$ consists of G-bundles on C equipped with a trivialization on $C \setminus \{x, y\}$. Note that even though the pair (x, y) is ordered, this is invariant under the symmetric group action. What is really interesting, is that we have two strata in C^2. We have the open stratum:

$$C^2 \setminus \Delta = \{x \neq y\} \tag{11.80}$$

where

$$\mathcal{G}\mathrm{r}_{G,C^2}\big|_{x,y} \simeq \mathcal{G}\mathrm{r}_{G,C}|_x \times \mathcal{G}\mathrm{r}_{G,C}|_y \ , \tag{11.81}$$

so we have:

$$\mathcal{G}\mathrm{r}_{G,C^2}\big|_{C^2\setminus\Delta} \simeq \mathcal{G}\mathrm{r}_{G,C}|^{\times 2}_{C^2\setminus\Delta} \ . \tag{11.82}$$

If we are looking in the other stratum :

$$\{x = y\} = \Delta \subset C^2 \ , \tag{11.83}$$

then we have

$$\mathcal{G}\mathrm{r}_{G,C^2}\big|_{x,y} \simeq \mathcal{G}\mathrm{r}_{G,C}|_x \ , \tag{11.84}$$

so we have

$$\mathcal{G}\mathrm{r}_{G,C^2}\big|_{\Delta} \simeq \mathcal{G}\mathrm{r}_{G,C} \ . \tag{11.85}$$

Example 74. For $G = \mathrm{GL}_1$ we have $\mathbb{Z} \times \mathbb{Z}$ away from the diagonal, and $\mathbb{Z}$ on the diagonal. I.e. we have $\mathbb{Z} \times \mathbb{Z} \times C^2 \setminus \Delta$ glued to $\mathbb{Z} \times C$ on the diagonal. The gluing map is the group law of the integers.

We can think of this as a dynamic way of writing a group. So given two group elements $(n, m) \in \mathbb{Z} \times \mathbb{Z}$ we might want to know how they add. So we think about putting n at a point x, and m at a point y, and having those points collide. Then the geometry of $\mathcal{G}\mathrm{r}_{\mathrm{GL}_1,C^n}$ encodes the addition law. I.e. the group structure is explicitly presented by the gluing map between $\mathbb{Z} \times \mathbb{Z}$ away from the diagonal and $\mathbb{Z}$ on the diagonal.

So in general we have this family $\mathcal{G}\mathrm{r}_{G,C^n}$, which has the following two key structures.

1. $\mathcal{G}\mathrm{r}_{G,C^n}|_{\{x_i\}_{i=1}^n}$ only depends on the *subset* $\{x_i\} \subset C$. I.e. for a surjection of finite sets $I \twoheadrightarrow J$ we get an associated diagonal map $\Delta_{I \twoheadrightarrow J}$, and we have that

$$\Delta^*_{I \twoheadrightarrow J} \mathcal{G}\mathrm{r}_{C^I} \simeq \mathcal{G}\mathrm{r}_{C^J} \ . \tag{11.86}$$

For example $\Delta^* \mathcal{G}\mathrm{r}_{C^2} \simeq \mathcal{G}\mathrm{r}_C$. We can rephrase this as saying that $\mathcal{G}\mathrm{r}_{C^{(-)}}$ defines a family over some space of finite subsets of C, called the Ran space of C:

$$\mathrm{Ran}\,(C) = \operatorname*{colim}_{I \twoheadrightarrow J} C^I \ , \tag{11.87}$$

i.e. the parameter space of finite subsets of C.

2. Factorization: $\mathcal{G}\mathrm{r}$ is *multiplicative* with respect to the disjoint union operation. I.e.

$$\mathcal{G}\mathrm{r}_{G,C^{m+n}}\big|_{\{x_i\}_{i=1}^m \sqcup \{y_i\}_{i=1}^n} \simeq \mathcal{G}\mathrm{r}_{G,C^m}|_{\{x_i\}_{i=1}^m} \times \mathcal{G}\mathrm{r}_{G,C^n}|_{\{y_i\}_{i=1}^n} \ . \tag{11.88}$$

Remark 75. In topology we had an honest multiplication map: for two maps from a pointed disk to BG, we can think of them as living in a larger disk. Now we have e.g. two points in C, and we can take the limit as they collide. The result comes from the gluing we have between two copies of the Grassmannian (away from Δ) and one copy (along Δ).

This is what is called a *factorization space*. This encodes the unusual algebraic structure on $\mathcal{G}\mathrm{r}$, which reduces to the $\mathbb{E}_2$-structure in the homotopy category. This defines a multiplication on objects on $\mathcal{G}\mathrm{r}$ (e.g. sheaves, etc.) which we can take limits of. For example, given $\mathcal{F}, \mathcal{G} \in \mathbf{Shv}\,(\mathcal{G}\mathrm{r}_G)$ and $x, y \in C = \mathbb{A}^1$ we can think that $\mathcal{F}$ lives at x and $\mathcal{G}$ lives at y:

$$\mathcal{F} \in \mathbf{Shv}\left(\mathcal{G}\mathrm{r}_{G,C}|_x\right) \qquad\qquad \mathcal{G} \in \mathbf{Shv}\left(\mathcal{G}\mathrm{r}_{G,C}|_y\right) \ , \tag{11.89}$$

and this defines a sheaf:

$$\mathcal{F} \boxtimes \mathcal{G} \in \mathbf{Shv}\left(\mathcal{G}\mathrm{r}_{G,C^2}|_{C^2 \setminus \Delta}\right) \ . \tag{11.90}$$

Now we want to take the limit as $x \to y$. I.e. we want to extend this construction over the diagonal. If these sheaves were, for example, coherent then we would not be able to do this. But if the sheaves are close enough to topological sheaves, then we are in luck: forming *nearby cycles* is a topological version of such a limit. So we can define $\mathcal{F} * \mathcal{G}$ as the nearby cycles of $\mathcal{F} \boxtimes \mathcal{G}$ on $\Delta \subset C^2$.

Therefore $\mathbf{Shv}\,(\mathcal{G}\mathrm{r})$ looks roughly like an $\mathbb{E}_2$/braided $\otimes$-category, but this nearby cycles construction doesn't quite give an associative tensor product. However, if we restrict to LG_+-equivariant sheaves on $\mathcal{G}\mathrm{r}$, then we get an associative, braided tensor-structure on:

$$\mathbf{Sph} = \mathbf{Shv}_{LG_+}\,(\mathcal{G}\mathrm{r}) \ . \tag{11.91}$$

We can think of this as follows. We can "stack" modifications vertically (this is the Hecke structure), or we can do modifications at different points and collide them. So there are three directions of multiplication: the associative vertical direction, and the braided multiplication from collision. By the Eckmann-Hilton argument, this defines a commutative product, i.e. a symmetric monoidal structure on the category. We already saw this in the physics: we are studying codimension 1 defects, so points in $C \times I$, and the multiplication coming from colliding points/small 3-balls in a larger 3-ball is commutative because we have enough dimensions to move them around one another.

11.5 Tannakian reconstruction of Sph

Recall Tannakian reconstruction started with a symmetric monoidal category $(\mathcal{C}, *)$. Then to this we associated a stack $\mathrm{Spec}\,(\mathcal{C}, *)$. For decent $\mathcal{C}$, the idea was that:

$$\mathcal{C} \simeq \mathbf{QC}\,(\mathrm{Spec}\,(\mathcal{C}, *)) \ . \tag{11.92}$$

There is a (possible more familiar) variation, where we have slightly more data. If we have $(\mathcal{C}, *)$ along with a fiber functor (faithful and exact $\otimes$-functor):

$$(\mathcal{C}, *) \to (\mathbf{Vect}_k, \otimes) \ , \tag{11.93}$$

then this gives us a map:

$$\begin{array}{c} \mathrm{Spec}\,(\mathbf{Vect}_k, \otimes) = \mathrm{Spec}\,(k) \\ \big\downarrow \\ \mathrm{Spec}\,(\mathcal{C}, *) \end{array} \ , \tag{11.94}$$

which is a flat cover. This implies that:

$$\mathrm{Spec}\,(\mathcal{C}, *) \xrightarrow{\simeq} \mathrm{pt}/\,\mathrm{Gal}\,(\mathcal{C}, F) \ , \tag{11.95}$$

where

$$\mathrm{Gal}\,(\mathcal{C}, F) = \mathrm{Aut}\,(F) \ . \tag{11.96}$$

So in our situation we need some forgetful functor:

$$\mathbf{Sph} = \mathbf{Shv}_{LG_+}\,(\mathcal{G}\mathrm{r}) \to \mathbf{Vect} \ . \tag{11.97}$$

The shortest way to say this, but maybe not the most insightful one, is that this is the cohomology functor. We will take a different approach. First we will consider a functor:

$$\mathbf{Sph} = \mathbf{Shv}_{LG_+}\,(\mathcal{G}\mathrm{r}_G) \to \mathbf{Shv}\,(\mathcal{G}\mathrm{r}_T) \ . \tag{11.98}$$

This is useful because:

$$\mathcal{G}\mathrm{r}_T \simeq \Lambda \ , \tag{11.99}$$

so we can just do Fourier series:

$$\mathbf{Shv}\,(\mathcal{G}\mathrm{r}_T)\,\mathbf{Shv}\,(\Lambda) \simeq \mathbf{Rep}\,(T^\vee) \tag{11.100}$$

where $T^\vee = \mathrm{Hom}\,(\Lambda, \mathbb{G}_m)$. So by relating $\mathcal{G}\mathrm{r}_G$ to $\mathcal{G}\mathrm{r}_T$ we will get a functor not just to $\mathbf{Vect}_k$, but

$$\mathbf{Sph} \to \mathbf{Rep}\,(T^\vee) \ . \tag{11.101}$$

How do we relate $\mathcal{G}\mathrm{r}_G$ to $\mathcal{G}\mathrm{r}_T \simeq \Lambda$? Inside of the Grassmannian we have the fixed points of the $\mathbb{G}_m$-action:

$$\Lambda \hookrightarrow \mathcal{G}\mathrm{r}_G \ . \tag{11.102}$$

So we want to take a sheaf on $\mathcal{G}\mathrm{r}_G$, and attach a vector space for every point of Λ, i.e. we want to attach a vector space for every fixed point at the center of a cell.

We will discuss parabolic restriction next, but the punchline is that we can relate a sheaf on $\mathcal{G}\mathrm{r}_G$ to a sheaf on $\mathcal{G}\mathrm{r}_T \simeq \Lambda$ by passing through $\mathcal{G}\mathrm{r}_B$:

$$\begin{tikzcd} & \mathcal{G}\mathrm{r}_B \arrow[dl] \arrow[dr] & \\ \mathcal{G}\mathrm{r}_G & & \mathcal{G}\mathrm{r}_T \end{tikzcd} , \tag{11.103}$$

where B is the Borel:

$$G \supset B \twoheadrightarrow T \ . \tag{11.104}$$

Example 75. If $G = \mathrm{GL}_n$ then B consists of upper-triangular matrices. Then diagonal matrices comprise T, which is given by the quotient of B by strictly upper-triangular matrices.

So we get a nice faithful $\otimes$-functor from $\mathbf{Sph} \to \mathbf{Rep}\,(T^\vee)$ which can be restricted to $\mathbf{Vect}_k$ to get a faithful $\otimes$-functor:

$$\begin{tikzcd} \mathbf{Sph} \arrow[rr] \arrow[dr] & & \mathbf{Rep}\,(T^\vee) \arrow[dl, "\text{Forget}"] \\ & \mathbf{Vect}_k & \end{tikzcd} . \tag{11.105}$$

But in the end, this picture will tell us that in fact $\mathrm{Spec}\,(\mathbf{Sph})$ naturally receives a map:

$$\begin{tikzcd} \mathrm{pt}/T^\vee \arrow[d] \\ \mathrm{Spec}\,(\mathbf{Sph}) \end{tikzcd} . \tag{11.106}$$

Lecture 28; transition

11.6 Parabolic induction/restriction

11.6.1 For a reductive group

We want a meaningful way to relate representations of a complex reductive group G to representations of of the maximal torus T. This is a version of highest weight theory. Recall that we can study finite-dimensional representations of G using highest weight theory. Recall this is as follows. Fix a Borel $B \subset G$, i.e. a subgroup $B \subset G$ such that G/B is a maximal projective quotient of G.[3] This quotient $\mathcal{B} = G/B$ is the *flag variety for* G. Inside of B we can consider the commutator subgroup

$$N \coloneqq [B, B] \subset B \ , \tag{11.107}$$

and then we can form the quotient which is a torus:

$$T = B/N \ . \tag{11.108}$$

[3]Equivalently the Borel can be defined as the maximal Zariski closed and connected *solvable* algebraic subgroup of G.

Example 76. If $G = \mathrm{GL}_n$ then B consists of upper-triangular matrices, N consists of strictly upper-triangular matrices, and $T = B/N$ consists of diagonal matrices.

Remark 76. Initially one might meet the torus T as a subgroup of G, but it is more useful to think that it is a quotient:

$$G \supset B \twoheadrightarrow T \ . \tag{11.109}$$

Now we can form pt/B, which defines a correspondence:

$$\begin{tikzcd} & \mathrm{pt}/B \arrow[dl] \arrow[dr] & \\ \mathrm{pt}/G & & \mathrm{pt}/T \end{tikzcd} , \tag{11.110}$$

which defines morphisms between linearizations of pt/G and pt/T (i.e. representations of G and T respectively) via push-pull. Explicitly, for $V \in \mathbf{Rep}(G)$, we can pull it back to pt/B to get:

$$V|_B \in \mathbf{Rep}(B) \ . \tag{11.111}$$

Then we can push this forward to pt/T. To see what this is, let us make this diagram a bit richer by taking the fiber product of the maps $\mathrm{pt}/B \to \mathrm{pt}/T \leftarrow \mathrm{pt}$ to get:

$$\begin{tikzcd} & & \mathrm{pt}/N \arrow[dl] \arrow[dr] & \\ & \mathrm{pt}/B \arrow[dl] \arrow[dr] & & \mathrm{pt} \arrow[dl] \\ \mathrm{pt}/G & & \mathrm{pt}/T & \end{tikzcd} . \tag{11.112}$$

Therefore pushing forward from pt/B to pt/T, at the level of underlying vector spaces this means that we are pushing forward from pt/N to pt, i.e. we are taking N-invariants. So starting from $V \in \mathbf{Rep}(G)$ we get:

$$(V|_B)^N \in \mathbf{Rep}(T) \ . \tag{11.113}$$

Taking the N-invariants of a representation V gives us the classical notion of the *highest weight vectors* of V. The action of T is the *highest weight.*

This is called *parabolic restriction.* Note that this is not the naive restriction from G to $T \subset G$, but rather we are thinking of T as a quotient of B, and using pt/B as a correspondence to relate them. This has an adjoint called *parabolic induction*, which turns a T-representation into a G-representation as follows. Note that the fibers of the map $\mathrm{pt}/B \to \mathrm{pt}/G$ are flag varieties G/B. Let $\mathbb{C}_\lambda$ be a 1-dimensional vector space on which T acts by some character λ. When we pull it back to pt/B we get a sheaf on:

$$\mathrm{pt}/B = G\backslash G/B \ , \tag{11.114}$$

so a sheaf of pt/G is equivalently a G-equivariant vector bundle $\mathcal{L}_\lambda$ on G/B. Then we push forward to get:

$$\Gamma(G/B, \mathcal{L}_\lambda) \in \mathbf{Rep}(G) \ . \tag{11.115}$$

This is the classical construction of *highest weight representations.*

Remark 77. We can think about this diagram a bit more abstractly. Gauge theory with gauge group G is about studying maps into pt/G, i.e. it is the theory of G-bundles on arbitrary spaces. When you are studying G-bundles, it is natural to look for is reductions to B, i.e. flags, and then look at the associated T-bundle. So this related the theory of G-bundles to the theory of T-bundles. We will eventually explain this as an interface between G-gauge theory and T-gauge theory.

Now we might wonder what happens if we perform parabolic induction followed by restriction, i.e. what symmetries of linearizations of pt/T are given by this procedure. By composing the correspondences, we see that the natural object that acts as symmetries is the fiber product $B\backslash G/B$:

$$\begin{tikzcd} & & B\backslash G/B \arrow[dl] \arrow[dr] & & \\ & \mathrm{pt}/B \arrow[dl] \arrow[dr] & & \mathrm{pt}/B \arrow[dl] \arrow[dr] & \\ \mathrm{pt}/T & & \mathrm{pt}/G & & \mathrm{pt}/T \end{tikzcd} , \tag{11.116}$$

i.e. it is the Bruhat cells, which is (set-theoretically) in bijection with the Weyl group. So the symmetries of this parabolic induction picture are given by the finite Hecke algebra.

11.6.2 For $\mathcal{G}\mathrm{r}_G$

The point of doing all of this is to relate G-gauge theory to something we already understand. The idea is that to study sheaves on $\mathcal{G}\mathrm{r}_G$, we should try to relate them to sheaves on $\mathcal{G}\mathrm{r}_T$. We already saw that this was a lattice $\mathcal{G}\mathrm{r}_T \simeq \Lambda$, and in fact we have $\mathcal{G}\mathrm{r}_T \simeq \Lambda \hookrightarrow \mathcal{G}\mathrm{r}_G$. But we should *not* just naively restrict.

Remark 78. Unless told otherwise, if you're trying to compare G to T, you should always use parabolic restriction, not naive restriction.

The basics idea is that we will have:

$$\begin{tikzcd} & \mathcal{G}\mathrm{r}_B \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{G}\mathrm{r}_G & & \mathcal{G}\mathrm{r}_T \simeq \Lambda \end{tikzcd} , \tag{11.117}$$

and then the basic functor we want will be something like

$$q_* p^! \colon \mathbf{Sph}_G \to \mathbf{Sph}_T \simeq \mathbf{Rep}\left(T^\vee\right) . \tag{11.118}$$

11.6.3 $\mathcal{G}\mathrm{r}_B$

We have seen that $\mathcal{G}\mathrm{r}_T$ is just a lattice, and then we discussed the structure of $\mathcal{G}\mathrm{r}_G$. Now we will try to understand $\mathcal{G}\mathrm{r}_B$. This will be very different from what we saw for a reductive group since B is solvable. By definition this is:

$$\mathcal{G}\mathrm{r}_B = B(K)/B(\mathcal{O}) . \tag{11.119}$$

E.g. we're looking at upper triangular matrices in Laurent series modulo upper triangular matrices in Taylor series. Really we're interested in the double quotient stack:

$$\underline{\mathcal{G}\mathrm{r}}_B \coloneqq B(\mathcal{O}) \setminus \mathcal{G}\mathrm{r}_B = B(\mathcal{O}) \setminus B(K) / B(\mathcal{O}) \ . \tag{11.120}$$

Recall $\mathcal{G}\mathrm{r}_G$ consists of G-bundles equipped with a trivialization on $C \setminus \mathrm{pt}$. $\mathcal{G}\mathrm{r}_B$ will similarly consist of B-bundles equipped with a trivialization on $C \setminus \mathrm{pt}$. The key observation is that a trivialization of a G-bundle is much strongly than a reduction of the structure group to B. The point is that a reduction to B is giving a full flag, and if you have a trivialization, then you can just take the coordinate subspaces as the flag. So given a G-bundle trivialized away from a point, this certainly has a reduction to B away from the point. I.e. we get a map from $C \setminus \mathrm{pt}$ to the thing parametrizing flags in $\mathbb{C}^n$, i.e. we get a map $C \setminus \mathrm{pt} \to G/B$. So for $\mathcal{P} \in \mathcal{G}\mathrm{r}_G$, we get a section of the associated G/B-bundle $\mathcal{P} \times^G G/B$ over $C \setminus \mathrm{pt}$. Now we have this beautiful thing called *the valuative criterion of properness*, which says that if I have a map from a punctured curve, $D^* \to G/B$, then it extends uniquely to the whole disk $D \to G/B$. Note that this is not a statement in families; it is a statement over a field. So the claim is as follows.

Claim 2. At the level of k-points, $\mathcal{G}\mathrm{r}_B \to \mathcal{G}\mathrm{r}_G$ is a bijection.

This is not a statement in families, i.e. this is not saying that these spaces are not homeomorphic. In fact we can realize this as follows. Whatever $\mathcal{G}\mathrm{r}_B$ is, it maps to both $\mathcal{G}\mathrm{r}_G$ and $\mathcal{G}\mathrm{r}_T$, which was totally disconnected ($\mathcal{G}\mathrm{r}_T \simeq \Lambda$). Therefore:

$$\mathcal{G}\mathrm{r}_B = \sqcup_{\lambda \in \Lambda} \mathcal{G}\mathrm{r}_{B,\lambda} \ , \tag{11.121}$$

i.e. we have torn $\mathcal{G}\mathrm{r}_B$ into pieces labelled by $\lambda \in \Lambda$.

The picture is the following. Recall the $\mathcal{G}\mathrm{r}_G$ has $G(\mathcal{O})$-orbits on $\mathcal{G}\mathrm{r}_G$ which contain $G(\mathcal{O})$-fixed points corresponding to the points of Λ. Then, set-theoretically, $\mathcal{G}\mathrm{r}_G$ is given as a disjoint union of pieces S_λ labelled by $\lambda \in \Lambda$, which are the inverse images in $\mathcal{G}\mathrm{r}_G$ of $\lambda \in \Lambda$. Again, since $T = B/N$ we have:

$$\begin{tikzcd} \mathrm{pt}/N \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt}/B \arrow[r] & \mathrm{pt}/T \end{tikzcd} \ . \tag{11.122}$$

Exercise 3. Use this to show that the S_λ are exactly the orbits of $LN = N(K)$.

These are called the *semi-infinite orbits*. These orbits are infinite-dimensional, but also of infinite codimension.

11.6.4 The geometric Satake theorem

The upshot of all of this is that we have a functor:

$$\begin{aligned} \mathbf{Sph}_G &\longrightarrow \mathbf{Sph}_T \simeq \mathbf{Rep}(T^\vee) \\ \mathcal{F} &\longmapsto \text{Cohomology of the !-restr. to } N(K)\text{-orbits} \end{aligned} \ . \tag{11.123}$$

I.e. we start with $\mathcal{G} \in \mathbf{Sph}_G$, and the result is a Λ-graded vector space. We can either stop here, or compose with Forget: $\mathbf{Rep}(T^\vee) \to \mathbf{Vect}$, to get an honest fiber functor. This is the correct fiber functor for the Geometric Satake Correspondence.

It is manifestly a tensor functor, because the operations that we composed to form this functor are all compatible with the factorization structure (colliding points) and with the symmetries realized by the double coset. It is also (pretty) manifestly faithful. The idea is that we're considering a sheaf $\mathcal{F} \in \mathbf{Shv}(G(\mathcal{O}) \setminus \mathcal{G}\mathrm{r})$, and then the sheaf is supported on the closure of some stratum. Choose some fixed point λ inside of this maximal orbit. If we make this measurement by restricting to S_λ, then we basically get the stalk at this point. The measurements are more complicated on the lower strata, but we can at least pick off the top-dimensional piece of the support of $\mathcal{F}$, and we can use this to check that the functor is faithful.

As a result we get the Geometric Satake Theorem. The basic geometric phenomena in the topology of $\mathcal{G}\mathrm{r}$ were identified by Lusztig, Ginzburg formulated the theorem as we know it today (and sketched a proof), then there is the factorization picture introduced by Drinfeld, and finally Mirkovic-Vilonen introduced these semi-infinite cells and completed the proof. cite

Theorem 9 (Geometric Satake)**.** $\mathbf{Sph}_G$ *is equivalent to* $\mathbf{Rep}(G^\vee)$ *(where* $G^\vee$ *is given as* Aut *of the fiber functor), where* $G^\vee$ *is the* Langlands dual group.

$G^\vee$ is a reductive algebraic group with maximal torus $T^\vee \subset G^\vee$ dual to $T \subset G$. The way we have constructed the dual group, it comes with a canonical torus. It has the same Weyl group: $W_{G^\vee} = W_G$, and in fact the root data for G are dual to the root data for $G^\vee$. The center of $G^\vee$ is identified with the Cartier dual of the fundamental group of G:

$$Z(G^\vee) \simeq \pi_1(G)^\vee = \pi_0(\mathcal{G}\mathrm{r})^\vee \ . \tag{11.124}$$

Example 77. $Z(\mathrm{GL}_n) = \mathbb{G}_m$, which is Cartier dual to $\mathbb{Z} = \pi_1(\mathrm{GL}_n)$. This is consistent with the fact that GL_n is self-Langlands-dual: $(\mathrm{GL}_n)^\vee = \mathrm{GL}_n$.

These features determine the dual group explicitly, but it isn't just abstractly a group: it is explicitly given via Tannakian reconstruction on $\mathbf{Sph}_G$.

Remark 79. The ring/field over which $G^\vee$ is constructed is the ring/field of coefficients for our sheaves. This could have all been done over $\mathbb{Z}$ instead of $\mathbb{C}$.

Remark 80. This theorem says that $\mathbf{Sph}_G \simeq \mathbf{Rep}(G^\vee)$. The way we have been presenting it, the information usually flows from the left to the right, i.e. from $\mathbf{Rep}(G^\vee)$ to $\mathbf{Sph}_G$. In modular representation theory the information flows in the other direction. This is kind of a side benefit, but we should have expected it anyway: this is a nonabelian analogue of Fourier theory, so it will be a two-way bridge just like the classical theory.

Remark 81. One might wonder if this theorem holds over the sphere spectrum. We can define $\mathbf{Sph}_G$ over the sphere, by talking about sheaves of spectra instead of sheaves of abelian groups. But now it is not an abelian category anymore, and it is not symmetric monoidal (it is only $\mathbb{E}_3$). See the last section of Lurie's ICM address. cite

11.6.5 Back to parabolic restriction on $\mathcal{G}\mathrm{r}_G$

Recall we have this idea that $\mathcal{G}\mathrm{r}_G = G(K)/G(\mathcal{O})$ is an analogue of the *building* of G over K a local field (e.g. for $K = \mathbb{Q}_p$). Or it is an analogue of the symmetric space $G_\mathbb{R}/K$ where

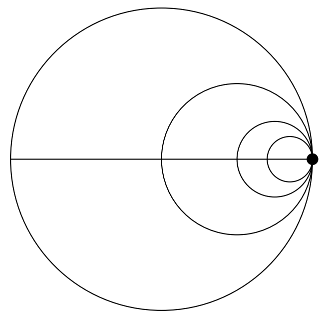


Figure 11.3: The Poincaré disk model of $\mathbb{H} = \mathrm{SL}_2 / \mathrm{SO}_2 = G_{\mathbb{R}}/K$. The point on the boundary corresponds to $B_{\mathbb{R}}$, the horizontal line is a T-orbit, and the horocycles (smaller circles tangent to the point at infinity) are the N-orbits.

now $K \subset G_{\mathbb{R}}$ is the maximal compact. So we have this picture of $\mathcal{G}\mathrm{r}$ in fig. 11.2, and we want to compare this to the building and symmetric space for SL_2, which are a 3-regular tree and hyperbolic space $\mathbb{H}$ respectively.

Poincaré disk and hyperboloid model

The most familiar picture is the real version, i.e. the space $G_{\mathbb{R}}/K$. So now $\mathbb{R}$ is playing the role of the local field, and therefore from the point of view of complex geometry this looks totally different from the building and $\mathcal{G}\mathrm{r}$. This is hyperbolic space:

$$\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \simeq \mathbb{H} . \tag{11.125}$$

E.g. we can think of this via the Poincaré disk model. The interior is $G_{\mathbb{R}}/K$ and the points at infinity comprise $\mathbb{RP}^1 = G_{\mathbb{R}}/B_{\mathbb{R}}$, i.e. it is the flag mangold for $G_{\mathbb{R}}$.

Now pick a point at infinity, i.e. a Borel $B_{\mathbb{R}} \subset G_{\mathbb{R}}$. For SL_2 this is a point in $\mathbb{RP}^1$, i.e. a line in $\mathbb{R}^2$, and $B_{\mathbb{R}}$ is given as the stabilizer of this line. There is a corresponding torus T, and the associated T-orbits are lines through the center of the Poincaré disk.

Now we would like to also draw the N-orbits. For SL_2,

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} . \tag{11.126}$$

The N-action fixes the point at infinity corresponding to $B_{\mathbb{R}}$, and the orbits are given by horocycles as in fig. 11.3.

So we have our analogue of $\mathcal{G}\mathrm{r}$, hyperbolic space $\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$, and we can map down to $\mathrm{SL}_2\mathbb{R}/N$, which is known as *the horocycle space*. Then we have the horocycle correspondence, which consists of points $x \in \mathbb{H}$ along with a horocycle through it, which forms $\mathrm{SL}_2(\mathbb{R})$ (modulo a sign):

$$\begin{tikzcd} & \mathrm{SL}_2(\mathbb{R}) / (\pm 1) \arrow[dl] \arrow[dr] & \\ \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2 & & \mathrm{SL}_2\mathbb{R}/N \end{tikzcd} . \tag{11.127}$$

The horocycle space can actually be described as

$$\mathrm{SL}_2\mathbb{R}/N \simeq \mathbb{R}^2 \setminus \{0\} . \tag{11.128}$$

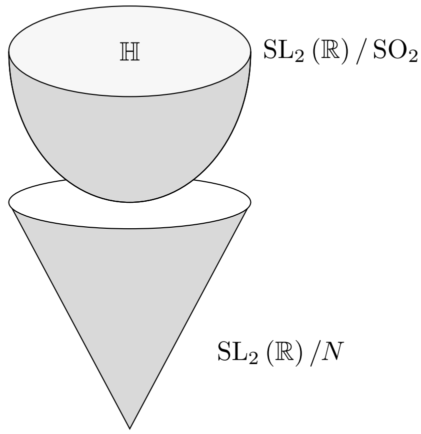


Figure 11.4: If we model $\mathrm{SL}_2 / \mathrm{SO}_2$ as a hyperboloid, then we can think of the horocycle space $\mathrm{SL}_2(\mathbb{R}) / N$ as the light cone.

This is easier to see if we model hyperbolic space as a hyperboloid. Then $\mathrm{SL}_2 \mathbb{R}/N$ is the light cone in $\mathbb{R}^3$ and $\mathbb{H}$ appears as a sheet as in fig. 11.4. Near ∞ $\mathbb{H}$ looks like the light cone modulo $\mathbb{R}^\times$, i.e. $\mathbb{RP}^1$. I.e. we have a map:

$$\begin{tikzcd} \mathrm{SL}_2(\mathbb{R}) / N \simeq \mathbb{R}^2 \setminus \{0\} \arrow[d, "T"] \\ \mathbb{RP}^1 = \mathrm{SL}_2(\mathbb{R}) / B \end{tikzcd} \tag{11.129}$$

where the fiber over the distinguished point is the torus T.

Given a function on $\mathbb{H}$, a natural measurement we can make is its integral along one of these horocycles. Or we can think about this as an integral transform. I.e. given $f \in L^2(\mathbb{H})$, we can pull and push through (11.127) to get a function on the space of horocycles $\mathrm{SL}_2(\mathbb{R}) / N$, i.e. a function on this light cone. Then we can restrict it to any ray inside this light cone, and each ray corresponds to a torus T. I.e. we start with a function on $\mathbb{H}$, and pull, push, and restrict to a get a function on T. Now we can use Fourier theory to decompose this function. This procedure is known as the Harish-Chandra transform.

To complete the analogy with $\mathcal{G}\mathrm{r}_G$, the horocycles are the analogues of the semi-infinite orbits $S_\lambda \subset \mathcal{G}\mathrm{r}$. The fiber functor for **Sph** was given by restricting to an orbit and pushing forward. Here this corresponds to restricting to a horocycle and averaging (integrating). Also recall that the points of Also note that, in fig. 11.3, the horocycles that pass through the fixed point at infinity ate exactly indexed by the T-orbit. This is the analogue of the statement from before that said that the set of $N(K)$-orbits was in bijection with $\mathcal{G}\mathrm{r}_T$.

Building

Recall the vertices of the building for PSL_2 are $\mathcal{O}$-lattices in K^2 modulo rescaling (i.e. up to homothety). The resulting graph was the 3-regular tree shown in figs. 9.1 and 9.2. Near infinity the tree look like $\mathbb{P}^1(K)$. The analogue of the horocycle picture is the following. Choose a point at infinity. Formally this means an *end* of the tree. This is an equivalence

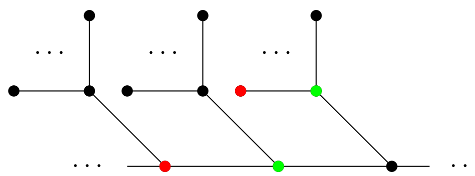


Figure 11.5: The apartment of the 3-regular tree from fig. 9.2. The green vertices are the same distance from the end, and the red vertices are the same distance from the end.

class of paths. Two paths are equivalent if they eventually agree at some point along on their path to infinity, i.e. if they only differ by finitely many vertices/edges.

Once we have chosen an end of the tree, we can consider the collection of all vertices at a fixed *distance* from the given end. We say that two vertices V_1 and V_2 have the same distance from the end if there are paths P_1 and P_2 (in the equivalence class of the end) which satisfy the following. Since P_1 and P_2 are in the equivalence class of the end, they eventually agree, i.e. they only differ by finitely many vertices. We require that the number of edges which are in P_1 and not P_2 matches the number of edges in P_2 and not P_1. The vertices of a fixed distance comprise a horocycle. This is more clear when we draw the *apartment* of the tree as in fig. 11.5. This straightened out apartment picture is the analogue of the upper-half space model. The particular point at infinity is $i\infty \in \mathbb{H}$, and the horocycles ($N_{\mathbb{R}}$-orbits) are horizontal lines in $\mathbb{H}$.

Lecture 29; May 20, 2021

11.7 Spectral decomposition

11.7.1 The Langlands dual group

Recall we thought of the geometric Satake correspondence as describing the $\otimes$-category of line operators for the theory $\mathcal{A}_G$:

$$\mathbf{Sph} \coloneqq \mathcal{A}_G\left(S^2\right) = \mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \ . \tag{11.130}$$

This acts by Hecke modifications on $\mathbf{Shv}\left(\mathrm{Bun}_G(C)\right)$. Then the result of geometric Satake is that

$$(\mathbf{Sph}, *) \simeq \mathbf{Rep}\left(G^\vee\right) \tag{11.131}$$

where $G^\vee$ is the Langlands dual group. This is a reductive algebraic group over k (where k is the field of coefficients of our sheaves).

This group $G^\vee$ had some coarse features such as:

$$\pi_1(G) \simeq Z\left(G^\vee\right)^\vee \ . \tag{11.132}$$

So if G is simply connected then $G^\vee$ is of adjoint type (has no center). For a maximal torus $T \subset G$, the dual torus $T^\vee$ is a maximal torus for $G^\vee$. In fact, the construction of $G^\vee$ comes with a dual torus, so $G^\vee$ is really a *split* reductive group.

Remark 82. $G^\vee$ actually comes with a *pinning*. So it comes with a torus $T^\vee \subset G^\vee$, but it also comes with a canonical cocharacter:

$$\mathbb{G}_m \subset T^\vee \subset G^\vee \ , \tag{11.133}$$

which we will called $2\rho^\vee$. This inclusion corresponds to the fiber functor:

$$\mathbf{Vect}^{\mathbb{Z}\text{-gr}} \leftarrow \mathbf{Vect}^{\Lambda\text{-gr}} \leftarrow \mathbf{Rep}G^\vee \ . \tag{11.134}$$

The point is that the fiber functor $\mathbf{Sph} \to \mathbf{Vect}$ factors as the cohomology functor to graded-vector spaces and Forget:

$$\mathbf{Vect} \xleftarrow{\text{Forget}} \mathbf{Vect}^{\mathbb{Z}\text{-gr}} \xleftarrow{\mathbb{H}^*} \mathbf{Sph} \ . \tag{11.135}$$

In particular, this canonical $\mathbb{G}_m \subset T^\vee$ gives us a canonical grading on representations, which picks out a Borel for us (all things that are positive).

Example 78. Starting with a torus T, the Langlands dual is just the dual torus $T^\vee$. The Langlands dual of GL_n is itself:

$$\mathrm{GL}_n^\vee = \mathrm{GL}_n \ . \tag{11.136}$$

Langlands duality exchanges adjoint and simply-connected groups, for example:

$$\mathrm{PGL}_n^\vee = \mathrm{SL}_n \ . \tag{11.137}$$

Type D_n is self-dual:

$$\mathrm{SO}_{2n}^\vee = \mathrm{SO}_{2n} \ . \tag{11.138}$$

On the other hand, type B_n gets exchanged with type C_n:

$$\mathrm{SO}_{2n+1}^\vee = \mathrm{Sp}_{2n} \ . \tag{11.139}$$

This is something to keep in mind: the groups SO_{2n+1} and Sp_{2n} don't appear to be related (e.g. they have very different geometry), yet the combinatorics of the groups are dual. This is why things involving Langlands duality tend to be very deep: the groups involved tend to be unrelated (in an obvious way).

11.7.2 Eigenobjects

Recall we introduced this category to act on things. Let C be a Riemann surface/algebraic curve. Then for $x \in C$ we get an action: $\mathbf{Sph} \circlearrowright \mathbf{Shv}\left(\mathrm{Bun}_G\left(C\right)\right)$. Explicitly, $V \in \mathbf{Sph}$ defines a sheaf on $\mathcal{H}\mathrm{ecke}_x$, which gives an integral transform from sheaves on Bun_G to itself via the correspondence:

$$\begin{tikzcd} & \mathcal{H}\mathrm{ecke}_x \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd} \ . \tag{11.140}$$

Recall we saw this in examples 70 and 71 for $G = \mathrm{GL}_n$. So we have an action of a commutative $\otimes$-category $\mathbf{Sph} \simeq \mathbf{Rep}\left(G^*\right)$ on a category $\mathbf{Shv}\left(\mathrm{Bun}_G\right)$, and we want to spectrally decompose. So we want to identify the eigenobjects, i.e. we are just trying to diagonalize a bunch of operators simultaneously.

An eigenobject is an object $\mathcal{F} \in \mathbf{Shv}\left(\mathrm{Bun}_G\right)$ along with an isomorphism:

$$V * \mathcal{F} \xrightarrow{\sim} E_x\left(V\right) \otimes \mathcal{F} \tag{11.141}$$

for $V \in \mathbf{Rep}(G^\vee) \simeq \mathbf{Sph}$. I.e. $E_x(-)$ is the "eigenvalue". This is a functor:

$$E_x\colon \mathbf{Rep}(G^\vee) \xrightarrow{\otimes} \mathbf{Vect} \ . \tag{11.142}$$

Using our Tannakian story we can write the eigenvalues down easily. The point is that $\mathbf{Rep}(G^\vee) = \mathbf{Shv}(\mathrm{pt}/G^\vee)$ and $\mathbf{Vect} = \mathbf{Shv}(\mathrm{pt})$, so such a functor is equivalent to a map $\mathrm{pt} \to \mathrm{pt}/G^\vee$. But there is only one such map up to isomorphism. Equivalently, such a map is given by a $G^\vee$-torsor E_x. Then given $V \in \mathbf{Rep}(G^\vee)$ we can send this to the associated vector bundle: $(V)_{E_x} \in \mathbf{Vect}$.

Now we can do two things: vary the point, or decategorify (take some traces). First we will vary the point.

11.7.3 Varying the point

Now we vary the point $x \in C$. So instead of acting by a single copy of $\mathbf{Rep}(G^\vee)$, now we will have an action of one such category for every point in some collection of points of C. More formally, we have the action of a restricted[4] tensor product:

$$\bigotimes_{x\in C}{}' \mathbf{Rep}(G^\vee) \circlearrowright \mathbf{Shv}(\mathrm{Bun}_G(C)) \ . \tag{11.143}$$

I.e. we are making modification at a whole bunch of points instead of just one.

We have been making a basic assumption that there is a topological field theory around. I.e. we're motivated by the idea that $\mathcal{A}_G(C) = \mathbf{Shv}(\mathrm{Bun}_G(C))$, and the above action should be by the category of line operators. The fact that this comes from a TFT means that the actions of the line operators depends locally constantly on $x \in C$. So this action factors through the action of $\int_C \mathbf{Rep}(G^\vee)$:

$$\begin{tikzcd} \bigotimes' \mathbf{Rep}(G^\vee) \arrow[dr] & \circlearrowright \mathbf{Shv} \\ & \int_C \mathbf{Rep}(G^\vee) \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd} \ . \tag{11.144}$$

In fact we said that:

$$\mathrm{Spec} \int_C \mathbf{Rep}(G^\vee) = \mathrm{Map}_{\mathrm{lc}}(C, \mathrm{pt}/G^\vee) \tag{11.145}$$

$$= \mathrm{Loc}_{G^\vee}(C) \ . \tag{11.146}$$

Remark 83. $\mathrm{Loc}_{G^\vee}(C)$ is something extremely simple in the topological/Betti setting. Explicitly, it is:

$$\mathrm{Loc}_{G^\vee}(C) = \{\pi_1(C) \to G^\vee\}/G^\vee \tag{11.147}$$

$$= \left\{A_1, \dots, A_g, B_1, \dots, B_g \in G^\vee \,\middle|\, \prod A_i B_i A_i^{-1} B_i^{-1} = 1\right\}/G^\vee \ , \tag{11.148}$$

since the fundamental group of a Riemann surface has this simple presentation of the A and B-cycles.

[4] This is defined analogously to a restricted product from earlier: all but finitely many "entries" are trivial.

The upshot of this realization, is that if we believe that this comes from a field theory:

$$\mathcal{A}_G(C) = \mathbf{Shv}(\mathrm{Bun}_G(C)) \ , \tag{11.149}$$

then the action of $\otimes' \mathbf{Rep} G^\vee$ on this factors as:

$$\begin{tikzcd} \bigotimes' \mathbf{Rep}(G^\vee) \arrow[r, phantom, "\circlearrowright"] \arrow[dr, twoheadrightarrow] & \mathbf{Shv}(\mathrm{Bun}_G(C)) \\ & \mathbf{QC}(\mathrm{Loc}_{G^\vee}(C)) \arrow[u, phantom, "\circlearrowright"] \end{tikzcd} . \tag{11.150}$$

As it turns out, there are some (very deep and hard) theorems along these lines for various versions of **Shv**.

Theorem 10 (Gaitsgory's vanishing theorem)**.** *This holds in the de Rham setting. I.e. the naive Hecke action* $\mathbf{QC}\left(\mathrm{Loc}^{dR}_{G^\vee}\right) \circlearrowright \mathcal{D}(\mathrm{Bun}_G(C))$ *factors through the corresponding global object.*

Theorem 11 (Nadler-yun)**.** *This holds in the Betti setting, where our sheaves have nilpotent singular support:* $\mathbf{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(C))$*.*

Remark 84. $\mathbf{Shv}_{\mathcal{N}}(\mathrm{Bun}_G(C))$ is the category we get by starting from locally constant sheaves on Bun_T, Bun_L (for L a Levi subgroup) and inducing.

Theorem 12 (Arinkin-Gaitsgory-Kazhdan-Raskin-Rozenblyum-Varshovsky)**.** *This holds in the restricted*[5] *setting (over* $\mathbb{F}_q$*).*

This tells us that $\mathbf{Shv}(\mathrm{Bun}_G(C))$ **spectrally decomposes over** $\mathrm{Loc}_{G^\vee}$, the *space of Langlands parameters*. I.e. $\mathcal{A}_G(C)$ is a sheaf of categories over $\mathrm{Loc}_{G^\vee}$. This is one of the main points of the Langlands correspondence.

The most concrete thing we can ask for is the fiber at some point $E \in \mathrm{Loc}_{G^\vee}$. This will be the collection of *E-Hecke eigensheaves.* This is a sheaf $\mathcal{F} \in \mathbf{Shv}(\mathrm{Bun}_G)$ equipped with an isomorphism:

$$V_x * \mathcal{F} \xrightarrow{\sim} (V_x)_{E_x} \otimes \mathcal{F} \tag{11.151}$$

for all $x \in C$ and $V \in \mathbf{Rep}(G^\vee)$. I.e. $\mathcal{F}$ is an eigenobject for the Hecke operators at every point on the curve, and the eigenvalue is $E \in \mathrm{Loc}_{G^\vee}$, i.e. we have our eigenvalues at each point ($G^\vee$-torsors) and they fit together into the locally constant sheaf E.

These are categorical analogues of the notion of *special* functions appearing in harmonic analysis. These are functions which tend to satisfy very nice systems of differential equations. In particular, eigenfunctions for commuting families of differential operators are special functions, e.g. the exponential function $e^{\lambda x}$. So the Hecke eigensheaves are analogues of $e^{\lambda x}$, where the eigenvalue λ is now replaced by a $G^\vee$-local system E on the curve C.

We talked about Hecke eigensheaves in the abelian case, where we could construct them using e.g. the Abel-Jacobi map or the Fourier-Mukai transform. But in the nonabelian setting there is no easy way to construct them. For any particular local system it is unclear that there are any. But this still leads us to the naive Geometric Langlands Conjecture.

[5]This is kind of the intersection of the de Rham and Betti world, i.e. it is some kind of "core" geometric Langlands.

11.8 Naive Geometric Langlands Conjecture

The naive Geometric Langlands Conjecture says roughly that there are enough Hecke eigensheaves, and that they span. I.e. we have:

$$\mathcal{A}_G(C) \simeq \mathbf{QC}(\mathrm{Loc}_{G^\vee}) \ , \tag{11.152}$$

where a skyscraper $\mathcal{O}_E \in \mathbf{QC}(\mathrm{Loc}_{G^\vee})$ corresponds to a Hecke eigensheaf $\mathcal{A}_E \in \mathcal{A}_G(C)$, which is unique up to scalar.

Remark 85. This is analogous to the situation one level down where exponentials spanning L^2 means we have a Fourier transform.

So for every $E \in \mathrm{Loc}_{G^\vee}$ we got $\mathcal{A}_E \in \mathcal{A}_G(C)$ unique up to scalar. Equivalently this is saying that:

$$\mathbf{QC}(\mathrm{Loc}_{G^\vee}) \circlearrowright \mathbf{Shv}(\mathrm{Bun}_G) \tag{11.153}$$

is a free rank 1 module. In other words this module is spanned by eigenvectors. More concretely, we can choose a suitable cyclic vector to act on, for example the Whittaker sheaf, and then this is saying that the induced morphism is an equivalence:

$$\mathbf{QC}(\mathrm{Loc}_{G^\vee}) \xrightarrow{\sim} \mathbf{Shv}(\mathrm{Bun}_G) \ . \tag{11.154}$$

This is the analogue of the skyscraper at the identity in Fourier theory.

This conjecture is not true as is, as has been known for a long time. The basic issue is that the stack $\mathrm{Loc}_{G^\vee}$ is singular, and $\mathbf{QC}(\mathrm{Loc}_{G^\vee})$ does not detect these singularities. So we can think hard and correct it, or we can restrict to an open dense locus inside $\mathrm{Loc}_{G^\vee}$ consisting of *irreducible* local systems. For $G = \mathrm{GL}_n$ this just means irreducible vector bundles with flat connection (i.e. no sub-bundles closed under parallel transport). In general a local system is irreducible if it doesn't have any reductions to a parabolic which are flat. So we have an open dense locus $\mathrm{Loc}^{\mathrm{irr}}_{G^\vee}$ consisting of irreducible local systems. This turns out to be a smooth stack, which is almost a variety. If E is irreducible, then:

$$\mathrm{Aut}(E) = Z(G^\vee) \ . \tag{11.155}$$

For example, if G is simply-connected, then $G^\vee$ is adjoint, so these are trivial. So in this case $\mathrm{Loc}^{\mathrm{irr}}_{G^\vee}$ forms a smooth variety.

In any case, the conjecture is expected to hold on this locus. The case of $G = \mathrm{SL}_2$ is due to Drinfeld and $G = \mathrm{GL}_n$ is due to Gaitsgory. cite

What does this restriction to irreducible local systems correspond to on the other side? Being an irreducible local system means there is no reduction to a parabolic $P^\vee \subset G^\vee$, which is equivalent to not being in the image of any maps: $\mathrm{Loc}_{P^\vee} \to \mathrm{Loc}_{G^\vee}$. In particular, it doesn't talk to these parabolic-induction/Eisenstein series correspondences

$$\begin{tikzcd} & \mathrm{Loc}_{P^\vee} \arrow[dl] \arrow[dr] & \\ \mathrm{Loc}_{G^\vee} & & \mathrm{Loc}_{G^\vee} \end{tikzcd} \ . \tag{11.156}$$

I.e. it doesn't talk to Eisenstein series.

On the other side of the Langlands correspondence, this means that we are looking for $\mathcal{F} \in \mathbf{Shv}(\mathrm{Bun}_G)$ such that parabolic restriction:

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_L \end{tikzcd} \tag{11.157}$$

is zero. This is the definition of being *cuspidal*.

Example 79. A cusp form is a modular form which is 0 at ∞. This turns out to be equivalent to having constant term given by 0. Recall the constant term is given by integrating over a horizontal line/horocycle, and this operation is exactly the parabolic restriction from $\mathrm{PSL}_2 \to T$.

So sheaves which are cuspidal on the $\mathcal{B}$-side should match with sheaves on the irreducible locus on the $\mathcal{A}$-side.

Why it can't hold

The naive geometric Langlands conjecture fails because of the following. What "is the" automorphic sheaf on Bun_G corresponding to

$$E = \mathrm{triv} \in \mathrm{Loc}_{G^\vee}(C)\ ? \tag{11.158}$$

The trivial local system is the worst point. It is the most singular/stacky (most automorphisms) point. One way to see that this is confusing, is that triv comes from $\mathrm{triv}_{T^\vee}$. But it also comes from $\mathrm{triv}_{L^\vee \subset G^\vee}$ for any Levi L. Now the theory of Eisenstein series gives us a formula:

$$\mathcal{A}_{\mathrm{triv}, L^\vee} \in \mathbf{Shv}(\mathrm{Bun}_G)\ , \tag{11.159}$$

by doing some version of this parabolic induction, which are all different. So we have a bunch of sheaves which all want the trivial local system as their eigenvalue.

Another closely related thing is to try the most obvious sheaf on Bun_G, namely the constant sheaf $\underline{k}$. Is it a Hecke eigensheaf? When we apply a Hecke operator at x, V_x, to $\underline{k}$ we get $V_x * \underline{k}$, which is the pushforward of the pullback of $\underline{k}$ restricted to some $G(\mathcal{O})$-orbit (or more generally tensoring it with some sheaf on $\mathcal{G}\mathrm{r}$). So the result is the cohomology of the sheaf $\mathcal{H}_V \in \mathbf{Shv}(\mathrm{Gr})$ tensored with $\underline{k}$. So for any point $x \in C$ we get the same thing: $\underline{k}$ tensored with this cohomology, which possibly lives in nonzero degree. This is called an *Arthur eigensheaf*. These are the kind of things we're missing.

11.8.1 Naive E-M/Montonen-Olive S-duality

Once we are feeling good about the naive geometric Langlands conjecture, we can go to town with it and make the naive guess that the entire theory $\mathcal{A}_G$ is equivalent to $\mathcal{B}_{G^\vee}^{\mathrm{naive}}$, the 4-dimensional $\mathcal{B}$-model (recall from section 10.7) on $\mathrm{pt}/G^\vee$. Recall this sends:

$$\mathcal{B}_{G^\vee}^{\mathrm{naive}}\left(M^3\right) = \mathcal{O}\left(\mathrm{Loc}_{G^\vee}(M)\right) \tag{11.160}$$

$$\mathcal{B}_{G^\vee}^{\mathrm{naive}}\left(\Sigma^2\right) = \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}(\Sigma)\right) \tag{11.161}$$

$$\mathcal{B}_{G^\vee}^{\mathrm{naive}}\left(N^1\right) = \mathbf{ShvCat}\left(\mathrm{Loc}_{G^\vee}(N)\right)\ . \tag{11.162}$$

Rather than saying this is a "naive" version of conjecture, we might say that this is a first approximation. Recall we built $G^\vee$ (and therefore $\mathcal{B}_{G^\vee}^{\text{naive}}$) out of the category of line operators. We might have started with local operators to get a zeroth approximation, but this would have been trivial in this (non-derived) case. This theory might not contain all of the information required to match the theory $\mathcal{A}_G$, but they are almost the same.

If $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$ are equivalent as field theories, this certainly means that the line operators and their actions must agree. On the $\mathcal{A}$-side we have $\mathcal{A}_G\left(S^2\right) = \mathbf{Sph}$, and on the $\mathcal{B}$-side we have:

$$\mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\left(S^2\right)\right) \ . \tag{11.163}$$

Since S^2 is simply-connected, $\mathrm{Loc}_{G^\vee}\left(S^2\right) \simeq \mathrm{pt}/G^\vee$, so up to some derived enhancement we get:

$$\mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\left(S^2\right)\right) = \mathbf{QC}\left(\mathrm{pt}/G^\vee\right) \ . \tag{11.164}$$

Even if we say this correctly (derivedly) we still have that

$$\mathbf{Rep}\left(G^\vee\right) \subset \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\left(S^2\right)\right) \ . \tag{11.165}$$

These are Wilson line operators. We can see the action explicitly as follows. Let C be a Riemann surface, $x \in C$, and $V \in \mathbf{Rep}\left(G^\vee\right)$. Then we get a vector bundle on $\mathrm{Loc}_{G^\vee}(C)$ as follows. To any local system we can consider the associated vector bundle for V, and take its fiber at x:

$$E \mapsto (V)_{E_x} \ . \tag{11.166}$$

I.e. there is a functor given by pullback:

$$\mathbf{Rep}\left(G^\vee\right) \xrightarrow{i_x^*} \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\right) \ , \tag{11.167}$$

where

$$i_x \colon \mathrm{Loc}_{G^\vee} \to \mathrm{pt}/G^\vee \tag{11.168}$$

takes the fiber at x. In other words, given $x \in C$ and $V \in \mathbf{Rep}\left(G^\vee\right)$ there is an operator:

$$V * (-) : \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\right) \to \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\right) \ . \tag{11.169}$$

These are Wilson line operators.

So the naive equivalence $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}^{\text{naive}}$ implies that line operators should match on the two sides:

$$\mathcal{A}_G\left(S^2\right) \simeq \mathcal{B}_{G^\vee}\left(S^2\right) \ . \tag{11.170}$$

This statement is geometric Satake. On the spectral/$\mathcal{B}$-side, the line operators are these very simple Wilson lines, and on the automorphic/$\mathcal{A}$-side we have these more complicated 't Hooft lines. Then geometric Satake tells us that they are the same.

Then the naive Geometric Langlands Conjecture says that we have an equivalence of categories, and for every point $x \in C$ we have compatible actions:

$$\begin{array}{ccc} \mathbf{Shv}\left(\mathrm{Bun}_G\right) & \simeq & \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\right) \\ \circlearrowright & & \circlearrowright \\ \mathbf{Sph} & \simeq & \mathbf{Rep}\left(G^\vee\right) \end{array} \ . \tag{11.171}$$

Lecture 30;
May 27, 2021

Chapter 12

Categorification

12.1 The basics

It is well-known that *categorification* is not really well-defined. Rather, it is an ill-defined inverse to something called *decategorification.* The basic idea is that, given a number, one might hope that it arises as the dimension of some vector space. Of course, only natural numbers come about this way, but given a natural number one might hope that it comes from a vector space, which carries strictly more information than the number itself. In this case the number is said to be the *decategorification* of the vector space.

Example 80. The Betti numbers of a topological space arise as the dimension of the homology groups. I.e. the decategorification of $H_i(X)$ is $b_i(X)$ for X a topological space.

If we any integer, so possibly negative, then we still hope that it comes from a vector space, only now we ask if it is the *Euler characteristic* of a *graded* vector space.

Then *categorification* is the ill-defined process of trying to replace a number by some natural vector space in such a way that it is the inverse of decategorification. I.e. a vector space *categorifies* a number if that number is the decategorification/dimension/Euler characteristic of that vector space.

This can be generalized as follows. Rather than just a vector space V, one might have another linear-algebraic datum such as an operator $\mathcal{O} \in \mathrm{End}(V)$. Then we might expect a given number to come about as $\mathrm{Tr}(\mathcal{O})$. The upshot is that the trace of an operator is not always an integer.

Example 81. If $\mathcal{O} = \mathrm{id}$, then $\mathrm{Tr}(\mathrm{id}) = \dim$, so this reduces to the above notion of decategorification being dim or χ.

Example 82. Let $G \circlearrowright V$ be a representation of a group. The character of V is a function χ defined on G, which sends $g \in G$ to the trace of g, where we are thinking of g as an operator on V. So rather than just having a number, we have a collection of numbers, indexed by $g \in G$, coming about as the trace of $g \colon V \to V$.

This collection of numbers $\{\mathrm{Tr}(g)\}_{g\in G}$ defines an element $\chi_V \in \mathbb{C}\frac{G}{G}$, i.e. it defines an element of a vector space and then we upgrade it to an actual representation $V \in \mathbf{Rep}(G)$.

So in this example we start with an element of a vector space, and then we upgrade it to an object of a category. Therefore we can recast this process as starting with a vector space, and upgrading it to a category. This is why this process is called *categorification*. The vector space is explicitly given as the Grothendieck group of the category, for example:

$$\mathbb{C}\frac{G}{G} = K_0\left(\mathbf{Rep}\left(G\right)\right) \ . \tag{12.1}$$

So we might start with a number, which is given as the dimension or Euler characteristic of a vector space, or we might start with a vector space which is given as some invariant of a category, such as K_0.

Remark 86. Just like positivity of a number is an indication that it is a shadow of some deeper structure (i.e. that it is $\dim\left(V\right)$, or that it arises via combinatorics), having a natural basis for a vector space can be thought of as a shadow of some deeper structure (i.e. that it is K_0 of a category). For example, if we have simple or projective objects in our category, these might descend to natural bases of the Grothendieck group.

Example 83. Spaces of functions are a natural source of vector spaces. These often arise from categories of sheaves. So geometry gives us natural categories and natural vector spaces which are then related by this decategorification procedure.

12.2 (De)categorification and TFT

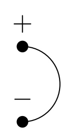
Let $\mathcal{Z}$ be a TFT. There is a natural operation to take, which is to cross a bordism with S^1 before evaluating $\mathcal{Z}$ on it, i.e. we are considering $\mathcal{Z}\left(\left(-\right)\times S^1\right)$. The general slogan is that:

$$\mathcal{Z}\left(\left(-\right)\times S^1\right) = \dim \mathcal{Z}\left(-\right) \ . \tag{12.2}$$

Let $\mathcal{Z}$ be a 1-dimensional; TFT. Recall from section 10.1 that:

$$\mathcal{Z}\left(\emptyset\right) = \mathbb{C} \qquad \mathcal{Z}\left(\bullet^+\right) = V \qquad \mathcal{Z}\left(\bullet^-\right) = V^\vee \ , \tag{12.3}$$

for V a vector space with dual $V^\vee$, so the bordism:

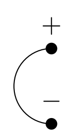


(12.4)

defines a linear map:

$$\mathbb{C} \to V \otimes V^\vee \ . \tag{12.5}$$

Similarly, the bordism:

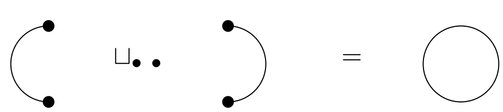


(12.6)

defines a linear map:

$$V \otimes V^\vee \to \mathbb{C} \ . \tag{12.7}$$

Then the bordism $\emptyset \to \emptyset$ defined by S^1 is the composition of these two bordisms:



(12.8)

and therefore gets assigned to the composition of those two maps:

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend right, "\dim(V)"'] & V \otimes V^{\vee} \arrow[r] & 1 \end{tikzcd} \quad , \tag{12.9}$$

i.e.

$$\mathcal{Z}\left(S^1\right) = \dim\left(V\right) \ . \tag{12.10}$$

This is a theorem when $\mathcal{Z}$ is valued in **Vect**, i.e. $V \in$ **Vect**, but if $\mathcal{Z}$ is higher dimensional, and valued in a more general target, then this will be our *definition* of the *dimension*. What do we mean by this? Let $\mathcal{Z}$ be an n-dimensional TFT. For a bordism M of dimension $\leq n$ we have

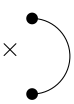
$$\mathcal{Z}\left(M\right) \in \mathcal{C} \tag{12.11}$$

for $\mathcal{C}$ some symmetric monoidal (higher) category. Now we have that:

$$\mathcal{Z}\left(M \times S^1\right) \in \mathrm{End}\left(1_{\mathcal{C}}\right) \ . \tag{12.12}$$

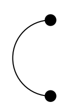
We can cross the above pictures with M to get a morphism:

$$\mathcal{Z}\left(M \times \right) : \mathcal{Z}\left(M\right) \otimes \mathcal{Z}\left(\overline{M}\right) \to 1_{\mathcal{C}} \ , \tag{12.13}$$



and a morphism:

$$\mathcal{Z}\left(M \times \right) : 1_{\mathcal{C}} \to \mathcal{Z}\left(M\right) \otimes \mathcal{Z}\left(\overline{M}\right) \ . \tag{12.14}$$



Then we can calculate the value $\mathcal{Z}\left(M \times S^1\right)$ as the composition:

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r] \arrow[rr, bend right] & \mathcal{Z}\left(M\right) \otimes \mathcal{Z}\left(\overline{M}\right) \arrow[r] & 1_{\mathcal{C}} \end{tikzcd} \ , \tag{12.15}$$

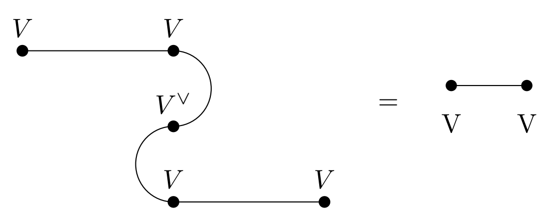
i.e.

$$\mathcal{Z}\left(M \times S^1\right) =: \dim\left(\mathcal{Z}\left(M\right)\right) : 1_{\mathcal{C}} \to 1_{\mathcal{C}} \ . \tag{12.16}$$

More formally, let $V \in \mathcal{C}$ be an object of a symmetric monoidal category. Then V is *dualizable* if there exists $V^{\vee} \in \mathcal{C}$ and morphisms:

$$1 \to V \otimes V^{\vee} \qquad\qquad V^{\vee} \otimes V \to 1 \tag{12.17}$$

such that the Mark of Zorro condition is satisfied. I.e. in pictures we have:



(12.18)

and in symbols we have:

$$(\mathrm{ev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}) = \mathrm{id} \ . \tag{12.19}$$

If V is dualizable, this implies there exists a morphism $\dim(V) \in \mathrm{End}(1_{\mathcal{C}})$ given by the composition:

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "\mathrm{coev}"] \arrow[rr, bend right, "\dim(V)"'] & V \otimes V^{\vee} \arrow[r, "\mathrm{ev}"] & 1_{\mathcal{C}} \end{tikzcd} \ . \tag{12.20}$$

This is the notion of dimension of a dualizable object of a symmetric monoidal (higher) category.

We also have an abstract notion of a *trace* $\mathrm{Tr}\colon \mathrm{End}(V) \to 1_{\mathcal{C}}$, which is given as the evaluation map, once we have identified:

$$\mathrm{End}(V) \simeq V \otimes V^{\vee} \ , \tag{12.21}$$

which is a consequence of dualizability of V.

12.3 Examples

Example 84. Let $\mathcal{C} = \mathbf{Vect}$. Then we can define the dual to be:

$$V^{\vee} \coloneqq \mathrm{Hom}(V,) \tag{12.22}$$

and then dualizability is the property that:

$$V \otimes V^{\vee} \simeq \mathrm{End}(V) \ . \tag{12.23}$$

There is a natural map

$$\mathrm{id}\colon 1 \to \mathrm{End}(V) \tag{12.24}$$

given by the inclusion of the identity. There is also always a map given by pairing:

$$V \otimes V^{\vee} \to 1 \ . \tag{12.25}$$

Therefore we get two less obvious maps in red:

$$\begin{tikzcd} & V \otimes V^{*} \arrow[r, "\mathrm{ev}"] \arrow[d, equal] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\sum e_i \otimes e^i"] \arrow[r, "\mathrm{id}"] \arrow[urr, bend right, "\mathrm{Tr(id)=dim}(V)"'] & \mathrm{End}(V) \arrow[ur, "\mathrm{Tr}"] & \end{tikzcd} \ . \tag{12.26}$$

Example 85. The dimension of a chain complex is the Euler characteristic.

Example 86. The dimension of an associative algebra A is more interesting. To understand the dimension we need to specify which category we are thinking of A as living in. We will think of A as an object of the Morita category of algebras. In other words, we are thinking of A via the category A-**mod**. I.e. the objects of the category are algebras, and the morphisms from A to B are A-B bimodules.

We claim that every associative algebra A is dualizable with dual A^{op}. The unit is $1 = k$, and ev and coev are as follows. $\mathrm{coev}\colon k \to A \otimes A^{\mathrm{op}}$ is a k-$A \otimes A^{\mathrm{op}}$ bimodule, which is equivalently an A-A bimodule. The algebra A itself defines such a bimodule because it has an action of A on both the left and the right. This is coev. Similarly, $\mathrm{ev}\colon A \otimes A^{\mathrm{op}} \to k$ is an $A \otimes A^{\mathrm{op}}$-$k$ bimodule, which is equivalently an A-A bimodule. The natural choice is again A itself.

Exercise 4. Show this duality data satisfies the Mark of Zoro relation.

The dimension of A is given by the composition, which is tensor product in the Morita category, so:

$$\dim(A) = A \otimes_{A \otimes A^{\mathrm{op}}} A =: \mathrm{HH}_*(A) \ . \tag{12.27}$$

This is the Hochschild homology of A. This has a nice universal property. There is a natural map:

$$\tau\colon A \to \mathrm{HH}_*(A) \tag{12.28}$$

given by $a \mapsto a \otimes 1$. This map satisfies:

$$\tau(ab) = \tau(ba) \ , \tag{12.29}$$

so τ is a *trace*. In fact it is the universal trace: if we have a map

$$\tau'\colon A \to M \tag{12.30}$$

which coequalizes multiplication in the two orders, then we have a unique map:

$$\begin{tikzcd} A \arrow[r, "\tau"] \arrow[dr, "\tau'"'] & \mathrm{HH}_*(A) \arrow[d, dashed] \\ & M \end{tikzcd} \ . \tag{12.31}$$

This Morita theory of algebras sits as a full subcategory of all categories, by sending A to A-**mod**. So whenever we have a description of a category $\mathcal{C}$ as $\mathcal{C} = A$-**mod**, we have a notion of dimension given by:

$$\dim(\mathcal{C}) = \mathrm{HH}_*(\mathcal{C}) := \mathrm{HH}_*(A) \ . \tag{12.32}$$

There is a definition of $\mathrm{HH}_*(\mathcal{C})$ even if $\mathcal{C}$ is not of the form A-**mod**.

Warning 1. This is not the same as the above notion of dimension given by the Grothendieck group $K_0(\mathcal{C})$. The Grothendieck group is a much more subtle invariant than $\mathrm{HH}_*(\mathcal{C})$, but HH_* is more concrete. For example, if $\mathcal{C}$ is k-linear then $\mathrm{HH}_*(\mathcal{C})$ is a k-vector space. Another benefit of HH_* is that it comes quite naturally from the field theory.

Remark 87. For most categories appearing in representation theory, it is usually the case that, once we make $K_0(\mathcal{C})$ into a k-vector space, it maps isomorphically to HH_*:

$$k \otimes_{\mathbb{Z}} K_0(\mathcal{C}) \xrightarrow{\sim} \mathrm{HH}_*(\mathcal{C}) \ . \tag{12.33}$$

For example, if $\mathcal{C} = \mathbf{Rep}(G)$ for finite group G, then $K_0(\mathbf{Rep}(G))$ is the representation ring, and $\mathrm{HH}_*(\mathbf{Rep}(G))$ is the vector space of class functions, which are identified once we tensor up with k.

Example 87. Let $\mathcal{C}$ be the category of vector space equipped with an endomorphism. This is the same as $k[x]$-**mod**, so $\mathrm{HH}_*(\mathcal{C}) = \mathrm{HH}_*(k[x])$. As with any commutative ring, in the non-derived world, the answer is just $\mathrm{HH}_*(k[x]) = k[x]$. If we are doing a derived version, there is a second copy of $k[x]$ in degree 1, so it is $k[x]$ tensored with an exterior algebra. This will be more clear when we see how to calculate HH_* geometrically.

12.4 Traces

So far we have considered dualizability of an associative algebra, up to Morita equivalence, and calculated the dimension. We might also just care that we have a map:

$$A \otimes A^{\mathrm{op}}\text{-}\mathbf{mod} \to \mathrm{End}(A\text{-}\mathbf{mod}) \ . \tag{12.34}$$

The point is that Morita morphisms from A-**mod** to B-**mod** are A-B bimodules ${}_AM_B$, where a (left) A-module gets tensored with M on the right, to produce a (left) B-module. So an $A \otimes A^{\mathrm{op}}$-module, i.e. an A-A bimodule, defines an endomorphism of A-**mod**. In the derived setting, with suitable adjectives on the functors, this map is an equivalence.

Then the induced map $\mathrm{End}(A\text{-}\mathbf{mod}) \to 1$, the evaluation map, is what one might call a trace:

$$\begin{tikzcd} A \otimes A^{\mathrm{op}}\text{-}\mathbf{mod} \arrow[r, "\mathrm{ev}"] \arrow[d, "\simeq"] & 1 \\ \mathrm{End}(A\text{-}\mathbf{mod}) \arrow[ur, "\mathrm{Tr}"] & \end{tikzcd} \ . \tag{12.35}$$

So if we have an endofunctor $F\colon A\text{-}\mathbf{mod} \circlearrowleft$, then we have its trace:

$$\mathrm{Tr}(F) \in 1 = k\text{-}\mathbf{mod} \ . \tag{12.36}$$

So the trace of an endofunctor is a vector space in such a way that:

$$\mathrm{Tr}(\mathrm{id}_{A\text{-}\mathbf{mod}}) = \dim(A) = \mathrm{HH}_*(A) \ . \tag{12.37}$$

So when you're a dualizable object, you not only get a dimension, but you can also take the trace of any endomorphism.

Classically we are saying that if ${}_AM_A$ is an A-bimodule, then we can attach to this its trace $\mathrm{Tr}(M)$, which is just M tensored with the diagonal bimodule:

$$\mathrm{Tr}(M) = M \otimes_{A\otimes A^{\mathrm{op}}} A =: \mathrm{HH}_*(A, M) \ , \tag{12.38}$$

which is also known as the *Hochschild homology of A with coefficients in M*. In these terms we have:

$$\dim(A) = \mathrm{HH}_*(A) = \mathrm{HH}_*(A, A) = \mathrm{Tr}(\mathrm{id}_{A\text{-}\mathbf{mod}}) \ . \tag{12.39}$$

12.5 dim in Bord

Recall a TFT is a symmetric monoidal functor from a bordism category to some kind of linear target, e.g. a linear (higher) category:

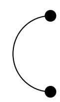
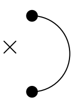
$$\mathcal{Z}\colon (\mathbf{Bord}_n, \sqcup) \to (\mathcal{C}, \otimes) \ . \tag{12.40}$$

Any symmetric monoidal functor has the property that:

$$\mathcal{Z}\left(\dim\left(-\right)\right) = \dim \mathcal{Z}\left(-\right) \ . \tag{12.41}$$

The notion of dimension is defined purely abstractly for any symmetric monoidal category, so this equality follows from the fact that the notion of having a dual, ev, and coev are all preserved by a symmetric monoidal functor.

In $\mathbf{Bord}^{\mathrm{or}}$ any manifold M is dualizable. The dual is the manifold with the opposite orientation: $\overline{M}$. The unit is the empty manifold of the same dimension, and coev and ev are the following bordisms:

$M \times$  $M \times$  . (12.42)

Composition is gluing (along $M \sqcup \overline{M}$) so the dimension is:

$$\dim\left(M\right) = M \times S^1 \ . \tag{12.43}$$

Therefore (12.41) implies that:

$$\mathcal{Z}\left(M \times S^1\right) = \dim\left(\mathcal{Z}\left(M\right)\right) \ , \tag{12.44}$$

which is our slogan from (12.2).

12.5.1 Traces in Bord

Suppose we have an endomorphism $f\colon M \to M$ of a bordism. Then we are supposed to get $\mathrm{Tr}\left(f\right)$, which will be a new manifold, in such a way that if $f = \mathrm{id}$ we get $\mathrm{Tr}\left(\mathrm{id}\right) = M \times S^1$. The trace turns out to be the mapping torus of f:

$$\mathrm{Tr}\left(f\right) = M \times I / \left(\left(0, x\right) \sim \left(1, f\left(x\right)\right)\right) \ . \tag{12.45}$$

I.e. we take the manifold crossed with an interval, and glue the points at 0 to their image under f at 1. Then since $\mathcal{Z}$ is a symmetric monoidal functor, we have that:

$$\mathcal{Z}\left(\mathrm{Tr}\left(f\right)\right) = \mathrm{Tr}\left(\mathcal{Z}\left(f\right), \mathcal{Z}\left(M\right)\right) \ , \tag{12.46}$$

where $\mathcal{Z}\left(f\right)$ is the map on $\mathcal{Z}\left(M\right)$ induced by f.

12.6 Geometric/Lagrangian interpretation of $\dim$

Our TFT $\mathcal{Z}$ often factored through a space of fields given by some category **Corr** of correspondences of spaces (e.g. topological spaces, algebraic varieties, etc.):

$$\begin{tikzcd} \mathbf{Bord}_n \arrow[r, "\text{fields}"] \arrow[rr, bend right, "\mathcal{Z}"'] & \mathbf{Corr} \arrow[r, "\text{linearize}"] & \mathcal{C} \end{tikzcd} \ . \tag{12.47}$$

For example we might take linearize by taking sheaves on the space of fields. When this is true we say the theory is *Lagrangian*.

Example 88. Starting from a bordism M, the space of fields might be $\mathrm{Loc}_{G^\vee}(M)$, and then we can take the linearization to be $\mathbf{QC}(\mathrm{Loc}_{G^\vee}(M))$.

The functors we have factored $\mathcal{Z}$ into are both symmetric monoidal, so we can apply the same idea as above in (12.41): that dim is preserved. In $\mathbf{Bord}_n$ the dimension is something like crossing with S^1, in the linear category $\mathcal{C}$ the dimension is something like HH_* or χ, and in this middle category of correspondences we will have some geometric version of dimension, or more generally nonlinear traces.

12.6.1 Nonlinear traces

Dimension (the loop space)

We're thinking of our category **Corr** as having objects given by spaces[1] and morphisms given by correspondences:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} . \tag{12.48}$$

Any $X \in \mathbf{Corr}$ is self-dual. The duality data is given by the correspondences we get from the diagonal map:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta"] & \\ \mathrm{pt} & & X \times X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ X \times X & & \mathrm{pt} \end{tikzcd} . \tag{12.49}$$

This diagonal map is the analogue of the diagonal bimodule when we were considering dualizability of algebras in the Morita category.

Now we can calculate the dimension of X. This is some correspondence from pt to itself, which can be any space. We know it is given by the composition of the ev and coev correspondences, which is given by the fiber product:

$$\begin{tikzcd} & & X \times_{X \times X} X \arrow[dl] \arrow[dr] & & \\ & X \arrow[dl] \arrow[dr, "\Delta"] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\ \mathrm{pt} & & X \times X & & \mathrm{pt} \end{tikzcd} , \tag{12.50}$$

which resembles the formula for Hochschild homology of an algebra.

There are many ways to think about $X \times_{X \times X} X$, but one way to think of this is as the self-intersection of the diagonal:

$$X \times_{X \times X} X = \Delta \cap \Delta . \tag{12.51}$$

[1] We might give a more technical definition of a space, but we really just mean something like a variety, stack, manifold, etc. as long as we have a notion of a colimit.

At a naive set-theoretic level this is just Δ, but if you are working with orbifolds, fiber products are more interesting. In general this is called the *derived loop space:*

$$\Delta \cap \Delta =: \mathcal{L}X := X \times_{X \times X} X \tag{12.52}$$

or the *inertia* of X. I.e.

$$\dim(X) = \mathcal{L}X \ . \tag{12.53}$$

The name "loop space" is consistent with what we have seen so far about Lagrangian field theories. Recall we're thinking of these spaces $X \in \mathbf{Corr}$ as spaces of fields, which we're thinking of as maps from a bordism M into some target space T:

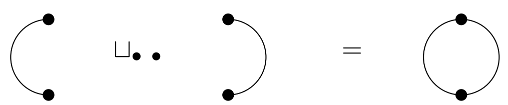
$$X = \mathrm{Map}(M, T) \ . \tag{12.54}$$

Taking the dimension in **Bord** is crossing with S^1, and the dimension of the space of fields on M is the space of fields on the dimension of M (the functor sending a bordism to the space of fields is symmetric monoidal). Therefore:

$$\begin{aligned}
\dim(X) &= \dim(\mathrm{Map}(M, T)) && (12.55)\\
&= \mathrm{Map}(\dim(M), T) && (12.56)\\
&= \mathrm{Map}\left(M \times S^1, T\right) && (12.57)\\
&= \mathcal{L}\,\mathrm{Map}(M, T) \ , && (12.58)
\end{aligned}$$

where here $\mathcal{L}$ denotes the usual loop space (space of maps from S^1).

We can see that the space of maps from S^1 to X is the same as (12.52) as follows. We think of S^1 as:



(12.59)

From this description we have that the space of maps from S^1 to X can be identified with the space of maps from $\mathrm{pt} \sqcup \mathrm{pt}$ to X (i.e. pairs of points in X) equipped with two different paths between these points. If we are in the locally constant setting, so our maps are locally constant ($X = \mathrm{Map}_{\mathrm{lc}}(M, T)$), then having a path between the images of $x, y \in M$ is the same as identifying them: $f(x) \sim f(y)$. So we have the space of: pairs of points of X equipped with two identifications. But these are exactly points of $\Delta \cap \Delta$: pairs of points of X which are identified once (so they're on the diagonal) and then identified again (so they're on the self-intersection of Δ).

Trace of a nontrivial map

Let $f\colon X \to X$ (for $X \in \mathbf{Corr}$). We want to calculate the trace of f. First of all we rephrase f as the correspondence:

$$\begin{tikzcd}
& Z \simeq X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
X & & X
\end{tikzcd} \ . \tag{12.60}$$

The claim is that:

$$\mathrm{Tr}(f) = Z|_{\Delta \hookrightarrow X \times X} = \Gamma(f) \cap \Delta \ , \tag{12.61}$$

where $\Gamma(f)$ denotes the graph of f, but this is just the fixed points of f:

$$\mathrm{Tr}(f) = X^f \ . \tag{12.62}$$

Table 12.1: The flavors of Tr in the categories **Bord**, **Corr**, and the target of the TFT $\mathcal{C}$.

Category	**Bord**	**Corr**	$\mathcal{C}$
Tr	mapping tori	fixed points	traces/HH_*
$\dim = \mathrm{Tr}\,(\mathrm{id})$	$(-) \times S^1$	$\mathcal{L}\,(-)$	χ, dim

Example 89. Let X be a finite set. The induced map on functions on X: $f_* \colon k\,(X) \to k\,(X)$ can be thought of as a matrix. Then the trace of this map is the sum of the diagonal entries of the matrix, which is the number of fixed points of f. So, classically, traces arise from counting fixed points. We will say this in a fancier way later: this appearance of traces is the same as in the Lefschetz fixed point formula.

In summary: if our field theory $\mathbf{Bord} \to \mathcal{C}$ factors as two symmetric monoidal functors:

$$\mathbf{Bord} \to \mathbf{Corr} \to \mathcal{C} \ , \tag{12.63}$$

then Tr is preserved, and the flavor of Tr in each category is in table 12.1.

The upshot of this is as follows. Consider a sheaf theory for objects of **Corr** which is symmetric monoidal, e.g.

$$\mathbf{Shv} \colon \mathbf{Corr} \to \mathbf{Cat} \ , \tag{12.64}$$

sending $X \mapsto \mathbf{Shv}\,(X)$ and a morphism:

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd} \tag{12.65}$$

to:

$$q_* p^! \colon \mathbf{Shv}\,(X) \to \mathbf{Shv}\,(Y) \ . \tag{12.66}$$

Note that this is just notation for the functor induced by the correspondence. If we ate in such a setting, then it automatically follows that it take traces to traces (and therefore dimensions to dimensions). In practice, the sheaf theories we care about don't tautologically satisfy this. E.g. we need:

$$\mathbf{Shv}\,(X \times X) \simeq \mathbf{Shv}\,(X) \otimes \mathbf{Shv}\,(X) \ , \tag{12.67}$$

which is sometimes not true in practice.

12.6.2 Relation to categorification

The upshot of this entire discussion about traces is that if a vector space V comes about as the value of a field theory $\mathcal{Z}$:

$$V = \mathcal{Z}\left(M \times S^1\right) \ , \tag{12.68}$$

then we know how to categorify it:

$$V = \dim\left(\mathcal{Z}\,(M)\right) \ , \tag{12.69}$$

so $\mathcal{Z}(M)$ categorifies V. Or more generally, if V is $\mathcal{Z}(\text{mapping torus})$, then we have a natural point of view from which V can be categorified.

12.7 Lefschetz fixed point formula

The subject of categorification became very popular in the 1990s with the school of Frenkel, Khovanov, and Baez-Dolan. But the idea of categorification, and some of its deeper instances, go back to the 1960s with work of Grothendieck: the Grothendieck-Lefschetz trace formula. So people in geometric representation theory have been doing categorification since the early 1970s, without quite calling it that.

cite

Consider a map $f\colon X \to X$ for X a topological space. Recall the Lefschetz fixed point formula is:

$$\sum \{\text{fixed points}\} = \sum (-1)^i \operatorname{tr}(f|_{H^i}) = \operatorname{Tr}(f, H^*(X)) \ . \tag{12.70}$$

This sum of fixed points is the same as the number of intersection points between Δ and the graph of f:

$$\sum \{\text{fixed points}\} = \#(\Delta \cap \Gamma_f) = \dim H^*(\Delta \cap \Gamma_f) \ . \tag{12.71}$$

We have to count these fixed points carefully, by weighting each fixed point by a different index (if the intersection point is not transverse), but this is the general shape of the formula. As we saw, taking fixed points is the correct notion of trace of a map $f\colon X \to X$ in the geometric world. So this formula is saying that cohomology takes the trace of $f\colon X \circlearrowleft$ in the geometric category to trace of the induced map on the cohomology of X.

The Grothendieck-Lefschetz trace formula is as follows. Let $X/\mathbb{F}_q$ be a scheme (or stack) over a finite field. Then we have that:

$$\#X(\mathbb{F}_{q^r}) = \operatorname{Tr}\left(F^r|_{H_c^*(X,\overline{\mathbb{Q}_\ell})}\right) \ , \tag{12.72}$$

where F is the Frobenius morphism and $H_c^*\left(-,\overline{\mathbb{Q}_\ell}\right)$ denotes compactly supported ℓ-adic cohomology. Morally, this is true because:

$$\#X(\mathbb{F}_{q^r}) = \#\left(X^{F^r}\right) \ . \tag{12.73}$$

I.e. the vector space and endomorphism

$$H_c^*(X) \circlearrowleft F \tag{12.74}$$

categorifies $\#X(\mathbb{F}_q)$.

We can think about $\#X(\mathbb{F}_q)$ as the pushforward of the constant function 1 on $X(\mathbb{F}_q)$ under the map:

$$\pi\colon X(\mathbb{F}_q) \to \mathrm{pt} \ . \tag{12.75}$$

I.e.

$$\#X(\mathbb{F}_q) = \pi_! 1 \ . \tag{12.76}$$

Similarly, cohomology is the pushforward of the constant sheaf on X:

$$H_c^*(X) = \pi_! \overline{\mathbb{Q}_\ell} \ . \tag{12.77}$$

This is the birth of *the function sheaf correspondence*.

Remark 88. We are being a bit sloppy about shriek versus star pushforward. There is something shrieky about integration, which is what we are doing when we push the constant function forward, and the fact that our cohomology is compactly supported also means that we should take $\pi_!$.

12.7.1 Function-sheaf correspondence

The function-sheaf correspondence is one of the deeper notions of categorification, and came out of the Weil conjectures. The idea is that there is a world of sheaves, and there is a world of functions. When we say functions, we mean functions on $X(\mathbb{F}_q)$. Given a function, we have an operation of integration which gives a number:

$$\text{function on } X(\mathbb{F}_q) \xmapsto{\int} \# \quad . \tag{12.78}$$

Similarly, compactly supported cohomology turns a sheaf into a vector space:

$$\mathbf{Shv} \xrightarrow{H_c^*} \mathbf{Vect} \quad . \tag{12.79}$$

But this is not just a vector space, it is a vector space with a Frobenius F:

$$\mathbf{Shv} \ni \mathcal{E} \xmapsto{H_c^*} V \circlearrowleft F \quad . \tag{12.80}$$

To get from a vector space to a number, we can take the trace:

$$V \circlearrowleft F \xmapsto{\mathrm{Tr}} \mathrm{Tr}(F) \quad . \tag{12.81}$$

The function-sheaf correspondence is saying that there is a notion of trace from sheaves to functions which makes the following diagram commute:

$$\begin{tikzcd} \text{function on } X(\mathbb{F}_q) \arrow[r, "\int"] & \# \\ \mathbf{Shv} \ni \mathcal{E} \arrow[u, "\mathrm{Tr}(F)"] \arrow[r, "H_c^*"] & V \circlearrowleft F \arrow[u, "\mathrm{Tr}(F)"] \end{tikzcd} \quad . \tag{12.82}$$

I.e. we can take the trace before or after passing globally.

Given a number, one might categorify it as a vector space with an operator, which is what the Lefschetz formula says, and then Grothendieck adds that you can apply this operation to other sheaves besides the constant sheaf, which gave the number of points. The operation $\mathrm{Tr}(F)$ from sheaves to functions is:

$$\mathrm{Tr}(F, \mathcal{E})(x) = \mathrm{Tr}(F, i_x^* \mathcal{E}) \tag{12.83}$$

for $x \in X(\mathbb{F}_q)$. In order for this to make sense, we need $\mathcal{E}$ to be a Weil sheaf. This means it is weakly equivariant, i.e. we have an isomorphism:

$$\mathcal{E} \xrightarrow{\sim} F^* \mathcal{E} \quad . \tag{12.84}$$

So we get a function on $X(\mathbb{F}_q)$ by taking the trace of F on the stalk of $\mathcal{E}$ at the given point.

This is an instance where we see that decategorification is a map (given a sheaf we get a function on $X(\mathbb{F}_{q^r})$ for every r) whereas categorification is an art. The idea of the function sheaf correspondence, is that interesting functions come from sheaves.

Let G be a reductive group. Work of Lusztig, Kazhdan-Lusztig, and Deligne-Lusztig in the 80s gives us the following slogan. The representation theory of the finite groups $G(\mathbb{F}_q)$ is geometric, meaning it has a categorification, i.e. that it comes from applying $\mathrm{Tr}(F)$ to sheaves. There are different manifestations of this principle that the structure of this representation theory has a geometric origin in the sheaf theory on the group G and its coset spaces. This is extremely important, for example, if you look at the classification of finite simple groups, almost all of them arise this way. So the characters of these groups can be calculated geometrically.

cite

12.7.2 Lusztig's character sheaves

The theory of character sheaves establishes that characters of complex $G(\mathbb{F}_q)$-representations come from sheaves on $\frac{G}{G}$, i.e. sheaves equivariant for conjugation. So starting with such a sheaf, we can take the trace of Frobenius to get a class function:

$$\mathrm{Tr}(F) : \frac{G}{G} \to \mathbb{C}\left(\frac{G(\mathbb{F}_q)}{G(\mathbb{F}_q)}\right) . \tag{12.85}$$

12.7.3 Kazhdan-Lusztig theory

Recall the finite Hecke algebra from example 46:

$$\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)} = \overline{\mathbb{Q}_\ell}\left[B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)\right] , \tag{12.86}$$

i.e. $\overline{\mathbb{Q}_\ell}$-valued functions on the flag variety $G(\mathbb{F}_q)/B(\mathbb{F}_q)$ which are constant on Bruhat orbits. This arises as the endomorphisms of the representation given by functions on the flag variety:

$$\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)} = \mathrm{End}\left(\overline{\mathbb{Q}_\ell}\left[G(\mathbb{F}_q)/B(\mathbb{F}_q)\right]\right) . \tag{12.87}$$

This is important to study because:

$$\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)}\textbf{-mod} \subset G(\mathbb{F}_q)\textbf{-mod} ,$$

i.e. the modules over the finite Hecke algebra comprise part of the representation theory of G. Specifically they are the "unitary principal series" representations. This covers most representations of this finite group.

Kazhdan-Lusztig theory says that $\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)}$ arises via categorification, and one can get deep insight by studying it this way. Specifically it says that $\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)}$ comes from $\mathrm{Tr}(F)$ on

$$\mathbf{Shv}(B\backslash G/B) . \tag{12.88}$$

Note that now we're thinking of G as a scheme over $\mathbb{F}_q$. The way people usually say it is that:

$$\underbrace{K_0}_{=\dim}(\mathbf{Shv}(B\backslash G/B)) = \mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)} . \tag{12.89}$$

The upshot is that we can study the finite Hecke algebra using all of this wonderful geometry. E.g. we have a Verdier duality operation on $\mathbf{Shv}(B\backslash G/B)$, and there is a natural self-dual basis given by intersection cohomology sheaves of orbits. This descends to the *Kazhdan-Lusztig* basis for $\mathcal{H}_{G(\mathbb{F}_q),B(\mathbb{F}_q)}$. This basis has miraculous properties, and most of the representation theory of this Hecke algebra can be reconstructed just by knowing this basis.

This is an instance where categorification helps you understand the representation theory of finite groups in a way that you cannot without it.

12.8 Geometric Satake

The geometric Satake correspondence basically says that the *Satake correspondence*, which we did not really talk about, is geometric. The Satake correspondence is about the spherical Hecke algebra:

$$Sph = \overline{\mathbb{Q}_\ell}\left[G(\mathcal{O})\backslash G(K)/G(\mathcal{O})\right] \ . \tag{12.90}$$

Here K is a local field. We can think of this as Laurent series $K = \mathbb{F}_q((t))$ or a finite extension $K/\mathbb{Q}_\ell$. We came across this as:

$$Sph = \mathrm{End}_{G(K)} \mathrm{Fun}\left(G(K)/G(\mathcal{O})\right) \ . \tag{12.91}$$

Recall we care about this because it naturally acted on:

$$Sph \circlearrowright \text{ functions on } G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_{\mathbb{A}}) \ . \tag{12.92}$$

So we can think of this as having to do with the local representation theory of G (it is the endomorphisms of this representation) and it also appears in the global context by studying automorphic functions.

A Hecke algebra has no reason to be commutative in general, but we can learn something about the commutativity in this context by categorifying to the spherical Hecke category:

$$\mathbf{Sph} = \mathbf{Shv}\left(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})\right) \ , \tag{12.93}$$

where we're in the positive characteristic setting: $K = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]] \subset K$. The decategorification is realized by trace of the Frobenius $\mathrm{Tr}(F)$:

$$\begin{tikzcd} Sph \\ \mathbf{Sph} \arrow[u, "\mathrm{Tr}(F)"] \end{tikzcd} \ . \tag{12.94}$$

The geometric Satake correspondence identifies:

$$\mathbf{Sph} \xrightarrow{\sim} \mathbf{Rep}(G^\vee) = \mathbf{QC}(\mathrm{pt}/G^\vee) \ . \tag{12.95}$$

Then we can deduce the classical Satake correspondence by decategorifying $\mathbf{Rep}(G^\vee)$ with $\mathrm{Tr}(F)$. The Frobenius on $\mathbf{QC}(\mathrm{pt}/G^\vee)$ is just the identity, at least if the group is split. Therefore $\mathrm{Tr}(F) = \dim$, which maps to the representation ring:

$$\begin{tikzcd} |\mathbf{Rep}(G^\vee)| \\ \mathbf{Rep}(G^\vee) \arrow[u, "\dim"] \end{tikzcd} \ . \tag{12.96}$$

But we know that **QC** commutes with dim, so we want to calculate the inertia/loop space:

$$\mathcal{L}\left(\mathrm{pt}/G^\vee\right) = \mathrm{pt}/G^\vee \times_{\mathrm{pt}/(G^\vee \times G^\vee)} \mathrm{pt}/G^\vee = \frac{G^\vee}{G^\vee} \ . \tag{12.97}$$

Therefore we have that the representation ring of $G^\vee$ is algebraic functions on $\frac{G^\vee}{G^\vee}$:

$$\left|\mathbf{Rep}\left(G^\vee\right)\right| = \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \ . \tag{12.98}$$

All together we have a diagram as in the function-sheaf correspondence:

$$\begin{tikzcd} Sph \arrow[r, "\sim"] & \left|\mathbf{Rep}\left(G^\vee\right)\right| = \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \\ \mathbf{Sph} \arrow[u, "\mathrm{Tr}(F)"] \arrow[r, "\sim"] & \mathbf{Rep}\left(G^\vee\right) \arrow[u, "\mathrm{Tr}(F)=\dim"'] \end{tikzcd} \ , \tag{12.99}$$

and the top line is the classical Satake isomorphism. Note that we can also think of this as Weyl-invariant functions on the torus:

$$\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = \mathcal{O}\left(H^\vee\right)^W \ . \tag{12.100}$$

One way this is useful is that we had a good reason for the category **Sph** to be commutative, and now this shows that the Hecke algebra *Sph* is too. Maybe a deeper thing to say, is that the Langlands dual group $G^\vee$ appears. So the Langlands interpretation of the Satake isomorphism is that:

$$Sph \simeq \left|\mathbf{Rep}\left(G^\vee\right)\right| \ , \tag{12.101}$$

where $G^\vee$ is the reductive group over $\mathbb{C}$ with root data dual of that of G. But we cannot recover $G^\vee$ from $|\mathbf{Rep}\left(G^\vee\right)|$: groups are in general not determined by their representation rings. But the geometric Satake correspondence actually constructs the dual group $G^\vee$ for us using Tannakian reconstruction. This is one of many reasons that the geometric Satake correspondence is more powerful than the classical one.

Chapter 13

The Langlands correspondence

13.1 Geometric Satake

Lecture 31; June 3, 2021

Recall that we can phrase geometric Satake in terms of the line operators in this "automorphic TFT", i.e. the assignment $\mathcal{A}_G\left(S^2\right)$. Then geometric Satake says that this can be identified with representations of the Langlands dual:

$$\left(\mathcal{A}_G\left(S^2\right), *\right) \simeq \mathbf{Rep}\left(G^\vee\right) \ . \tag{13.1}$$

In fact, this was our definition of $G^\vee$. Recall that, physically, $\mathcal{A}_G\left(S^2\right)$ consists of 't Hooft line operators, so they're 1-dimensional defect of this four-dimensional gauge theory. They're a kind of generalization of Dirac monopoles. We saw these in electro-magnetism, where we had a worldline of a monopole in four-dimensions, where gauge fields are allowed to have singularities. So if we have a line in 4-dimensions, the link is a 2-sphere and prescribing a bundle on S^2, or really its Chern class, is prescribing the charge of the monopole itself. In the abelian case the charge was an integer, but now the charge is an element of the representation ring $|\mathbf{Rep}\left(G^\vee\right)|$. There is something more concrete, which is similar to what we had in the abelian case, which is a 't Hooft loop operator. Given a loop in an 4-manifold, the link of this is now $S^2 \times S^1$, so the elements of $\mathcal{A}_G\left(S^2 \times S^2\right)$ are the possibly "labels" for a loop. From our discussion of categorification we now know that:

$$\mathcal{A}_G\left(S^2 \times S^1\right) = \dim \mathcal{A}_G\left(S^2\right) \tag{13.2}$$

$$= |\mathbf{Rep}\left(G^\vee\right)| \ . \tag{13.3}$$

So, classically, a 't Hooft monopole is something you have on a closed loop in four dimensions which is labelled by a representation of the dual group $G^\vee$, but now it's just a vector space of representations.

Really this is what appears in the following setting. If we study this theory on a three manifold M^3 we get some vector space $\mathcal{A}_G\left(M^3\right)$. And if I give you some knot $K \subset M^3$, and a label $V \in |\mathbf{Rep}\left(G^\vee\right)|$, then we can construct a monopole operator which acts on the vector space $\mathcal{A}_G\left(M^3\right)$. So these are the 't Hooft loop operators. Representations $V \in |\mathbf{Rep}\left(G^\vee\right)|$ also label Wilson loops in the much simpler $\mathcal{B}$-theory $\mathcal{B}_{G^\vee}$. Recall that this theory studies flat $G^\vee$-connections on 3-manifolds. And given a flat $G^\vee$-connection and a knot $K \subset M^3$ there is a classical observable called the Wilson loop operator which takes the trace of the holonomy of the flat connection in a representation V along a knot K. So this defines a function on the space of local systems $\mathrm{Loc}_{G^\vee}\left(M^3\right)$.

13.2 Unramified Langlands correspondence for function fields

In the Langlands program, we will replace the 3-manifold M^3 with different objects playing the role of a 3-manifold for us. As we have seen, we can replace M with:

1. a curve over a finite field $C/\mathbb{F}_q$, and
2. a number field F, i.e. a finite extension of $\mathbb{Q}$.

The most important feature of the Langlands correspondence is this algebra $|\mathbf{Rep}(G^\vee)|$ acting. Instead of a knot $K \subset M^3$, we have either a point $x \in C(\mathbb{F}_q)$ in the first case, or a prime $p \in \mathcal{O}_F$. Recall the representation ring came from decategorifying the geometric Satake correspondence. We also saw that it came from a more subtle decategorification. Namely we saw that $|\mathbf{Rep}(G^\vee)|$ appears as $\mathrm{Tr}(F)$, the trace of the Frobenius, on:

$$\mathbf{Sph} = \mathbf{Shv}\left(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})\right) \ . \tag{13.4}$$

For the Frobenius to make sense, we are thinking e.g. that:

$$K = \mathbb{F}_q((t)) \supset \mathbb{F}_q[[t]] = \mathcal{O} \ . \tag{13.5}$$

The result is functions rather than sheaves:

$$Sph = |\mathbf{Sph}| = \mathrm{Fun}\left(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})\right) \ . \tag{13.6}$$

This is a long way of saying that the classical Satake isomorphism says:

$$\mathcal{H}_{G(K),G(\mathcal{O})} \simeq |\mathbf{Rep}(G^\vee)| \ . \tag{13.7}$$

In fact, classical Satake works the same for $K = \mathbb{F}_q((t))$ or K a finite extension of $\mathbb{Q}_p$ (mixed characteristic local field).

Now we recall why we classically study this Hecke algebra. Recall we have this identification:

$$\begin{aligned} \mathrm{Bun}_G(C) &\simeq G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_{\mathbb{A}}) && (13.8)\\ &= G(F)\backslash {\prod}' G(K_x)/G(\mathcal{O}_x) \ . && (13.9) \end{aligned}$$

I.e. it is a product of affine Grassmannians $G(K_x)/G(\mathcal{O}_x)$ modulo this big global piece $G(F)$. Because we can write it this way, this means it has a symmetry. Namely, functions on this space:

$$\mathrm{Aut}_C \coloneqq \mathrm{Fun}_c(\mathrm{Bun}_G(C)) \tag{13.10}$$

automatically have an action of the spherical Hecke algebra $|\mathbf{Sph}|$ for every point $x \in C$. In other words, the tensor product acts on Aut_C:

$$\bigotimes_{x\in C}{}' |\mathbf{Rep}(G^\vee)| \circlearrowright \mathrm{Aut}_C \ . \tag{13.11}$$

Remark 89. This is the everywhere unramified story. We will have to do some stuff to get the ramified version.

In this setting, the Langlands correspondence is saying the following. So we have this vector space Aut_C with this huge action. The idea is to try to match this with something which has the same symmetry. Do we know any other vector spaces that have this big action? Again we are supposed to think Wilson loops: whenever you naturally see functions labeled by representations of a group, if you look at spaces of flat connections then whenever you see a loop then you can take this Wilson function. So consider the space:

$$\mathrm{Loc}_{G^\vee}(C) = \{\mathrm{Gal}(F) \to G^\vee\}/G^\vee \ , \tag{13.12}$$

where we are quotienting out by conjugation on the right, and really we should think of a Weil group of F rather than the Galois group, meaning we should treat the Frobenius as if it generates a copy of $\mathbb{Z}$ (rather than $\widehat{\mathbb{Z}}$). There are some subtle issues here, e.g. this is not an algebraic variety in an obvious sense. In any case, for every point $x \in C$, there is a natural map:

$$\mathrm{Loc}_{G^\vee}(C) \to \frac{G^\vee}{G^\vee} \ . \tag{13.13}$$

Namely we can just forget everything about the specific representation besides the image of the Frobenius F_x associated to $x \in C$. I.e.

$$\rho \mapsto \rho(F_x) \ . \tag{13.14}$$

Dually this map is equivalent to a map:

$$\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = |\mathbf{Rep}(G^\vee)| \to \mathcal{O}(\mathrm{Loc}_{G^\vee}(C)) \ , \tag{13.15}$$

which are exactly what we were calling Wilson loops.

So, in other words, Aut_C and $\mathcal{O}(\mathrm{Loc}_{G^\vee}(C))$ both carry an action of this giant tensor product:

$$\mathrm{Aut}_C \circlearrowleft \bigotimes_{x\in C}{}' |\mathbf{Rep}(G^\vee)| \circlearrowright \mathcal{O}(\mathrm{Loc}_{G^\vee}(C)) \ . \tag{13.16}$$

On the right, we really just have a homomorphism, and the action just comes from multiplication. On the left it comes from these convolution/Hecke operators. So we found something which has the same symmetries as the space of automorphic forms Aut_C, and now we can ask:

Question 8. $\mathrm{Aut}_C \simeq \mathcal{O}(\mathrm{Loc}_{G^\vee}(C))$?

Before asking for it to be an isomorphism, we can ask the following. Again, the action on $\mathcal{O}(\mathrm{Loc}_{G^\vee}(C))$ really comes from a homomorphism:

$$\bigotimes_{x\in C}{}' |\mathbf{Rep}(G^\vee)| \to \mathcal{O}(\mathrm{Loc}_{G^\vee}(C)) \ . \tag{13.17}$$

So the first thing we can ask is whether this action factors as:

$$\begin{tikzcd}
\mathrm{Aut}_C \arrow[loop right, dashed] & & \mathcal{O}(\mathrm{Loc}_{G^\vee}(C)) \\
\bigotimes'_{x\in C} |\mathbf{Rep}(G^\vee)| \arrow[u, phantom, "\circlearrowright"] \arrow[urr] & &
\end{tikzcd} \ . \tag{13.18}$$

So we can ask whether the actions of these representation rings at every point "cohere" into, or satisfy the same relations as, they do inside of local systems on the curve. I.e. we are asking the following.

Question 9. Does Aut_C sheafify/spectrally decompose over $\mathrm{Loc}_{G^\vee}(C)$? I.e. does the $\mathcal{A}$-side sheafify over the $\mathcal{B}$-side?

The answer is yes. This is the "automorphic to Galois direction of the Langlands correspondence" and was shown by Vincent Lafforgue. There is a more abstract proof in [AGK+20b, AGK+20a, AGK+21]. cite

So now that we have this sheafification, we can ask: which module is this? This is still open, but we can at least make the conjecture, as stated in [AGK+20b, AGK+20a, AGK+21].

Conjecture 1 (Unramified Langlands correspondence for function fields)**.**

$$\mathrm{Aut}_C \simeq \omega\left(\mathrm{Loc}_{G^\vee}(C)\right) \tag{13.19}$$

as sheaves over $\mathrm{Loc}_{G^\vee}(C)$.

Remark 90. ω denotes the dualizing sheaf. The point is that the smart thing to do is to take volume forms rather than functions. This is analogous to the difference between taking **QC** versus **IndCoh**.

13.3 Ramified Langlands correspondence

One of the reasons we need to move to the ramified situation is to accommodate the real/original setting for the Langlands correspondence where we have a number field (rather than a 3-manifold or a function field). Ramification roughly involves passing from plain bundles to bundles equipped with data at certain points. In the number field setting there just isn't a version of the everywhere unramified setting: the prime at infinity always looks different from the others, and so there is a point on the curve which looks different from all of the others.

When we talk about ramification, there are kind of two stages/generations: a preliminary step and then a more sophisticated step. In the first step we basically ignore S, i.e. we try to understand what is happening away from the special points where out bundles have some extra structure. The second step is where we zoom in around one of the ramification points and try to understand what is going on there. This is the local Langlands correspondence.

13.3.1 Ramification on the automorphic side

When we pass to the ramified setting on the automorphic side, instead of $\mathrm{Bun}_G(C)$, we consider:

$$\mathrm{Bun}_G(C, S)\ , \tag{13.20}$$

where S is a finite subset of C, which consists of bundles equipped with some extra structure at the points in S.

More precisely, we can write this using the adélic description. Recall we have:

$$\mathrm{Bun}_G(C) = G(F)\backslash G(\mathbb{A})/\prod_{x\in S} G(\mathcal{O}_x)\ . \tag{13.21}$$

So now we will mod out by less on the right to get:

$$\mathrm{Bun}_G\left(C,S\right)=G\left(F\right)\backslash G\left(\mathbb{A}\right)/\left(\prod_{x\notin S}G\left(\mathcal{O}_x\right)\times\prod_{x\in S}K_x\right)\ ,\tag{13.22}$$

where $K_x\subset G\left(\mathcal{O}_x\right)$. If we don't mod out by anything on the right, i.e. we have $G\left(F\right)\backslash G\left(\mathbb{A}\right)$, then this consists of bundles with a trivialization near every point $x\in C$.

Once we have phrased the function field setting in this way, there is a natural analogue for a number field. For F a number field we write:

$$[G]_F=G\left(F\right)\backslash G\left(\mathbb{A}\right)\tag{13.23}$$

$$[G]_{F,K_S}=G\left(F\right)\backslash G\left(\mathbb{A}\right)/\left(\prod_{p\notin S}G\left(\mathcal{O}\right)\times\prod_{p\in S}K_p\right)\ ,\tag{13.24}$$

for S some collection of primes containing the prime at infinity.

In this setting, *automorphic forms* are functions (L^2 in the number field case) on $[G]_{F,K_S}$, and it is said that K_S is the *level* of the automorphic form.

13.3.2 Spectral side

So in the ramified story we have automorphic forms in the function field setting:

$$\mathrm{Aut}_{C,S,K_S}=\mathrm{Fun}_c\,\mathrm{Bun}_G\left(C,K_S\right)\tag{13.25}$$

and in the number field setting:

$$\mathrm{Aut}_{F,S,K_S}=L^2\left([G]_{F,K_S}\right)\ .\tag{13.26}$$

Then we can ask: what acts?

Remark 91. The physical picture of this is as follows. Consider a Riemann surface C. Then $\mathrm{Bun}_G\left(C,K_S\right)$ consists of bundles with some data at these points. From the physics point of view, we are thinking of points $x\in C$ as codimension 2 singularities in 4-dimensions. So we have surface defects/solenoids. This first agnostic stage of ramification (i.e. when we look away from S) we don't specify what is happening at these points, because we just want to do what we did before away from S.

Away from S, we still have the same local factor, i.e. it still looks like a product over $x\in S$ of $G\left(K\right)/G\left(\mathcal{O}\right)$, and then we still quotient on the left by this global group $G\left(F\right)$. Algebraically this is saying that for $x\notin S$ we still get an action of the same Hecke algebra:

$$\mathcal{H}_{G(K),G(\mathcal{O})}=\left|\mathbf{Rep}\left(G^\vee\right)\right|\ .\tag{13.27}$$

Function fields

Briefly restricting to the function field setting (we will go back to the number field setting in the next subsection) this is saying that $\mathrm{Aut}_{C,S}$ has an action of this big (restricted) tensor product:

$$\bigotimes_{x\notin S}{}^{\prime}\left|\mathbf{Rep}\left(G^\vee\right)\right|\circlearrowright\mathrm{Aut}_{C,S}\ .\tag{13.28}$$

Now we can ask the same question as before: where else does this algebra appear? I.e. what is something that has the same symmetries as $\mathrm{Aut}_{C,S}$? The same kind of picture as before naturally arises. Namely there is a natural quotient map of rings:

$$\bigotimes_{x \notin S} |\mathbf{Rep}\left(G^{\vee}\right)| \twoheadrightarrow \mathcal{O}\left(\mathrm{Loc}_{G^{\vee}}\left(C \setminus S\right)\right) \ , \tag{13.29}$$

and we can ask if the action factors as:

$$\begin{tikzcd} \mathrm{Aut}_{C,S} \arrow[loop right, dashed] & \mathcal{O}\left(\mathrm{Loc}_{G^{\vee}}\left(C \setminus S\right)\right) \\ \circlearrowright & \\ \bigotimes_{x \notin S} |\mathbf{Rep}\left(G^{\vee}\right)| \arrow[uur, two heads] & \end{tikzcd} \quad . \tag{13.30}$$

I.e. away from S we get the same spectral decomposition as before. Again this is a theorem of V. Lafforgue. Namely that, for any level structure K_S, $\mathrm{Aut}_{C,S}$ sheafifies over $\mathrm{Loc}_{G^{\vee}}\left(C \setminus S\right)$. cite
So these local systems are allowed to be ramified at S. At unramified points the local Galois group $\mathrm{Gal}\left(F\right)$ is acting through a small quotient $\mathbb{Z}\left\langle F\right\rangle$, i.e. a copy of $\mathbb{Z}$ generated by the Frobenius. At a ramified point we might have a complicated monodromy around this point. So at the points of S we are just not saying anything about the action factoring through this small quotient.

Remark 92. This is an amazing theorem, but we don't actually know *which* sheaf it is. To pin this down we really need to understand what is happening at the ramification points, which is the subject of local Langlands correspondence.

Number fields

Now we return to the number field setting. So let $[F : \mathbb{Q}] < \infty$ be a finite extension of $\mathbb{Q}$. Then we have this space of automorphic forms: Aut_{F,S,K_S} which depends on F, a finite set of primes S, and some level structure K_S. This carries an action of:

$$\bigotimes_{p \notin S}{}' |\mathbf{Rep}\left(G^{\vee}\right)| \ , \tag{13.31}$$

and we might ask for the same kind of answer. As it turns out, this same kind of answer is wrong. We might have guessed the following. Consider some version of the Galois group, and look at conjugacy classes of homomorphisms:

$$\left\{\mathrm{Gal}_F \to G^{\vee}\right\}/G^{\vee} \ . \tag{13.32}$$

Then we might have hoped to map from the restricted tensor product to functions on this space:

$$\bigotimes_{p \notin S}{}' |\mathbf{Rep}\left(G^{\vee}\right)| \to \mathcal{O}\left(\left\{\mathrm{Gal}_F \to G^{\vee}\right\}/G^{\vee}\right) \ , \tag{13.33}$$

and factor the action. As it turns out, this is not correct: it does not factor.

Langlands proposed a conjectural definition of a group, called the Langlands group of F, Lang_F such that this picture *does* hold, once we have replaced the Galois group with the Langlands group. So there is supposed to be a map:

$$\mathrm{Lang}_F \to \mathrm{Gal}_F \ , \tag{13.34}$$

and so we are supposed to look at homomorphisms:

$$\mathrm{Lang}_F \to G^\vee \ . \tag{13.35}$$

Some will factor through $\mathrm{Gal}(F)$, but not all of them. So we need a bigger space of parameters.

Remark 93. This is one reason it can be hard to pin down a statement of the Langlands correspondence: it involves representations of a version of the Galois group that hasn't been fully defined.

Another point of view is that, inside of these spaces of automorphic forms, there are smaller spaces:

$$\begin{tikzcd}
\text{automorphic forms} \\
\text{algebraic automorphic forms} \arrow[u, hook] \\
\text{cohomological automorphic forms} \arrow[u, hook]
\end{tikzcd} . \tag{13.36}$$

These spaces are more accessible than the space of all automorphic forms, but they still capture many of the reasons people care about the Langlands correspondence. Roughly, the difference between ordinary automorphic forms is that $L^2([G]_F)$ is replaced by $H^*([G]_F)$, possibly with twisted coefficients. For example we might look at $H^*(\Gamma\backslash G_{\mathbb{R}}/K)$, rather than L^2 functions on this space, and this is basically just group cohomology $H^*(\Gamma)$. We won't go into detail about algebraic automorphic forms, but roughly speaking the difference between all automorphic forms and algebraic/cohomological automorphic forms is a condition at the prime at ∞.

Then there is something called the Clozel conjectures, which are a version of the Langlands conjectures.

Conjecture 2 (Clozel)**.** *Algebraic automorphic forms correspond to Galois representations.*

I.e. if we put a condition on $\mathrm{Aut}_{F,S}$, then we have a picture like before:

$$\begin{tikzcd}
\mathrm{Aut}^{\mathrm{alg}}_{F,S} \arrow[loop right, dashed] & \mathcal{O}\left(\{\mathrm{Gal}_F \to G^\vee\}/G^\vee\right) \\
\bigotimes'_{p\notin S} |\mathbf{Rep}(G^\vee)| \arrow[u, circlearrowright] \arrow[ur]
\end{tikzcd} . \tag{13.37}$$

We are still sweeping a bit under the rug because we need to be talking about ℓ-adic representations, and we haven't specified ℓ. The solution, is that instead of just Galois representations $\mathrm{Gal} \to G^\vee$, we have compatible collections of ℓ-adic Galois representations for all ℓ. This is a version of what people call *motives.* Equivalently the Clozel conjectures deal with representations of some kind of Motivic Galois group. The classical picture to have in mind is that we have, say, L-functions of modular forms, which are in $\mathrm{Aut}^{\mathrm{alg}}_{F,S}$, and then on the other side one should think of elliptic curves, which give rise to Motivic Galois representations.

Remark 94. When we consider functions on $[G]_F$, we need to specify the field k which they are valued in. When we consider Galois representations $\mathrm{Gal}(F) \to G^\vee$, the group $G^\vee$ is defined over the same field k. For simplicity fix $G^\vee = \mathrm{GL}_n$. So we are considering representations:

$$\mathrm{Gal}(F) \to \mathrm{GL}_n(k) \ . \tag{13.38}$$

The problem is that the Galois group is a profinite groups, i.e. a type of topological group. And the only reasonable notion of a representation of a topological group is a continuous representation.

We might guess that we can just take $k = \mathbb{C}$, but the topology of $\mathrm{Gal}(F)$ and $\mathbb{C}$ don't align well: the only continuous representations $\mathrm{Gal}(F) \to \mathrm{GL}_n(\mathbb{C})$ are ones with finite image. This is already important (it is the story of the Artin conjectures), but they don't cover all of the ones we are interested in. For example, the representations that come from an elliptic curve are not finite image representations. In fact, the finite image representations that come from algebraic geometry just come from the cohomology of 0-dimensional varieties.

To get more representations, we should look for representations valued in something like $\mathrm{GL}_n\left(\overline{\mathbb{Q}_\ell}\right)$. Of course $\overline{\mathbb{Q}_\ell}$ is isomorphic to $\mathbb{C}$, but it has some topology so these ℓ-adic Galois representations are no longer required to have finite image. But now we have some extra complications. For example, if the representation comes as the cohomology of an algebraic variety over $\mathbb{F}_p$, then we might have that $\ell = p$, which needs to be treated separately. Then the question becomes: what is ℓ? In the local situation, so we have a single p, we can just choose $\ell \neq p$. But if we're doing something global, so we're trying to do all primes at once, then we can't pick an ℓ which is different from all primes p. So we are led to the idea of a compatible family of representations. I.e. instead of considering a single ℓ-adic representation, consider a collection of compatible Galois representations for all ℓ.

This seems like a weird notion, but it comes from algebraic geometry. Let X be a scheme. Then we might try to define its cohomology. So we need to pick a field of coefficients. As it turns out, there is not a good algebraic cohomology theory with rational coefficients. I.e. there is no functor landing in $\mathbb{Q}$-vector spaces satisfying the appropriate conditions. But we can do this for $\mathbb{Q}_\ell$ for every ℓ. Each of these have an action of Gal, so we have one for every ℓ. So really we have a family of ℓ-adic Galois representations for all primes ℓ (or maybe $\ell \neq p$ if $X/\mathbb{F}_p$). Really what we would like to study is some kind of Motivic cohomology of X: $H^*_M(X)$, which is some kind of $\mathbb{Q}$-vector space. This doesn't actually exist, but it is a well-defined object of a $\mathbb{Q}$-linear category, called the category of motives. Then given a motive and a prime ℓ, we get a $\mathbb{Q}_\ell$-vector space by taking ℓ-adic étale cohomology.

13.4 L-functions

With the same starting data, there is something more concrete we can do. So we have this space of automorphic forms Aut with an action of this global Hecke algebra:

$$\mathcal{H}\mathrm{ecke} = \bigotimes_{p \notin S}{}' \left|\mathbf{Rep}\left(G^\vee\right)\right| \ . \tag{13.39}$$

Spectral decomposition is just a fancy version of diagonalization. So when we do spectral decomposition, the first thing we can ask for is Hecke eigenfunctions. An automorphic form $f \in \mathrm{Aut}$ is, in particular, a joint eigenfunction of all of the Hecke operators.[1] One thing we

[1] There are some extra analytic growth conditions that we won't focus on.

can ask, is how to capture the eigenvalue for the action of $\mathcal{H}$ecke on Aut. Technically it is a character, i.e. a homomorphism:

$$\mathcal{H}\text{ecke} \to k \ . \tag{13.40}$$

But this is not very concrete. Recall:

$$\mathcal{H}\text{ecke} = \bigotimes \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \ . \tag{13.41}$$

We can choose a basis for invariant functions on $G^\vee$, and try to describe the eigenvalues of these operators in this basis. But there is a more elegant approach. Suppose that $G^\vee$, or more generally pick a homomorphism:

$$V : G^\vee \hookrightarrow \mathrm{GL}_n \ . \tag{13.42}$$

This corresponds to a map:

$$\bigotimes \mathcal{O}\left(\frac{\mathrm{GL}_n}{\mathrm{GL}_n}\right) \to \bigotimes \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \ . \tag{13.43}$$

Now we've turned this into a question about class functions on GL_n. Everyone has a favorite way of measuring a matrix, which is the characteristic polynomial. This is why we translate the question to one in terms of GL_n: an arbitrary $G^\vee$ doesn't have a clear way to encode the information of an element. Let $F \in \mathrm{GL}_n$. Then we attach the *characteristic polynomial*, which is:

$$\det(1 - tF) \ . \tag{13.44}$$

A fancier way of saying it, is that we're looking at the graded trace of F acting on the exterior algebra of the standard representation:

$$\det(1 - tF) = \mathrm{gr}\,\mathrm{tr}\left(F, \wedge^\bullet \mathbb{C}^n\right) \ . \tag{13.45}$$

The point is that the coefficients of the polynomial are the traces of the matrix on exterior powers of the standard representation. Really we are thinking of V as the representation filling the place of $\mathbb{C}^n$. For us, we start with $F \in G^\vee$, map it to a matrix in GL_n, and then take the characteristic polynomial:

$$\det(1 - tF) = \mathrm{gr}\,\mathrm{tr}\left(F, \wedge^\bullet V\right) \ . \tag{13.46}$$

Now we consider the reciprocal of the characteristic polynomial. This is something that is natural to do from the point of view of number theory, but in any case it clearly contains the same information. This also has a fancier description:

$$\frac{1}{\det(1 - tF)} = \mathrm{gr}\,\mathrm{tr}\left(F, \mathrm{Sym}^\bullet V\right) \ . \tag{13.47}$$

This is just a version of the geometric series formula:

$$\frac{1}{1-t} = 1 + t + \cdots \ . \tag{13.48}$$

Remark 95. This is some version of Koszul duality. Koszul duality is a fancy version of this, but it still involves a passage between symmetric and exterior algebras.

Let W be a vector space with an action of:

$$\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = |\mathbf{Rep}\,(G^\vee)| \ . \tag{13.49}$$

If $v \in W$ is an eigenvector, then the eigenvalue is a homomorphism

$$\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \to k \ , \tag{13.50}$$

which is the same as a point:

$$F_v \in G^\vee /\!\!/ G^\vee = H^\vee / W \ . \tag{13.51}$$

Then we can write this fancy characteristic polynomial:

$$\frac{1}{\det\left(1 - F_v\right)} \tag{13.52}$$

which is another way of encoding the data of this eigenvalue.

Then the *L-function* of an automorphic form f is a version of (the reciprocal of) the characteristic polynomial of the Hecke eigenvalue. I.e. the L-function is:

$$L\left(f, V, s\right) := \prod_{p \notin S} \frac{1}{\det\left(1 - p^{-s} F_p\right)} \tag{13.53}$$

where $F_p \in \frac{G^\vee}{G^\vee}$ is the eigenvalue of the action of:

$$|\mathbf{Rep}\,(G^\vee)| = \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \tag{13.54}$$

associated to p.

Notice that the definition of the L-function only depends on the conjugacy classes $F_p \in \frac{G^\vee}{G^\vee}$. So if we have a Galois representation:

$$\rho \colon \ \mathrm{Gal}\left(F\right) \to G^\vee \tag{13.55}$$

then we can define an L-function:

$$L\left(\rho, V, s\right) := \prod_{p \notin S}{}' \frac{1}{\det\left(1 - F_p p^{-s}\right)} \ . \tag{13.56}$$

In these terms, the Langlands correspondence loosely says that an automorphic form f should get matched to a Galois representation ρ if their L-functions agree:

$$L\left(f, V, s\right) = L\left(\rho, V, s\right) \ . \tag{13.57}$$

On the right, $L\left(\rho, V, s\right)$ is a version of the characteristic polynomial of F_p for all p. On the left, $L\left(f, V, s\right)$ is a version of the characteristic polynomial of the Hecke eigenvalue.

Example 90. The L-function of the trivial 1-dimensional representation is the Riemann ζ-function:

$$\prod \frac{1}{1 - p^{-s}} = \zeta\left(s\right) = \sum \frac{1}{n^s} \ . \tag{13.58}$$

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13.5 Global Langlands

transition

Recall the general outline of the Langlands program is as follows. We have some space of automorphic forms over a field F, and then we have some subset S of primes of $\mathcal{O}_F$ (or points of the corresponding curve) where things are ramified. In any case, we can look where we do know what is going on, i.e. where it is unramified, i.e. at primes $p \notin S$, and for every such prime we get an action:

$$\mathrm{Aut} \circlearrowleft |\mathbf{Rep}\,(G^\vee)| = \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \ . \tag{13.59}$$

Then we can try to diagonalize this action, i.e. find the eigenvalues. An eigenvalue for the action of $\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right)$ (corresponding to a prime p) is a homomorphism from this ring to the ground field, which is the same as a point, i.e. a conjugacy class:

$$\lambda_p \in G^\vee /\!\!/ G^\vee = \operatorname{Spec} \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = H^\vee / W \tag{13.60}$$

where $H^\vee$ is the Cartan of $G^\vee$ and W is the Weyl group. Note that this is technically different from the stack $\frac{G^\vee}{G^\vee}$.

This is just to say that if we have $f \in \mathrm{Aut}$ which is a Hecke eigenform, i.e. an eigenfunction for $\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right)_p$ (for every $p \notin S$) then we can attach the data:

$$\left\{\lambda_p \in G^\vee /\!\!/ G^\vee\right\}_{p \notin S} \ . \tag{13.61}$$

Example 91. Let f be classical modular form:

$$f \in L^2\left(\mathrm{SL}_2\left(\mathbb{Z}\right) \backslash \mathrm{SL}_2\left(\mathbb{R}\right) / \mathrm{SO}_2\right) \ . \tag{13.62}$$

Among these are Hecke eigenforms, which can be described in terms of the behavior of their Fourier coefficients.

Then the Langlands correspondence tells us that the same kind of data can be attached to a rep of the Galois group. So given a Galois representation and $p \notin S$, we get a map:

$$\left\{\mathrm{Gal}\left(F\right) \to G^\vee\right\} \to \frac{G^\vee}{G^\vee} \tag{13.63}$$

where a representation $\rho\colon \mathrm{Gal}\left(F\right) \to G^\vee$ maps to $\rho\left(F_p\right)$, where F_p is the Frobenius for p. I.e. we get the data:

$$\left\{\rho\left(F_p\right) \in G^\vee /\!\!/ G^\vee\right\}_{p \notin S} \ . \tag{13.64}$$

Inside of the space of all automorphic forms we have a subspace of algebraic automorphic forms.[2] To say something about all automorphic forms we need something called the Langlands group. So now the rough dream of the Langlands correspondence is that there should be a match between algebraic automorphic forms and Galois representations. And the way they are matched is with their numerics: the Hecke eigenvalues should be identified with the eigenvalues of the Frobenius for ρ:

$$\left\{\lambda_p\right\}_{p \notin S} = \left\{\rho\left(F_p\right)\right\}_{p \notin S} \ . \tag{13.65}$$

[2] The condition we put to reduce to this subspace is a condition on the infinite prime.

The way this is usually stated is not in terms of these numbers, but in terms of functions called the L-function, which can be constructed from the above numerical data. So given a representation:[3]

$$V : G^\vee \to \mathrm{GL}_n \ , \tag{13.66}$$

we can define the L-function of f to be:

$$L(f, V, s) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s} V(\lambda_p))} \ . \tag{13.67}$$

Now, given a Galois representation ρ, we can write the same formula to define the L-function of ρ:

$$L(\rho, V, s) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s} V(\rho(F_p)))} \ . \tag{13.68}$$

And so the basic idea is that f and ρ should be matched if

$$L(f, V, s) = L(\rho, V, s) \ . \tag{13.69}$$

The motivation is that $L(\rho, V, s)$ are very important, and the fact that they can be written as $L(f, V, s)$ yields a huge amount of arithmetic properties. For example, it can help us evaluate $L(\rho, V, s)$ at certain values of s.

Example 92. The Riemann ζ-function is the L-function for the trivial representation:

$$\zeta(s) = \prod \frac{1}{1 - p^{-s}} = \sum \frac{1}{n^s} \ . \tag{13.70}$$

So we might have a description of these functions as an infinite product and we want to use this to regularize them. This regularization is part of this vague theme of reciprocity/magical glue. Everything we have said has treated the primes $p \notin S$ (or points of the corresponding curve) as totally independent and we're not cohering them together in any way. From the adélic point of view the special sauce/glue that brought them together was the global factor $G(F)$ on the left:

$$[G] = G(F) \backslash \prod G(K_x) / \prod G(\mathcal{O}_x) \ . \tag{13.71}$$

On the level of L-functions, that is what this analytic continuation is. In particular, the Langlands conjectures imply that $L(\rho, V, s)$ is an *entire* function of s, for ρ irreducible (and $\dim > 1$).

13.6 The local Langlands correspondence

13.6.1 General shape

Recall we have a 4-dimensional TFT $\mathcal{A}_G$, and we're feeding in a number field, which we're thinking about as an analogue of a 3-manifold. Therefore we expect $\mathcal{A}_G$ to send the number field to a vector space:

$$\mathcal{A}_G\left(M^3\right) \in \mathbf{Vect} \ . \tag{13.72}$$

[3] Recall we map to GL_n so we have a notion of a characteristic polynomial.

This is the vector space of unramified automorphic forms.

In practice, we don't just have a 3-manifold, but rather a 3-manifold M with a bunch of knots/link inside. These are primes in S in the number field setting. Then we excise the link to get:

$$M_0 \coloneqq M^3 \setminus \text{ link } . \tag{13.73}$$

Rather than a vector space, we get an object of the following category:

$$\mathcal{A}_G\left(M_0\right) \in \mathcal{A}_G\left(\partial M_0\right) \simeq \bigotimes_{\pi_0(\partial M_0)} \mathcal{A}_G\left(T^2\right) \ , \tag{13.74}$$

since the boundary of the tubular neighborhood of the link is a union of tori.

So in a ramified situation, we think that we're living in some category:

$$\mathcal{A}_G\left(M_0\right) \in \mathcal{A}_G\left(\partial M_0\right) \ , \tag{13.75}$$

there is some equivalence of categories:

$$\mathcal{A}_G\left(\partial M_0\right) \simeq \mathcal{B}_{G^\vee}\left(\partial M_0\right) \tag{13.76}$$

coming from EM-duality, called local Langlands, and then this local Langlands correspondence matches an object of $\mathcal{A}_G\left(\partial M_0\right)$ with an object of $\mathcal{B}_{G^\vee}\left(\partial M_0\right)$:

$$\mathcal{A}_G\left(M_0\right) \in \mathcal{A}_G\left(\partial M_0\right) \simeq \mathcal{B}_{G^\vee}\left(\partial M_0\right) \ni \mathcal{B}_{G^\vee}\left(M_0\right) \ . \tag{13.77}$$

I.e. we have global states which are compatible with and equivalence of the local categories.

Remark 96. Recall unramified *geometric* Langlands fixes a compact curve Σ over $\mathbb{C}$ and then we attached this category of sheaves $\mathbf{Shv}\left(\mathrm{Bun}_G\left(\Sigma\right)\right)$, and said this should be equivalent to $\mathbf{QC}$ of local systems on the curve:

$$\mathbf{Shv}\left(\mathrm{Bun}_G\left(\Sigma\right)\right) \simeq \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\left(\Sigma\right)\right) \ . \tag{13.78}$$

To pass to the ramified setting we remove n disks from Σ, which we will write as Σ_0. We can think of this surface with boundary as a bordism from the empty 1-manifold to the disjoint union of n copies of S^1. So the TFT sends this to a map:

$$\mathcal{A}_G\left(\Sigma_0\right) : \mathcal{A}_G\left(\emptyset\right) = 1 \to \bigotimes_1^n \mathcal{A}_G\left(S^1\right) \ , \tag{13.79}$$

i.e. we can think that this defines an object:

$$\mathcal{A}_G\left(\Sigma_0\right) \in \mathcal{A}_G\left(\partial\Sigma_0\right) = \bigotimes_1^n \mathcal{A}_G\left(S^1\right) \ . \tag{13.80}$$

Now the point is that if we start with a 3-manifold and excise a link, the boundary is a surface. Which is geometric Langlands. I.e. if we're flexible enough about our context, local Langlands is the same as geometric Langlands. The punchline will be that the local Langlands correspondence is a version of geometric Langlands on some specific curve.

Given a local field, e.g. $\mathbb{Q}_p$, we attach a category: $\mathcal{A}_G(\mathbb{Q}_p)$, and the local Langlands correspondence is an equivalence of categories:

$$\mathcal{A}_G(\mathbb{Q}_p) \simeq \mathcal{B}_{G^\vee}(\mathbb{Q}_p) \ . \tag{13.81}$$

Then the collection of automorphic forms ramified at p should be an object of the category $\mathcal{A}_G(\mathbb{Q}_p)$, and it should match with some spectral data (the Galois representations ramified at p).

The other thing to say is that if we're really looking at a curve over a finite field, then we might hope to do what we did before, which was take the trace of the Frobenius. So if we do the geometric story, so it's categorified, then to get the arithmetic story we should take the trace of the Frobenius. This gives a good hint towards what should be attached to a local field.

13.6.2 The local categories

Let's try to understand what kind of beasts these categories are. We want to populate these categories. To do this, let's try to remember how we got into this mess. Recall we had this automorphic space:

$$[G]_F = G(F) \backslash {\prod_p}' G(K_p) / \prod_{p\notin S} G(\mathcal{O}_p) \times \prod_{p\in S} K_p \tag{13.82}$$

where F is either a function field of a curve or a number field. In the geometric situation, this corresponds to looking at G-bundles on a curve $C/\mathbb{F}_p$ with some level structure at $p \in S$. Then our notion of automorphic forms was:

$$\mathrm{Aut} = \mathcal{O}([G]_F) \ . \tag{13.83}$$

If we look at the factor at p:

$$G(K_p)/H \tag{13.84}$$

and consider functions on this, then this is acted on by:

$$H\backslash G(K_p)/H = \mathcal{H} \ . \tag{13.85}$$

In other words:

$$\mathrm{Aut} \in \mathcal{H}_{G(K),H}\text{-}\mathbf{mod} \ . \tag{13.86}$$

The point of this category is that it fits inside of:

$$\mathcal{H}_{G(K),H}\text{-}\mathbf{mod} \hookrightarrow \mathbf{Rep}(G(K)) \ , \tag{13.87}$$

the point being that:

$$\mathcal{H}_{G(K),H} = \mathrm{End}_{G(K)}(\mathrm{Fun}(G(K)/H)) \ . \tag{13.88}$$

Taking H-invariants defines a map in the other direction. Therefore Aut corresponds to some object

$$\widetilde{\mathrm{Aut}} \in \mathbf{Rep}(G(K)) \ . \tag{13.89}$$

The process of forming $\widetilde{\mathrm{Aut}}$ is just removing the quotient by H. I.e. we have that Aut is functions on:

$$G(F) \backslash {\prod}' G(K_p) / \prod \times H \ , \tag{13.90}$$

and then $\widetilde{\mathrm{Aut}}$ is functions on:

$$G(F) \backslash {\prod_p}' G(K_p) / \prod \ . \tag{13.91}$$

The upshot is that if we don't quotient out by H, then the entire group $G(K_p)$ acts on this space.

Let's consider the more extreme version: modding out by nothing on the right gives:

$$G(F) \backslash {\prod_p}' G(K_p) \ , \tag{13.92}$$

which has an action on the right of $G(K_p)$ for all primes p. Then we quotient out by stuff on the right to get a space that isn't so big.

Geometrically we are looking at Bun_G, and then we are asking for *full level structure* at the point $x \in C$. This means we're taking the bundle and trivializing it on a disk at x.

So if I have a bad prime, with a weird level structure, then we can just pass to a full level structure, i.e. pass to something with an action of the whole group $G(K)$. An analogue of this that we have already seen is as follows. People study automorphic forms on $\mathrm{SL}_2 \mathbb{Z}\backslash\mathbb{H}$, but this is the same as:

$$\mathrm{SL}_2 \mathbb{Z}\backslash\mathbb{H} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \ , \tag{13.93}$$

but then we can just study:

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) \ , \tag{13.94}$$

which has an action of the full group:

$$G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ . \tag{13.95}$$

It is somehow much more aesthetic to not mod out by anything on the right and get an honest representation of a Lie group.

This suggests that our category $\mathcal{A}_G$ at a prime p should be the category of representations of $G(\mathbb{Q}_p)$, or $G(K_p)$. With no level structure we had:

$$\mathrm{Fun}(\mathrm{Bun}_G) \in \mathbf{Vect} \ , \tag{13.96}$$

and now we have:

$$\mathrm{Fun}(\mathrm{Bun}_G, \infty \text{ level at x}) \in \mathbf{Rep}(G(K_x)) \ . \tag{13.97}$$

So the local Langlands correspondence should be some kind of spectral decomposition of $G(K)$-representations. These groups e.g. $G(\mathbb{Q}_p)$ or $G(\mathbb{F}_q((t)))$ are topological groups, so these representations should be continuous. We can capture this more concretely by asking for smooth $G(K)$-representations. A smooth $G(K)$-representation V is the same as a continuous representation of $G(K)$ where V is given the discrete topology. Or equivalently it means that for all $v \in V$ there exists an open subgroup $H \subset G(K)$ fixing v.

The point is that this category of representations sheafifies over some space of Langlands parameters. So we can think of this as a version of Pontrjagin duality for the group $G(K)$. If we wanted to do some kind of Fourier theory for nonabelian groups such as $G(\mathbb{Q}_p)$, this is what the Langlands program does.

13.6.3 Spectral side

Question 10. What is the spectral side?

There is a naive guess, which is something like $\mathrm{Loc}_{G^\vee}$. This will be the stack of Langlands parameters. In a geometric situation this will be some kind of stack of local systems, but in general, for some version of the Galois group (Weil, Weil-Deligne), this is the collection of morphisms:

$$\mathrm{Gal}(K) \to G^\vee \ , \tag{13.98}$$

i.e. the space of $G^\vee$ local systems on $\mathrm{Spec}(K)$. Recall the picture we have in physics, the state-operator correspondence, which says to take the boundary of the tubular neighborhood (the link) of the singularity, and then the possible insertions at the singularity correspond to the value of the theory on the link. I.e. the insertions are linear objects (functions, sheaves, etc.) on the space of fields on the link.

What we're trying to get at, is we don't know what is going on at the actual singularity, but away from the red point, we're doing what we had before: there's nothing bad going on. So if you convince yourself that in the good/old/unramified situation you're supposed to have $G^\vee$ local systems, then in the ramified situation we're doing exactly that on the link, i.e. we remove the singularity.

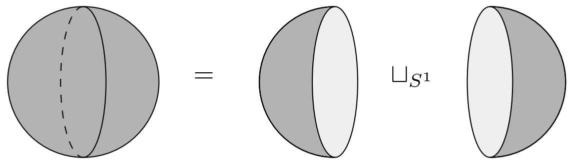
In number theory, the way people got around this was very roundabout. The historical development is a weird chicken and egg situation, where a lot of the results in local Langlands came about because they were inspired by a result in global Langlands. E.g. we know some result about L-functions, and then we can try to use this information to determine what is happening at the bad primes.

So for a curve C, or $\mathrm{Spec}\,\mathcal{O}_F$, consider a prime p where we have ramification. So we know something complicated is happening here, but we also know that nearby primes/points are unramified. There may be other (finitely many) ramification points, but generically we're unramified. So at one of these unramified points, we have an action of our favorite commutative operators: the spherical Hecke algebra. So every point away from p we just get local systems. Then we want to take the limit as we approach the ramification point. So this works fine in the geometric setting, but in the number field setting, it's hard to approach p from other primes. E.g. it's hard to find a prime that's very close to 5. This makes this picture very difficult to formulate precisely. But in the geometric setting, we can imagine doing Hecke modifications at a nearby point, and taking a limit. Zooming in around a ramification point, we're looking at a punctured disk. In the number field situation this is something like $\mathrm{Spec}\,\mathbb{Q}_p$. Then we're saying that at every point of this punctured disk we have the usual unramified story. In other words we get a $G^\vee$-local systems on this punctured disk:

$$\mathrm{Loc}_{G^\vee}\left(D^\times\right) \ . \tag{13.99}$$

Remark 97. Even in ordinary algebraic geometry this doesn't really make sense: these formal punctured disks don't have many points. This makes sense in a complex-analytic picture, but in a complex-algebraic picture, what do we mean by a point of $D^\times$? In algebraic geometry there is a history of making sense of things like this, involving words like “nearby cycles”.

From the physics point of view, this is just the statement that line operators: $\mathcal{Z}\left(S^2\right)$, act on surface operators: $\mathcal{Z}\left(S^1\right)$. So we're thinking of having a surface defect in 4-dimensions which intersects transversely with the curve, i.e. the intersection is a single point. This is the ramification point. Then we can imagine that at nearby points we have line defects, so our Hecke operators, and we can take the limit as the line defect collides with the surface defect. Writing S^2 as



(13.100)

we see that:

$$\mathcal{Z}\left(S^2\right) = \mathrm{End}\left(1_{\mathcal{Z}(S^1)}\right) \ , \tag{13.101}$$

where $1_{\mathcal{Z}(S^1)}$ is the trivial surface defect.

Now we can state that the local Langlands correspondence relates/spectrally decomposes smooth representations of $G(K)$ to/over $\mathrm{Loc}_{G^\vee}(K)$. The following is a concrete result to this effect.

Theorem 13 (Harris-Taylor, Henniart)**.** *Irreducible, smooth representations for* $\mathrm{GL}_n(K)$*, for* K *a local field, are in bijection with isomorphism classes of rank* n *Galois representations over* K*,* $\mathrm{Loc}_{\mathrm{GL}_n}/\sim$*.*

cite

This is saying that $\mathrm{Loc}_{\mathrm{GL}_n}/\sim$ is playing the role of a Pontrjagin dual, of sorts, of $\mathrm{GL}_n(K)$. But what about more general groups? For general groups people have found, experimentally and now formally, that irreducible representations of $G(K)$ map to $\mathrm{Loc}_{G^\vee}$. I.e. this is the automorphic to spectral direction, i.e. we can attach a Langlands parameter to an irreducible representation. The idea is that you should be able to attach Galois representation to a $G(K)$ representation, but this map is not 1-to-1. This is a theorem due to V. Lafforgue-Genestier in the function-field setting, and Fargues-Scholze in the number-field setting. The fibers of this map are called *L-packets*. These are L-indistinguishable (cite) representations. I.e. representations that L-functions can't tell apart. So this isn't a 1-to-1 correspondence, but these packets are small.

Why do we have this ambiguity? Really we should be attaching a sheaf on $\mathrm{Loc}_{G^\vee}$: on one side we have representations of $G(K)$, and the other side is some kind of category of sheaves on $\mathrm{Loc}_{G^\vee}$. But $\mathrm{Loc}_{G^\vee}$ is a stack because local systems have automorphisms. So even if a sheaf is supported on a single representation, $\rho\colon \mathrm{Gal} \to G^\vee$, there is more data: $\mathrm{Aut}(\rho)$, the centralizer of ρ, has to act. So we're getting:

$$\mathbf{Shv}(\mathrm{pt}/Z(\rho)) = \mathbf{Rep}(Z(\rho)) \ . \tag{13.102}$$

The refined version should say that the L-packets should form a subset of the representations of this centralizer $Z(\rho)$, which are trivial on the identity component. I.e. it is representations of $\pi_0(Z(\rho))$. This tells us why we should expect these packets: we were trying to think of things too set-theoretically.

Out of this picture we get a full subcategory:

$$\mathbf{Rep}(G(K)) \hookrightarrow \mathbf{Coh}(\mathrm{Loc}_{G^\vee}) \ . \tag{13.103}$$

This is more refined version of the local Langlands correspondence. This is shown for GL_n in [BZCHN21].

Remark 98. This is finally something that looks like a geometric Langlands correspondence. We said a local field is something like a 2-manifold, and, in geometric Langlands, the spectral side is some kind of coherent sheaves on $\mathrm{Loc}_{G^\vee}$ of the 2-manifold. But the automorphic side doesn't quite look the same yet.

If we look at $K = \mathbb{C}((t))$, then this is the analogue of a 1-manifold. So we should do something one level higher. I.e. we're doing local geometric Langlands. Then we consider the loop group:

$$LG = G(K) \ , \tag{13.104}$$

and instead of considering ordinary representations, we consider $G(K)$-categories. For example:

$$G(K) \circlearrowright \mathbf{Shv}(G(K)/H) \tag{13.105}$$

for any subgroup H. The 2-category of $LG = G(K)$-categories is what the TFT attaches to a circle:

$$\mathcal{A}_G\left(S^1\right) = \mathbf{Rep}_{\mathbf{Cat}}(G(K)) \ . \tag{13.106}$$

Then we can formulate local geometric Langlands as saying that the 2-category $\mathcal{A}_G\left(S^1\right)$ is identified with quasi-coherent sheaves of categories over $\mathrm{Loc}_{G^\vee}(D^\times)$. If we do this over $\mathbb{F}_q((t))$ instead, then this has this Frobenius action, and when we take a trace, to decategorify, then we get local Langlands for $\mathbb{F}_q((t))$.

The resulting version of local Langlands is actually nicer than what we formulated above: we get an equivalence rather than a full subcategory. We were a bit too naive about what to put on the automorphic side. There is a natural bigger category to choose, which will match exactly with $\mathbf{Coh}(\mathrm{Loc}_{G^\vee})$. I.e. $\mathbf{Rep}(G(K))$ is too small. The following example illustrates this.

Example 93. Let $G = T$. Rather than looking at all representations, we can look at unramified ones. This just means a representation of $T(K)$, only now we take double cosets for $T(\mathcal{O})$, but this is abelian (so double cosets are the same as single cosets) so we're considering:

$$\mathrm{Gr}_T = T(K)/T(\mathcal{O}) \simeq \Lambda \ , \tag{13.107}$$

where Λ is the character lattice of T. Then we have:

$$\Lambda\text{-}\mathbf{mod} \simeq \mathbf{QC}(T^\vee) \ . \tag{13.108}$$

This looks great, but this is actually not everything. Consider unramified local systems: $\mathrm{Loc}_{T^\vee}^{\mathrm{unram}}$, i.e. local systems that factor through the Frobenius. So one of these is determined by the image of the Frobenius in the dual group, i.e. an element of $T^\vee$. But as a stack, this still has automorphisms:

$$\mathrm{Loc}_{T^\vee}^{\mathrm{unram}} = \frac{T^\vee}{T^\vee} = T^\vee \times \mathrm{pt}/T^\vee \ . \tag{13.109}$$

But when we look at sheaves on this, we get a full subcategory:

$$\Lambda\text{-}\mathbf{mod} \simeq \mathbf{QC}(T^\vee) \hookrightarrow \mathbf{QC}\left(\mathrm{Loc}_{T^\vee}^{\mathrm{unram}}\right) \ . \tag{13.110}$$

I.e. we only got a full subcategory before (rather than equivalence) because we ignored the stabilizer. So in order to get an equivalence we need this extra factor:

$$\begin{tikzcd} \Lambda\textbf{-mod} \arrow[d, hook] & \simeq & \mathbf{QC}\left(T^{\vee}\right) \arrow[d, hook] \\ \Lambda\textbf{-mod} \otimes \mathbf{Rep}\left(T\right) & \simeq & \mathbf{QC}\left(\mathrm{Loc}_{T^{\vee}}^{\mathrm{unram}} = \frac{T^{\vee}}{T^{\vee}}\right) \end{tikzcd} . \tag{13.111}$$

We can think of this as a failure of decategorification. When we decategorify, there is a vertical/naive decategorification and a horizontal/smart decategorication of

$$\mathbf{Rep}\left(T^{\vee}\right) = \mathbf{QC}\left(\mathrm{pt}/T^{\vee}\right) . \tag{13.112}$$

The vertical/naive version takes dim of this as a plain category to get:

$$\dim\left(\mathbf{Rep}\left(T^{\vee}\right)\right) = \left(\left|\mathbf{Rep}\left(T^{\vee}\right)\right|, \bullet\right) = \mathcal{O}\left(T^{\vee}\right) . \tag{13.113}$$

Now we can pass to modules over this:

$$\mathcal{O}\left(T^{\vee}\right)\textbf{-mod} = \mathbf{QC}\left(T^{\vee}\right) . \tag{13.114}$$

This is the vertical/naive trace.

On the other hand, we can take the horizontal/smart version, which takes dim of $\mathbf{Rep}\left(T^{\vee}\right)$ as a $\otimes$-category, which is more like the Hochschild homology of an algebra, i.e. we will end up with a category. Geometrically we can calculate this by commuting $\mathbf{QC}$ with dim, i.e. we are taking the loop space of $\mathrm{pt}/T^{\vee}$

$$\mathbf{QC}\left(T^{\vee}/T^{\vee} = \mathcal{L}\left(\mathrm{pt}/T^{\vee}\right)\right) . \tag{13.115}$$

This is the horizontal/smart trace.

Note these do *not* agree: (modules over the) vertical trace are a subcategory of the horizontal trace:

$$\mathbf{QC}\left(T^{\vee}\right) \hookrightarrow \mathbf{QC}\left(T^{\vee}/T^{\vee}\right) . \tag{13.116}$$

So if we start with geometric Langlands, formulate local geometric Langlands, follow the same rules we have been doing all along, then we will try to decategorify in order to figure out what the local Langlands correspondence is. But we can do these two versions of the trace. And the punchline is that the horizontal trace gives you something better. This occurs already in V. Lafforgue, where it leads to the concept of excursion operators. This is the same comment as before:

$$\mathcal{O}\left(\frac{G^{\vee}}{G^{\vee}}\right)\textbf{-mod} \neq \mathbf{QC}\left(\frac{G^{\vee}}{G^{\vee}}\right) . \tag{13.117}$$

This idea is made precise in [AGK+20b, AGK+20a, AGK+21].

13.6.4 Full/categorical local Langlands

Let's go back to the arithmetic setting (so $K = \mathbb{F}_p\left(\left(t\right)\right)$, $\mathbb{Q}_p$, etc.). Then the B-side is something like $\mathbf{QC}\left(\mathrm{Loc}_{G^{\vee}}\right)$. Then we want this to be equivalent to some category which contains $\mathbf{Rep}\left(G\left(K\right)\right)$ as a full subcategory. The answer is (topological) sheaves on Bun_G, so the shape of the statement is:[4]

$$\mathbf{Shv}\left(\mathrm{Bun}_G\right) \simeq \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}\right) \ . \tag{13.118}$$

So we just have to figure out what Bun_G is. We know:

$$\mathbf{Rep}\left(G\left(K\right)\right) = \mathbf{Shv}\left(\mathrm{pt}/G\left(K\right)\right) \ , \tag{13.119}$$

i.e. this is sheaves on the trivial bundle locus in some space. So there will be some bigger space called Bun_G, and if just look at the trivial bundle, this will have automorphisms $G\left(K\right)$. Then representations of $G\left(K\right)$ will appear as sheaves living on this trivial bundle locus. But we're missing the rest of this space, and this is what the geometric Langlands point of view proposes.

There are two versions of this Bun_G.

Over a function field it is easier to say:

$$\mathrm{Bun}_G = G\left(K\right)/_F G\left(K\right) \ , \tag{13.120}$$

where $/_F$ denotes modding out by Frobenius twisted conjugation, i.e. by the action:

$$h * g = hgFh^{-1} \ . \tag{13.121}$$

This comes from following our nose: if we think about $\mathrm{Bun}_G\left(D^\times\right) = \mathrm{pt}/G\left(K\right)$ and then ask for the fixed points of the Frobenius. The set of isomorphism classes:

$$G\left(K\right)/_F G\left(K\right) \tag{13.122}$$

consists of what are known as Kottwitz G-isocrystals.

This is expected to be roughly equivalent to a version of Fargues-Scholze. They study sheaves on Bun_G of a specific curve, called the Fargues-Fontaine curve. So this is really taking to heart the idea that local Langlands should be geometric Langlands on a specific curve. I.e. that the local field K should correspond to some curve. In the case where $K = \mathbb{Q}_p$, this curve is roughly

$$\mathrm{Spec}\,\mathbb{Q}_p/\mathbb{Z} \tag{13.123}$$

where $\mathbb{Z}$ acts by the Frobenius. We can think of this as a punctured disk of sorts modulo the Frobenius. In order to say this correctly you need this whole perfectoid machinery. Lecture 33; June 24, 2021

[4] Recall $\mathbf{QC}$ should be $\mathbf{IndCoh}_{\mathrm{Nilp}}$.

Chapter 14

Derived algebraic geometry

14.1 Topological field theories and $\mathbb{E}_n$-algebras

For a physicist, there are roughly two sources of topological field theories. The first consists of theories which we can just write down as being manifestly topological, for example finite group examples and Chern-Simons theory. But this is not how most interesting topological theories arise. The second source consists of theories arising from topological twisting.

The idea of topological twisting to is as follows. Start with an "honest" (Euclidean) quantum field theory. This is some kind of functor defined on a category of Riemannian manifolds. Then some special quantum field theories have supersymmetry (SUSY). This means we have an action of some interesting super Lie algebra. There are specific operators Q, called BRST operators, which are odd, degree 1 endomorphisms on our spaces of states. So we have an operator with $Q^2 = 0$ and therefore we can pass to cohomology.

Most QFTs are not SUSY, and most SUSY theories are not topological. But one often finds that the theory has the structure of the so-called stress tensor T being Q-exact. Note that this is a structure rather than a property. The stress tensor is the object (in physics) which describes the reaction to changes in geometry/metric. So if this dependence on change of geometry is Q-exact then, at the level of cohomology, the theory doesn't depend on the geometry. Really, the stress tensor as controls the variation of our field theory $\mathcal{Z}$ over moduli spaces of Riemannian manifolds. For example, there is a whole space of Riemannian bordisms from $S^1 \sqcup S^1$ to S^1. So $\mathcal{Z}$ sends this to a family of maps $\mathcal{Z}\left(S^2\right)^{\otimes 2} \to \mathcal{Z}\left(S^1\right)$. The fact that this is Q-exact means that this dependence on the geometry is locally constant at the level of cohomology.

When we have a TFT $\mathcal{Z}$, if Σ is a bordism from $S^1 \sqcup S^1$ to S^1, i.e. a surface with three punctures, then we expect that $\mathcal{Z}(\Sigma)$ is a locally constant function on the moduli space of all such bordisms, valued in

$$\mathrm{Hom}\left(\mathcal{Z}\left(S^1\right)^{\otimes 2}, \mathcal{Z}\left(S^1\right)\right) \ . \tag{14.1}$$

On the chain level, we can rewrite this as saying that:

$$\mathcal{Z} \in \mathrm{Hom}\left(\mathcal{Z}\left(S^1\right)^{\otimes 2}, \mathcal{Z}\left(S^1\right)\right) \otimes C^*(M) \ . \tag{14.2}$$

Or equivalently it defines a map:

$$C_*(\mathcal{M}) \otimes \mathcal{Z}(M_{\mathrm{in}}) \to \mathcal{Z}(M_{\mathrm{out}}) \ . \tag{14.3}$$

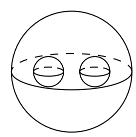
Another way to write the same thing is as follows. Let $|\mathcal{M}|$ be the moduli space of Riemannian bordisms. This is the underlying homotopy type/anima/∞-groupoid of the space $\mathcal{M}$. Then this is the same as a map:

$$|\mathcal{M}| \to \operatorname{Hom}\left(\mathcal{Z}\left(M_{\text{in}}\right), \mathcal{Z}\left(M_{\text{out}}\right)\right) \ . \tag{14.4}$$

On the categorical level we have to be more careful about what we mean by being local constant, i.e. we need to distinguish between the de Rham and Betti setting. We might look at a map from the de Rham space of $\mathcal{M}$ to operations, which is what one might call a de Rham field theory. The Betti version considers maps from $|\mathcal{M}|$ to operations. This doesn't make a difference at the level of functions/vector spaces, but locally constant sheaves have two versions: a flat connection doesn't automatically give you parallel transport.

14.1.1 $\mathbb{E}_n$-algebra structure on local operators

We will assume we are in this locally-constant/Betti situation. We would like to focus on the structure we get on local operators. Recall we had this idea that the space of local operators $A = \mathcal{Z}\left(S^{n-1}\right)$ acts on $\mathcal{Z}\left(M^{n-1}\right)$ for any $(n-1)$-manifold M^{n-1}. Now we will revisit this in a derived setting. When we wanted to compose local operators, we drew pictures like:



which gave multiplications:

$$A^{\otimes 2} \to A \ . \tag{14.5}$$

For $n > 1$, the balls could move around so we got commutative products (on the non-derived level). A better way to say this, is:

$$H_0\left(\operatorname{Conf}_2\left(D^n\right)\right) = \mathbb{Z} \ , \tag{14.6}$$

i.e. the configuration space of two points in D^n is connected. In general the field theory gives us a map:

$$\operatorname{Conf}_k\left(D^n\right) \to \operatorname{Hom}\left(A^{\otimes k}, A\right) \ . \tag{14.7}$$

When we pass to the derived setting, this is the definition of what is called an $\mathbb{E}_n$-algebra. I.e. A is an algebra over the *operad of little n-disks*. This tells us that there is a map:

$$C_*\left(\operatorname{Conf}_k\left(D^n\right)\right) \to \operatorname{Hom}\left(A^{\otimes k}, A\right) \ . \tag{14.8}$$

Topologically, we can just say that there is a map:

$$\left|\operatorname{Conf}_k\left(D^n\right)\right| \to \operatorname{Hom}\left(A^{\otimes k}, A\right) \ . \tag{14.9}$$

So this gives a notion of an $\mathbb{E}_n$-algebra in any symmetric monoidal higher category.

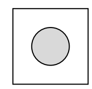
Remark 99. We can define the notion of an $\mathbb{E}_n$-algebra in any symmetric monoidal higher category, but sometimes the category is assumed to be the category of chain complexes, and one just says $\mathbb{E}_n$-algebra.

Example 94. An $\mathbb{E}_n$-algebra in (ordinary/discrete/non-derived) **Vect** is just an associative algebra for $n = 1$ and a commutative algebra for $n > 1$.

Example 95. An $\mathbb{E}_n$-algebra in (ordinary/discrete/non-derived) **Cat** is a monoidal category for $n = 1$, a braided monoidal category for $n = 2$, and a symmetric monoidal category for $n > 2$.

14.1.2 Origin of $\mathbb{E}_n$-algebras

The "primeval" example of an $\mathbb{E}_n$-algebra is as follows. Let X be a pointed topological space. Consider the space of n-fold based loops in X, $\Omega^n X$. We should think of this as:

$$\mathrm{Map}\left(\left(\mathbb{R}^n, \mathbb{R}^n \setminus D^n\right), (X,*)\right) = \mathrm{Map}\Bigg($$

$$, (X,*)\Bigg) . \qquad (14.10)$$

I.e. the space of maps from $\mathbb{R}^n$ to X such that the complement of $D^n \subset \mathbb{R}^n$ goes to the basepoint. We get a composition map by putting multiple disks inside of a larger disk. So we get a map:

$$\left|\mathrm{Conf}_k\, D^n\right| \times \left(\Omega^n X\right)^k \to \Omega^n X \ . \tag{14.11}$$

I.e. $\Omega^n X$ is an $\mathbb{E}_n$-algebra in the category of spaces. This is a fancy version of the statement that $\pi_n(X)$ is commutative for $n > 1$:

$$\pi_0\left(\Omega^n X\right) = \pi_n(X) \ . \tag{14.12}$$

Theorem 14 (Recognition theorem)**.** *If Y is a sufficiently connected space then identifications of Y with some n-fold loop space, $Y \xrightarrow{\sim} \Omega^n X$, correspond to $\mathbb{E}_n$-structures on Y.*

14.1.3 Additivity

The most important idea about $\mathbb{E}_n$-algebras, is that they satisfy a version of additivity. The version we will use is called Dunn-Lurie additivity, which says that:

$$\mathbb{E}_k + \mathbb{E}_\ell = \mathbb{E}_{k+\ell} \ . \tag{14.13}$$

We have seen a basic version of this before: if we have two consistent associative operations then they are forced to be equal and commutative. So this is true in greater generality: an $\mathbb{E}_k$-algebra within $\mathbb{E}_\ell$-algebras is an $\mathbb{E}_{k+\ell}$-algebra. One way to think of this is that it tells us what $\mathbb{E}_n$ means: it is an object equipped with n compatible associative products.

14.1.4 Categorical manifestation

An $\mathbb{E}_n$-algebra can be thought of as an n-truncated n-category. As it turns out, this is actually an equivalence.

An $\mathbb{E}_1$-algebra is a derived version of an associative algebra. I.e. this is a monoid, which is the same thing as a category with a single object $(\mathcal{C}, *)$. To $(\mathcal{C}, *)$ we can attach the algebra $\mathrm{End}(*)$. In the other direction, to an algebra A we can attach the category BA with a single object $*$ with endomorphisms given by $\mathrm{End}(*) = A$. This is the correspondence between $\mathbb{E}_1$-algebras and categories with a single object:

$$\mathrm{End}(*) \leftarrow\!\shortmid (\mathcal{C}, *) \tag{14.14}$$
$$A \mapsto BA \ . \tag{14.15}$$

Similarly we can consider a 2-category with one object $*$ and one 1-morphism 1_*. Then we can send this to an associative algebra $A = \mathrm{End}(1_*)$. But this single morphism 1_* is the unit for composition of 1-morphisms. The composition of 1_* and itself gives us a new associative structure on A, i.e. by additivity A is in fact an $\mathbb{E}_2$-algebra. So this 2-truncated 2-category gave us an $\mathbb{E}_2$-algebra.

We could have also done something in between. Given a 1-truncated 2-category $(\mathcal{C}, *)$ (so a 2-category with a single object) we can consider $\mathcal{A} = \text{End}(*)$ which is a monoidal ($\mathbb{E}_1$) category. As it turns out 1-truncated 2-categories are equivalent to $\mathbb{E}_1$-categories.

The idea is that if we have an n-category, and it is n-truncated (i.e. a single n-morphism all the way down). Then we will find that:

$$A = \text{End}\left(1_{1_{\cdots_1}}\right) \tag{14.16}$$

has n compatible multiplications. So the n in $\mathbb{E}_n$ is the same as the n in n-category. This is summarized in the periodic table of higher categories, table 14.1.

Table 14.1: Periodic table of higher categories. BTC stands for braided tensor category.

cat# / truncated#	1	2	3
1	$\mathbb{E}_1$-algebra	$\mathbb{E}_1$-category	$\mathbb{E}_1$ 2-category
2		$\mathbb{E}_2$-algebra	$\mathbb{E}_2$-category (BTC)
3			$\mathbb{E}_3$-algebra

Rather than looking at these small truncated categories with a single object, we can talk about a category *generated* by a single object. We would like to have a dictionary between associative ($\mathbb{E}_1$) algebras and pointed ($\mathbb{E}_0$) categories. For $(\mathcal{C}, *)$ an $\mathbb{E}_0$-category, we can attach the associative algebra End$(*)$. In the other direction, given an associative algebra A, we can send it to the category A-**mod**, with the distinguished object $A \in A$-**mod**. The functors $\text{End}\left(*_{(-)}\right)$ and $(-)$-**mod** are adjoint. This is a different way of writing the same dictionary as above (BA can be recovered from A-**mod**) but now the categories have things like colimits, so the categories of interest in representation theory fit into this framework.

Example 96. Suppose $\mathcal{C}$ is a monoidal ($\mathbb{E}_1$) category. Then we have an associative algebra:

$$\text{End}(1_{\mathcal{C}}) = A \ . \tag{14.17}$$

But this is actually an $\mathbb{E}_2$-algebra, because $1_{\mathcal{C}}$ is the unit. And we have an embedding:

$$A\text{-}\mathbf{mod} \hookrightarrow \mathcal{C} \ . \tag{14.18}$$

If A is an $\mathbb{E}_2$-algebra, then A-**mod** is in fact monoidal. I.e. taking $-$-**mod** drops you down one $\mathbb{E}_n$ level.

Example 97. If A is an $\mathbb{E}_n$-algebra, then A-**mod** is an $\mathbb{E}_{n-1}$-category. So we can take this category and look for modules for it: A-**mod-mod**, which is an $\mathbb{E}_{n-2}$-2-category. We can continue this process n-times to get an n-category:

$$A\text{-}\mathbf{mod}\cdots\text{-}\mathbf{mod} = A - \mathbf{mod}^n \ . \tag{14.19}$$

Remark 100. This is all to say that n-categories are basically just fancier/more general versions of $\mathbb{E}_n$-algebras. Like how a category is a fancy version of an associative algebra, where we only have these kind of partially defined multiplications.

14.2 Hochschild cohomology

For A an associative algebra, we attached its center/Hochschild cohomology of A:

$$Z\left(A\right) = \mathrm{HH}^*\left(A\right) = \mathrm{End}_{A\text{-}\mathbf{mod}\text{-}A}\left(A\right) \ . \tag{14.20}$$

Gerstenhaber discovered that Hochschild cohomology has the extra structure of a Gerstenhaber algebra. We will discuss the derived version first.

We know that $\mathrm{HH}^*\left(A\right)$ is associative because it is End of something. But a bimodule determines a functor given by tensoring:

$$\left(-\right) \otimes {}_A B_A \colon A\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod} \ . \tag{14.21}$$

So A is the unit bimodule, i.e. this is the Bernstein center of A-**mod**, and this contributes a second associative product. So we have one associative multiplication from being End, but this was the unit in a monoidal category, so we get a second direction of multiplication, i.e. it is an $\mathbb{E}_2$-algebra. This is known as the Deligne conjecture, which says that the Hochschild chain complex of an associative algebra A is naturally an $\mathbb{E}_2$-algebra. Now we can see this is a trivial consequence of this Dunn-Lurie additivity.

So for $\mathcal{C}$ a category, it makes sense to say that $\mathcal{C}$ is A-linear for A any $\mathbb{E}_2$-algebra: we can just ask for an $\mathbb{E}_2$ map:

$$A \to \mathrm{End}\left(\mathrm{id}_{\mathcal{C}}\right) \ . \tag{14.22}$$

Another way to say this is that we can take A, consider the monoidal category A-**mod**, and then we can ask for an action of A-**mod** on $\mathcal{C}$.

Recall we said that any category $\mathcal{C}$ spectrally decomposes over

$$\mathrm{Spec}\, Z\left(\mathcal{C}\right) = \mathrm{Spec}\,\mathrm{End}\left(\mathrm{id}_{\mathcal{C}}\right) \ . \tag{14.23}$$

But now this is Spec of an $\mathbb{E}_2$-algebra. Now we will focus on the more concrete structure at the level of cohomology that Gerstenhaber discovered.

14.2.1 Gerstenhaber algebras

Recall that being an $\mathbb{E}_n$-algebra (in chain complexes) means that we have actions:

$$C_*\left(\mathrm{Conf}_k\left(\mathbb{R}^n\right)\right) \otimes A^{\otimes k} \to A \ . \tag{14.24}$$

F. Cohen gave an explanation at the cohomological level, i.e. we go underived. The structure is a map cite

$$H_*\left(\mathrm{Conf}_k\left(\mathbb{R}^n\right)\right) \to H^*\left(A\right)^{\otimes k} \to H^*\left(A\right) \ . \tag{14.25}$$

A graded vector space with this extra structure is known as a P_n-algebra. I.e.

$$P_n = H_*\left(\mathbb{E}_n\right) \ . \tag{14.26}$$

Concretely, a P_n-algebra $B = H^*\left(A\right)$ has a degree 0 commutative multiplication, and in degree $1-n$ we get another operation, denoted by $\{\}$.

$$\mathrm{Conf}_2\left(\mathbb{R}^n\right) \simeq S^{n-1} \ , \tag{14.27}$$

so we have two classes:

$$H_0\left(S^{n-1}\right) \oplus H_{n-1}\left(S^{n-1}\right) \ . \tag{14.28}$$

The theorem is that the degree 0 component is the multiplication, and the degree $n-1$ contribution is the Poisson bracket $\{\}$, of degree $1-n$. Poisson means it is a (super) derivation.

Example 98. A $\mathbb{P}_1$-algebra is just an ordinary Poisson algebra.

Example 99. A $\mathbb{P}_3$-algebra is the same as an ordinary Poisson algebra where the underlying vector space is graded, and the degree of the bracket is -2.

Example 100. A $\mathbb{P}_2$-algebra is what is known as a Gerstenhaber algebra. I.e. it is equipped with a degree -1 Poisson bracket.

The theorem is then that $\mathrm{HH}^*(A)$ carries a natural $\mathbb{P}_2$-structure.

Example 101. The main example of a Gerstenhaber algebra is the Schouten-Nijenhuis bracket. Consider the exterior algebra $\wedge^\bullet T_M$. There is the wedge product, but then there is also the bracket $\{\} = [,]$, which is uniquely defined by saying that:

$$T_M \otimes T_M \to T_M \tag{14.29}$$

is the usual Lie bracket, only now it is of degree -1.

Theorem 15 (Hochschild-Konstant-Rosenburg (HKR))**.**

$$\mathrm{HH}^*(C^\infty(M)) \simeq \wedge^\bullet T_M = \operatorname{Sym} T[-1] \tag{14.30}$$

where $\{,\}$ corresponds to $[,]$.

This is also true in algebraic geometry, i.e. for X an affine variety and $\mathcal{O}(M)$ (instead of $C^*\infty(M)$).

Then Deligne asked: does this bracket lift to an $\mathbb{E}_2$-structure? This is the Deligne conjecture.

Example 102. Consider an $\mathbb{E}_3$-algebra. The commutative (up to homotopy) product comes from the OPE, i.e. colliding points. The bracket comes from putting one operator at the origin, and put the other operator on a sphere linking the origin. So this is really just the statement that the bracket is coming from the fundamental class in $H_2(S^2)$.

There is another cool way of drawing it using the Hopf link. Imagine taking one operator and moving it in a circle, and the other operator and moving it in a circle. This torus represents the same class in H_2 of the configuration space of two points in $\mathbb{R}^3$.

Let $\mathcal{Z}$ be an n-dimensional TFT. The punchline is that $\mathcal{Z}(S^{n-1})$ is an $\mathbb{E}_n$-algebra. Concretely this means that

$$H^*\left(\mathcal{Z}\left(S^{n-1}\right)\right) \tag{14.31}$$

is (graded) Poisson. So when we take Spec, we get a Poisson variety: $\operatorname{Spec} H^*\left(\mathcal{Z}\left(S^{n-1}\right)\right)$, which is often in fact symplectic (i.e. the bracket is nondegenerate).

Example 103. Recall $\wedge^\bullet T_M = \operatorname{Sym} T_M[-1]$. Then we have:

$$T^*[1]M = \operatorname{Spec}(\wedge^\bullet T_M)\ , \tag{14.32}$$

which is an odd Poisson variety, i.e. a variety with a bracket of degree -1. In other words, HKR says that:

$$\operatorname{Spec} \mathrm{HH}^*(M) = T^*[1]M\ . \tag{14.33}$$

In fact it is a 1-shifted symplectic manifold. See [PTVV13, CPT+17] for more on shifted symplectic geometry.

In practice, we just need out of shifted symplectic geometry what we needed out of ordinary symplectic geometry: that functions have a Poisson bracket. I.e. whatever an n-shifted symplectic manifold is, functions on it should be $\mathbb{P}^n$. And when we phrase it this way, it becomes clear: shifted symplectic manifolds are there to be quantized, i.e. lifted from $\mathbb{P}^n$ to $\mathbb{E}^n$.

14.3 Quantization

This leads us to a different point of view on $\mathbb{E}_n$-algebras: they are quantizations of $\mathbb{P}_n$-algebras. We should think about this as an analogue of the idea that $\mathbb{E}_1$-algebras are quantizations of $\mathbb{P}_1$-algebras, which means the following. For $A_\hbar$ an associative algebra, depending on some parameter $\hbar$ such that $A_{\hbar=0}$ is commutative, then $A_{\hbar=0}$ acquires a Poisson bracket:

$$\{a, b\} = \frac{\overline{a}\overline{b} - \overline{b}\overline{a}}{\hbar} \operatorname{mod} \hbar \ . \tag{14.34}$$

This is the classical way in which quantum and classical mechanics interact: in classical mechanics we have ordinary Poisson brackets, and if we have an associative algebra which becomes commutative at some parameter value then we naturally get a Poisson bracket. And the Poisson bracket tells you the first order data of deforming commutative to associative. Poisson is some kind of "tangent to associative at commutative".

Now this same kind of story generalizes to $\mathbb{E}_n$. For A an $\mathbb{E}_n$-algebra, the cohomology $H^*(A)$ is commutative. Then this acquires a bracket $\{\}$ and becomes a $\mathbb{P}_n$-algebra. This bracket is the obstruction to being commutative.

Really what we want to say is that for A an $\mathbb{E}_n$-algebra we do get a canonical Lie bracket. So there is an exact sequence of sorts:

$$\mathrm{Comm} \ \to \mathbb{E}_n \to \mathbf{Lie}\,[1-n] \ . \tag{14.35}$$

Just like how we had:

$$\begin{tikzcd} \mathrm{CommAssoc} \arrow[r, "{[,]}"] & \mathbf{Lie} \arrow[l, bend right, "U"'] \end{tikzcd} \ . \tag{14.36}$$

The upshot of this is that local operators in TFT, for $n > 1$, are almost commutative, i.e. $\mathbb{E}_n$, which we can think of as quantizations of $\mathbb{P}_n$-algebras. So we need something outside of ordinary algebraic geometry:

$$\mathcal{M}^0_{\mathcal{Z}} = \operatorname{Spec} \mathcal{Z}\left(S^{n-1}\right) \ . \tag{14.37}$$

This is almost an object of ordinary algebraic geometry, since it is almost the Poisson variety:

$$\operatorname{Spec} H^*\left(\mathcal{Z}\left(S^{n-1}\right)\right) \ , \tag{14.38}$$

but we really need some version of $\mathbb{E}_n$-algebraic geometry to accommodate $\mathcal{M}^0_{\mathcal{Z}}$. John Francis' thesis. Cite

Example 104 ($\mathcal{B}$-model)**.** Let $X = \operatorname{Spec}(R)$ be smooth and affine. Then local operators are given by:

$$\mathcal{B}_X\left(S^1\right) = \mathrm{HH}^*(R) \simeq \wedge^\bullet T_X \ , \tag{14.39}$$

so the moduli space of vacua is:

$$\mathcal{M}_{\mathcal{B}_X} = \operatorname{Spec} \mathcal{B}_X \left(S^1\right) = T^* [1] X \ , \tag{14.40}$$

or really some noncommutative object which has this as its classical image.

So we start with a variety X, construct the category $\mathbf{QC}(X)$, use this to construct a two-dimensional TFT $\mathcal{Z}$, and from this we can try to construct a variety:

$$\operatorname{Spec} \mathcal{Z} \left(S^1\right) \ . \tag{14.41}$$

One might expect to get X back. Indeed, in the non-derived world, Spec of the center is

$$\operatorname{Spec} Z(R) = \operatorname{Spec}(R) = X \ . \tag{14.42}$$

But in the derived world, $Z(R)$ is nondegenerately $\mathbb{E}_2$, so Spec of it looks more like $T^* [1] X$. So the $\mathcal{B}$-model $\mathcal{B}_X$ of some dimension naturally sheafifies over $T^* [n] X$.

The idea is that $T^* [1] X$ is symplectic. So it can be deformation quantized to the $\mathbb{E}_2$-algebra $\mathrm{HH}^* = \wedge^\bullet T_X$. But we really want to geometrically quantize X to get $\mathbf{QC}(X)$. HH^* consists of the observables, and $\mathbf{QC}(X)$ consists of the states. The relationship between them is that the geometric quantization $\mathbf{QC}(X)$ is linear over the deformation quantization $\mathrm{HH}^* = \wedge^\bullet T_X$. I.e. the states form a module over the observables. This is a noncommutative version of saying that $X \hookrightarrow T^* [1] X$ is a Lagrangian.

Remark 101. We have been sweeping the issue of the tangential structure on our bordisms under the rug. A TFT is defined on bordisms which come with some tangential structure. We have been implicitly working with oriented theories, i.e. where the bordisms come with an orientation. This is in contrast to, say, a framed theory where the bordisms come with a framing, i.e. a trivialization of their stabilized tangent bundle. So in the context of the $\mathcal{B}$-model, we might be trying to define on oriented surfaces, rather than framed surfaces, since there just aren't many trivialized tangent bundle (e.g. S^2 has no framing). Now given a category we would like to assign to the point, this alone only determines a framed theory. To upgrade from a framed theory to an oriented theory we need to give the category assigned to a point a Calabi-Yau structure. The data of such a structure is equivalent to a trivialization of a geometric object (the canonical bundle).

We can see this as follows. Hochschild homology of the assignment to a point is the assignment to the *framed* circle with the cylinder framing (i.e. the rotation-invariant framing coming from $S^1 \times \mathbb{R}$). Hochschild *co*homology of the assignment to a point is the assignment to the framed circle with the *blackboard* framing (i.e. the framing coming from the annulus). The upshot of this framing is that it naturally fits into the pair of pants. Note that HH_* is really $\dim = \operatorname{Tr}(\mathrm{id})$, which naturally has an S^1 action. On the other hand, $\operatorname{End}(\mathrm{id}) = \mathrm{HH}^*$ is naturally $\mathbb{E}_2$. So the circle has different framings, and they correspond to these different notions of decategorification. Recall, however, that the circle only has a single orientation. So the upshot of this whole discussion, is that to upgrade a framed theory to an oriented theory, one needs to at least provide an identification between HH_* and HH^*.

Now recall that HKR says:

$$\wedge^\bullet T_M = \mathrm{HH}^* \qquad\qquad \wedge^\bullet \Omega_M [n] = \mathrm{HH}_* \ . \tag{14.43}$$

To get an isomorphism between the two, you contract with a volume form. So to get an identification between HH_* and HH^* we need to choose a volume form. But the choice of a volume form is the same as a trivialization of the canonical bundle.

So when we talk about 2-dimensional oriented field theories, we're really talking about *framed* $\mathbb{E}_2$*-algebras* or *algebras over the oriented little disc operad* which are roughly $\mathbb{E}_2$-algebras with an S^1-action, i.e. they have the two distinguishing properties of HH_* and HH^* from above.

Chapter 15

States and observables in TFT

Recall we have been exploring the general idea that we are studying the *derived algebraic geometry of TFTs*. I.e. we start with a TFT and extract a lot of structure out of it. E.g. we can extract the ring of local operators and it turns out to even be commutative. Now we want to revisit this in the derived setting. In the derived setting, operators aren't really commutative, but rather they form an $\mathbb{E}_n$-algebra. I.e if $\mathcal{Z}$ is an n-dimensional TFT, then $\mathcal{Z}\left(S^{n-1}\right)$ is an $\mathbb{E}_n$-algebra. Lecture 34; August 13, 2021

The most important thing to know about $\mathbb{E}_n$-algebras is Dunn-Lurie additivity. This roughly says that the data of an $\mathbb{E}_n$ structure is equivalent to the data of n compatible $\mathbb{E}_1$ structures. I.e. at the level of operads:

$$\mathbb{E}_n = \mathbb{E}_k + \mathbb{E}_\ell \tag{15.1}$$

where $k + \ell = n$

We also pointed out a geometric feature: the cohomology[1] of an $\mathbb{E}_n$-algebra is a $\mathbb{P}_n$-algebra. A $\mathbb{P}^n$-algebra is a commutative algebra equipped with a bracket $\{\,,\}$ of degree $1-n$. When n is odd, a $\mathbb{P}_n$-algebra behaves like an ordinary Poisson algebra, but everything is graded and the bracket $\{\,,\}$ has nonzero degree. If n is even, then a $\mathbb{P}_n$-algebra is a Gerstenhaber algebra, and now there are different signs. It is a kind of "super" version of Poisson.

Now we will see how much algebraic geometry can be squeezed out of this. We have talked a lot about squeezing algebraic geometry out of the structure of operators, but this was ordinary (commutative) algebraic geometry, and now we want to get some kind of symplectic geometry.

Another way to say this, is that we would like to understand states and observables in TFT. The states arise via deformation quantization and the observables arise via geometric quantization, so equivalently we would like to understand deformation quantization and geometric quantization in the context of a general TFT.

[1]For us, an $\mathbb{E}_n$-algebra is just a cochain complex together with operations given by these little disks colliding. When we say cohomology, we are just taking the cohomology as a cochain complex. On the level of cohomology, the Poisson bracket takes an operator at one point, and integrates it on a little sphere around it.

15.1 Geometric and deformation quantization

This is a topic one could give a whole course about. We want to highlight the key features.

Consider quantum mechanics, i.e. 1-dimensional QFT. In quantum mechanics, there are two main players:

- the Hilbert space $\mathcal{H}$ of states, and
- the associative algebra A of observables.

Really we want to consider *pure states*, which are the rays in this space. The space of states is the value of the 1-dimensional theory on a point:

$$\mathbf{QM}\,(\mathrm{pt}) = \mathcal{H}\ . \tag{15.2}$$

The associative algebra of observables A acts on $\mathcal{H}$. So all together this consists of an algebra and a module:

$$A \circlearrowright \mathcal{H}\ . \tag{15.3}$$

Everything in QM get formulated in terms of these. In terms of **QC**:

$$\mathbf{QC}\,(\mathrm{pt} \sqcup \mathrm{pt}) = \mathrm{End}\,(\mathcal{H}) \tag{15.4}$$

We can think of these two points as coming to us as the link of a singular point in 1-dimensional spacetime. So these are exactly the local operators we have been developing.

We get **QM** via quantization of classical mechanics. For us, a (nondegenerate[2]) classical mechanical system is captured by a symplectic manifold M. This is the *phase space* of the system. So quantization is some procedure which takes in M and produces the space of states $\mathcal{H}$ and algebra of observables A in such a way that $A \circlearrowright \mathcal{H}$. *Deformation quantization* is the process of forming the observables (A) from M. *Geometric quantization* is the process of forming the states ($\mathcal{H}$) from M.

Remark 102. It is sometimes said that deformation quantization is a science (or at least a functor), whereas geometric quantization is more of an art.

15.1.1 Deformation quantization

Let M be a symplectic manifold. Then the ring of functions:

$$A_0 = (\mathcal{O}\,(M)\,, \{-\}) \tag{15.5}$$

is naturally a Poisson algebra. Given a Poisson algebra, we can ask for a deformation $A_\hbar$ in $\hbar$, i.e. an $\hbar$-family of associative algebras such that:

$$A_\hbar|_{\hbar=0} \simeq A_0 \tag{15.6}$$

is commutative, and

$$\frac{d}{d\hbar}\,[-] = \{-\}\ . \tag{15.7}$$

[2]If we instead considered the (larger) class of Poisson manifolds, we would be considering possibly degenerate classical mechanical systems.

I.e. we are saying the following. Given $a_0, b_0 \in A_0$, choose a lift to

$$a, b \in A_\hbar \ , \tag{15.8}$$

and then $[a, b]$ is divisible by $\hbar$, i.e. it vanishes at 0. Then we are saying that the Poisson bracket is given by:

$$\{a_0, b_0\} = \frac{[a, b]}{i\hbar} \operatorname{mod} \hbar \ . \tag{15.9}$$

Remark 103. The idea is that, within the world of associative algebras, Poisson algebras form the "tangent space" to commutative algebras: the first-order data of deforming a commutative algebra to an associative one is the Poisson bracket.

As it turns out, every Poisson algebra has a deformation quantization, and there is some parameterization of all of the deformations of the Poisson algebra. The collection of these deformation quantizations looks like deformations within the Poisson world (which can be given an explicit cohomological description). But in fact there is a distinguished deformation. In other words, we know that symplectic manifolds can ***always*** be canonically deformation quantized. Kontsevich solved this for general Poisson algebras. For symplectic it was solved earlier by . cite

15.1.2 Geometric quantization

Deformation quantization is a very concrete construction. Geometric quantization is a bit more roughly defined. Now we are looking for the space of states: a module over the deformation quantization. It should be a very small module, e.g. irreducible. Something like the smallest nonzero module. There are many desiderata, which we won't get into, but the idea is that geometric quantization is roughly "functions on half the variables of M".

Example 105. Consider a particle in $\mathbb{R}^n$. The phase space of the system is $T^*\mathbb{R}^n$, i.e. this the given symplectic manifold. The geometric quantization is:

$$L^2(\mathbb{R}^n) \ . \tag{15.10}$$

In general, if we have $M = T^*X$ (or if M is polarized) then we can write down a Hilbert space:

$$L^2(X) \ . \tag{15.11}$$

Note however that there are many variants of what one might mean by a polarization.

Example 106. If M is a compact Kähler manifold (e.g. $\mathbb{P}^n$) then there is a natural symplectic form.[3] Now "half" the variables means we are looking at holomorphic functions, i.e. something like:

$$\Gamma(M, \mathcal{L}) \tag{15.12}$$

for a line bundle $\mathcal{L}$ such that the first Chern class is the symplectic form: $c_1(\mathcal{L}) = \omega$.

[3]Note this is *not* a holomorphic symplectic form. The Kähler structure gives rise to a (non-degenerate skew-symmetric) real 2-form.

Pairing

There are many subtleties. For example, the elements of $L^2(X)$ are not naturally functions, but rather "half densities": things which can be integrated after pairing. So, in general, to define the kind of functions we're interested in we will need to choose a square root of the bundle of volume forms. Such a choice is analogous to the choice of a spin structure. This is all just to say that there are some choices involved in forming the geometric quantization, and it is not immediately clear that the construction is independent of the choices.

Symplectic invariance

If $M = T^*X$, it is tempting, for an algebraic geometer, to guess that the geometric quantization is $\mathcal{O}(X)$. This is a terrible picture: the geometric quantization should be independent of the polarization.

Example 107. $L^2(\mathbb{R}^n)$ (the geometric quantization of $T^*\mathbb{R}^n$) *is* independent of the polarization: the Hilbert space associated to this polarization is isomorphic to the Hilbert space associated to any other polarization. E.g. the Fourier transform sends the zero section to the fiber over 0, and provides an isomorphism between the Hilbert space of L^2 functions on the zero-section, and the Hilbert space of L^2 functions on the fiber over 0. I.e. it takes functions on half of the variables to functions on the other half and says they're isomorphic.

The idea is that L^2 functions are actually *microlocal objects*, i.e. they have a support theory over the cotangent bundle of whatever they are defined on. They "live" on the whole symplectic space we start with. So even though we are describing the space as $L^2(\mathbb{R}^n)$, functions on the zero section, it doesn't actually depend on the choice of polarization. However, $\mathcal{O}(X)$ *does depend on your choice of polarization.*

One way to make this precise is as follows. Given a distribution on $\mathbb{R}^n$, we can assign its wave-front set, which is a subset of $T^*\mathbb{R}^n$ given by the points and directions where the distribution fails to be an honest function. There is something else called the *Wigner distribution*: for any $f \in L^2$, we can lift $|f|^2$ to a distribution on $T^*\mathbb{R}^n$.

15.2 Analogue of deformation and geometric quantization in (topological) QFT

The main message is that n-dimensional QFT (and in particular, for us, TFT) bears a similar relation to something called *n-shifted* symplectic geometry. These shifted symplectic manifolds play the role of the semi-classical phase space in the setting of higher-dimensional QFT.

Remark 104. The definition of an n-shifted symplectic structure is rather young, see [PTVV13], but related ideas have been around for a while, e.g. Albert Schwarz. Cite

Recall that a symplectic manifold M gave us a Poisson algebra:

$$(\mathcal{O}(M), \{-\}) \ . \tag{15.13}$$

Similarly, we will want an analogous thing from a shifted symplectic manifold: an *n-shifted symplectic variety* will be a variety such that $\mathcal{O}(M)$ is a $\mathbb{P}_n$-algebra. So now we are in the world of some sort of derived geometry, since the space of functions needs a grading:

the Poisson bracket is of nonzero degree. So M is really a derived variety. Recall that a symplectic structure is a closed 2-form ω which is nondegenerate. Nondegeneracy is equivalent to ω defining an isomorphism:

$$\omega\colon TM \xrightarrow{\sim} T^*M \ . \tag{15.14}$$

An *n-shifted symplectic structure* defines an isomorphism:

$$\omega\colon TM \xrightarrow{\sim} T^*M\,[n] \ . \tag{15.15}$$

We won't worry about the analogue of the "closed" condition in the definition of a shifted symplectic form. See [PTVV13] for details.

sign

Remark 105. For M a smooth manifold, the tangent space only lives in one cohomological degree 0, so you can't have this kind of isomorphism with the shifted cotangent bundle.

We will try to relate shifted symplectic varieties to TQFTs. So we are doing *algebraic* geometry, and we are trying to relate it to *topological* field theory.

Remark 106. The adjectives "algebraic" and "topological" should kind of correspond. I.e. rather than trying to study differential geometry and relate it to a more broad class of QFTs, we are restricting ourselves to the algebraic and topological settings.

In higher dimensional field theory, we can do this backwards. I.e. there is a natural dequantization. :et $\mathcal{Z}$ be an n-dimensional TFT. We will think about this as the theory of states. So its assignments will be spaces of states. We will see symplectic geometry emerge by looking at observables. The easiest observables to write down are local operators, i.e. $\mathcal{Z}\left(S^{n-1}\right)$, which is an $\mathbb{E}_n$-algebra. Whenever we have an $\mathbb{E}_n$-algebra, we can take its cohomology to get a $\mathbb{P}_n$-algebra. So we can take Spec of this, to get an n-shifted Poisson variety:

$$\operatorname{Spec} H^*\mathcal{Z}\left(S^{n-1}\right) \ . \tag{15.16}$$

Recall we call this $\mathcal{M}^0$: the moduli space of vacua. For "nondegenerate" field theories, we might hope for $\mathcal{M}^0$ to be n-shifted symplectic, rather than just Poisson.

What is the relation between the $\mathbb{P}_n$-variety $\mathcal{M}^0$ and this slightly non-commutative ($\mathbb{E}_n$) algebra $\mathcal{Z}\left(S^{n-1}\right)$? As it turns out, $\mathcal{Z}\left(S^{n-1}\right)$ is what is called the *deformation quantization* of the $\mathbb{P}_n$-variety $\mathcal{M}^0$:

$$DQ\colon \mathcal{M}^0 \rightsquigarrow \mathcal{Z}\left(S^{n-1}\right) \ . \tag{15.17}$$

Remark 107. A version of the Rees construction shows that we really have a family of algebras $A_\hbar$, for $\hbar \in \mathbb{A}^1$, such that:

$$A_{\hbar=0} = H^*\left(\mathcal{Z}\left(S^{n-1}\right)\right) \qquad\qquad A_{\hbar=1} = \mathcal{Z}\left(S^{n-1}\right) \ . \tag{15.18}$$

Now we can formulate the question: when does an n-shifted Poisson variety have a deformation quantization given by an $\mathbb{E}_n$-algebra? This can be solved by the same techniques as the Kontsevich deformation quantization from before. This is saying that we can see local operators in a field theory $\mathcal{Z}$ come from the deformation quantization of the Poisson variety $\mathcal{M}^0$.

The general idea is that the observables in the theory $\mathcal{Z}$ are given by the deformation quantization of functions on mapping spaces:

$$\operatorname{Map}\left(\text{spacetime}, \mathcal{M}^0\right) \ . \tag{15.19}$$

So we started with a classical space, a Poisson manifold, and we are considering something which looks like a classical field theory: the fields are maps into this Poisson manifold. Then we can ask for the deformation quantization of the collection of functions on these mapping spaces, and these are the observables of the theory $\mathcal{Z}$. So there is a very tight relation between this shifted symplectic geometry, and the structure of a higher-dimensional field theory.

This was the theory of deformation quantization: an attachment of an algebra to a (shifted) symplectic manifold. The geometric quantization problem, on the other hand, is a much more nebulous one. Pavel Safronov recently worked out a version of shifted geometric quantization [Saf20]. The process of geometric quantization is supposed to attach the space of states to the (shifted) symplectic manifold, rather than the algebra of observables (which is the deformation quantization).

Let $\mathcal{M}$ be an n-shifted symplectic manifold and let M be a k-manifold. Then $\mathrm{Map}\,(M, \mathcal{M})$ is $(n-k)$-shifted symplectic. So we can form the deformation quantization of this, as before, to get the algebra of observables of the theory on M. The *geometric quantization* of $\mathrm{Map}\,(M, \mathcal{M})$ is the space of states of the theory on M. So we are building a TFT $\mathcal{Z}$ associated to $\mathcal{M}$ by defining $\mathcal{Z}\,(M)$ to be the geometric quantization of $\mathrm{Map}\,(M, \mathcal{M})$.

The observables of $\mathcal{Z}$ on some spacetime M is the factorization homology of the local operators:

$$\int_M \mathcal{Z}\left(S^{n-1}\right) \ . \tag{15.20}$$

Now we will say a bit more about what factorization homology actually is, and what it has to do with observables. Given an n-manifold M, we expect the theory to assign a number: $\mathcal{Z}\,(M) \in \mathbb{C}$. We want to get at this number by starting with some richer structure. This is the theory of Ward identities/conformal blocks. This appears in physics when we try to write down a partition function and write down all the constraints we can think of using local operators. Rather than think of $\mathcal{Z}\,(M)$, we will consider a richer object. Let M^n be closed and remove a point $x \in M$. This is now a manifold with boundary S^{n-1}, i.e. it now defines a bordism from S^{n-1} to $\emptyset$, so it induces a map:

$$\mathcal{Z}\,(M \setminus \{x\}) : \mathcal{Z}\left(S^{n-1}\right) \to \mathbb{C} = \mathcal{Z}\,(\emptyset) \ . \tag{15.21}$$

Note that S^{n-1} bounds the n-disk, so we have that $\mathcal{Z}\,(D^n) \in \mathcal{Z}\left(S^{n-1}\right)$, and in fact $\mathcal{Z}\,(M \setminus \{x\})$ sends $\mathcal{Z}\,(D^n)$ to the value of the partition function on M:

$$\begin{array}{ccc} \mathcal{Z}\left(S^{n-1}\right) & \longrightarrow & \mathbb{C} \\ & & \\ \mathcal{Z}\,(D^n) & \longmapsto & \mathcal{Z}\,(M) \end{array} \ . \tag{15.22}$$

So rather than being a number attached to a closed n-manifold, let's think of the partition function as being a function defined on all states on S^{n-1}. This is called the *one-point function of the theory* $\mathcal{Z}$. It is often said that, given $\mathcal{O} \in \mathcal{Z}\left(S^{n-1}\right)$, we can take the one-point function of $\mathcal{O}$ at x:

$$\langle \mathcal{O}_x \rangle \in \mathbb{C} \ . \tag{15.23}$$

We can do the same construction for two points $x, y \in M$ leading to a functional:

$$\mathcal{Z}\left(S^{n-1}_x\right) \otimes \mathcal{Z}\left(S^{n-1}_y\right) \to \mathbb{C} \ . \tag{15.24}$$

We can do this for any number of points, leading to what are called the n-point functions. We can take this to the extreme to get:

$$\mathcal{Z}\colon \bigotimes_{x\in M} \mathcal{Z}\left(S^{n-1}\right) \to \mathbb{C} \ . \tag{15.25}$$

This giant functional satisfies a lot of relations. I.e. it factors through a much smaller quotient, called the factorization homology:

$$\begin{tikzcd} \bigotimes_{x\in M} \mathcal{Z}\left(S^{n-1}\right) \arrow[rr] \arrow[dr] & & \mathbb{C} \\ & \int_M \mathcal{Z}\left(S^{n-1}\right) \arrow[ur] & \end{tikzcd} \ . \tag{15.26}$$

There is a very natural thing we can say about this quotient relation: it is exactly encoded by the $\mathbb{E}_n$-algebra structure of $\mathcal{Z}\left(S^{n-1}\right)$. Explicitly, $\int_M \mathcal{Z}\left(S^{n-1}\right)$ is a colimit in a category of little disks, but we will not go into the details of this. We should just think that if we have an observable $\mathcal{O}_x$ at x (state on the sphere around x) and an observable $\mathcal{O}_y$ at y (state on the sphere around y), and a sphere containing both of these smaller spheres, then we already have a product/composition which takes $\mathcal{O}_x$ and $\mathcal{O}_y$, and gives me a state on this bigger sphere. Then this factorization is saying that the map eq. (15.25) factors through these inclusions from little disks into bigger disks.

Warning 2. We are ignoring some details regarding the tangential structure. I.e. we might have done some trivialization on the tangent bundle in order to identify the boundary of these disks around every point with the same sphere. So if M is an oriented manifold, then we would have to keep track of the $\mathrm{SO}(n)$ action on these spheres.

Remark 108. In algebraic geometry you don't say this using disks, but rather points. This is where the *Ran space* comes up (the space of finite subsets of your space).

Example 108. So given an $\mathbb{E}_n$-algebra A, then we get a functorial assignment of vector spaces to manifolds M:

$$\int_M A \in \mathbf{Vect} \ . \tag{15.27}$$

For $M = D^n$ we get

$$\int_{D^n} A = A \ . \tag{15.28}$$

Example 109. Let A be an associative (i.e. $\mathbb{E}_1$) algebra. Then factorization homology over the circle can be calculated by covering the circle by two disks:

$$\int_{S^1} A = A \otimes_{A\otimes A^{\mathrm{op}}} A = \mathrm{HH}_*(A) \ , \tag{15.29}$$

which is the same thing as Hochschild homology. As expected, the Hochschild homology of an associative algebra is just a chain complex (i.e. it doesn't have a multiplication, unlike Hochschild cohomology).

Notice that the way we wrote HH_* as a tensor product broke the symmetry of the circle. If we just think of this as the factorization homology over the circle, then the circle action is clear. This circular rotation is where cyclic homology comes from. In general (orientation preserving) diffeomorphisms of the manifold act on the factorization homology.

Remark 109. Factorization homology is a kind of adélic version of an $\mathbb{E}_n$-algebra.

So we had this partition function eq. (15.25) factors through the much nicer object $\int_M \mathcal{Z}\left(S^{n-1}\right)$ consists of the observables of the theory $\mathcal{Z}$ on M.

Example 110. If A is actually commutative, and we look at $\mathcal{M} = \operatorname{Spec} A$, then factorization homology is just functions on maps:

$$\int_M A = \mathcal{O}\left(\operatorname{Map}\left(M, \mathcal{M}\right)\right) \ . \tag{15.30}$$

So factorization homology is trying to be functions on mapping spaces, but it is the kind of non-commutative/quantum version of it. I.e. this is a deformation of functions on mapping spaces, i.e. the deformation quantization of $\operatorname{Map}(M, \mathcal{M})$.

For M a finite homotopy type and A commutative, we can tensor $A \otimes M$, which is the same as:

$$A \otimes M = \mathcal{O}\left(\operatorname{Map}_{\mathrm{lc}}\left(M, \operatorname{Spec}(A)\right)\right) \ . \tag{15.31}$$

I.e. take a copy of A at each vertex, quotient by relations for edges, relations between relations for faces, etc. and you'll find you need the commutative algebra structure to do that.

So we started with an n-dimensional TFT $\mathcal{Z}$ and constructed an $\mathbb{E}_n$-algebras $A = \mathcal{Z}\left(S^{n-1}\right)$. Then, by a formal procedure, we constructed a vector space:

$$\int_{M^n} A \in \mathbf{Vect} \ . \tag{15.32}$$

We expect $\mathcal{Z}(M^n) \in \mathbb{C}$, but really what we get is an element of a vector space:

$$\mathcal{Z}(M^n) \in \left(\int_{M^n} A\right)^* \ . \tag{15.33}$$

This space $\left(\int_{M^n} A\right)^*$ is called the space of conformal blocks.

Remark 110. This is kind of answering: 'how much can we find out about $\mathcal{Z}(M)$ given only knowledge of the algebra of local operators?'

This is an idea that came about in *conformal* field theory (hence the name conformal blocks), but makes sense in other settings.

The same trick works in higher codimension. Now take N^{n-1}. We expect $\mathcal{Z}(N) \in \mathbf{Vect}$, and the idea is that this vector space is a module over $A = \mathcal{Z}\left(S^{n-1}\right)$. Actually it is a module in many ways: for any $x \in N$ we can remove $\{x\} \times \{1/2\} \in N \times I$, making this a bordism from:

$$S^{n-1} \sqcup N \to N \ , \tag{15.34}$$

i.e. it induces an action of $A = \mathcal{Z}\left(S^{n-1}\right)$ on $\mathcal{Z}(N)$:

$$A \otimes \mathcal{Z}(N) \to \mathcal{Z}(N) \ . \tag{15.35}$$

This action is locally constant as you vary the point x, but really we can do better. We can remove arbitrarily many at once to get:

$$\bigotimes_{D^{n-1} \hookrightarrow N} A \circlearrowright \mathcal{Z}(N) \ . \tag{15.36}$$

This factors through something much smaller. We know A is an $\mathbb{E}_n$-algebra. In particular it is $\mathbb{E}_{n-1}$. So we can take the factorization homology of this on the $(n-1)$-manifold N:

$$\int_N A \ . \tag{15.37}$$

But we only used the $\mathbb{E}_{n-1}$ structure, meaning there is still an $\mathbb{E}_1$ (i.e. associative) algebra structure left on $\int_N A$. This is a quotient by the same relations as before, but only in the N-direction:

$$\bigotimes_{D^{n-1}\hookrightarrow N} A \twoheadrightarrow \int_N A \ . \tag{15.38}$$

Then the claim is that:

$$\int_N A \circlearrowright \mathcal{Z}(N) \ . \tag{15.39}$$

We can continue into higher codimension to find that the entire theory $\mathcal{Z}$ is linear over $\int A$, i.e. for any P:

$$\text{algebra of observables on } P = \int_P A \circlearrowright \mathcal{Z}(P) = \text{ space of states on } P \ , \tag{15.40}$$

since $\int_P A$ is exactly the sort of beast which acts on $\mathcal{Z}(P)$ for any P.

Remark 111. Note we are only using local operators so far. We could do more refined versions of this with line operators, etc.

One way to say this geometrically is as follows. For M an ordinary variety, it always makes sense to talk about $\mathcal{O}(\mathrm{Map}(-,\mathcal{M}))$. In noncommutative geometry, this doesn't work as well. So, for example, we might start with a noncommutative algebra which happens to be $\mathbb{E}_n$. We don't have the same operations as before, but we do have a version of functions on the mapping space: we can just define it to be

$$\mathcal{O}(\mathrm{Map}(-,\mathrm{Spec}(A))) := \int_- A \ , \tag{15.41}$$

which makes sense whenever the thing we're plugging in is a manifold of dimension $\leq n$.

So we started from $\mathcal{M}$ n-shifted Poisson, and then deformation quantized that to get an $\mathbb{E}_n$-algebra A. Out of this we get observables on any manifold P^k, for $K \leq n$, as this factorization homology $\int_P A$, which is a noncommutative deformation of something we can write down classically: $\mathcal{O}(\mathrm{Map}(P,\mathcal{M}))$.

So this is all fine, but then there is the more complicated question of geometric quantization. I.e. how can we write down modules for these algebras of observables? I.e. how can we extract the theory $\mathcal{Z}$ from the shifted Poisson variety $\mathcal{M}$?

15.3 Singular support of coherent sheaves

This is an instance of the general story we have been telling in the case of the $\mathcal{B}$-model. We will consider the 2-dimensional $\mathcal{B}$-model on an algebraic variety X. Then we will say a bit about the 3-dimensional case, and then we will end up back in geometric Langlands. The punchline will be that the boundary conditions in Geometric Langlands have to do with Hamiltonian actions of reductive groups.

Recall the $\mathcal{B}$-model is the 2-dimensional theory studying maps into X. Explicitly it sends the point to some version of the derived category of coherent sheaves on X:

$$\mathcal{B}_X(\mathrm{pt}) = D^b\mathbf{Coh}(X) \ . \tag{15.42}$$

If $X = \mathrm{Spec}(R)$, then we're looking at the derived category of finite-dimensional R-modules:

$$D^b\mathbf{Coh}(X) = D^b(R\text{-}\mathbf{mod}_{\mathrm{fd}}) \ . \tag{15.43}$$

The local operators for $\mathcal{B}_X$ are given as:

$$\mathrm{End}\left(\mathrm{id}_{\mathcal{B}_X(\mathrm{pt})}\right) = \mathrm{HH}^*(R) = \mathrm{HH}^*(X) \ , \tag{15.44}$$

where

$$\mathrm{HH}^*(R) = \mathrm{End}_{R\otimes R^{\mathrm{op}}}(R) = \mathrm{End}(\mathrm{id}_{R\text{-}\mathbf{mod}}) \ . \tag{15.45}$$

As we have seen, this is an $\mathbb{E}_2$-algebra since we have two associative multiplications: one coming from being endomorphisms, and one coming from the unit property of the identity.

Also recall Hochschild-Kostant-Rosenberg (HKR), which can be realized at the level of vector spaces:

$$\mathrm{HH}^*(X) = \mathrm{Sym}\, T_X[-1] \ . \tag{15.46}$$

Usually HKR is stated as saying that, if X is smooth, then T_X only lives in degree 0 so $T_X[-1]$ only lives in degree 1, so this is the exterior algebra on sections of the tangent bundle. It will be interesting when X is not smooth and we see symmetric algebras appearing in $\mathrm{HH}^*(X)$ as well.

We will be interested in this moduli space:

$$\mathcal{M}^0 = \mathrm{Spec}\, H^*(\text{local operators}) = \mathrm{Spec}(\mathrm{Sym}\, T_X[-1]) = T^*X[1] \ . \tag{15.47}$$

The cool thing is that this space is naturally 1-shifted symplectic, since it is a shifted cotangent bundle.

One thing to be cautious about, is that if X is not affine, then we're just seeing the global functions on this thing: we're not quite seeing $T^*X[1]$, we're seeing the affinization.

So we started with $\mathcal{B}_X$, we get

$$\mathcal{M}^0 = T^*X[1] \ , \tag{15.48}$$

and then we can deformation quantize this to get the $\mathbb{E}_2$-algebra:

$$\mathrm{HH}^*(X) \ . \tag{15.49}$$

The honest derived Hochschild cohomology $\mathrm{HH}^*(X)$ is not quite this commutative algebra of functions, it is an $\mathbb{E}_2$-algebra which quantizes it. So it is what one might call 1-shifted *differential operators* on X. By 1-shifted differential operators we just mean the thing we get by quantizing the shifted cotangent bundle, much like how ordinary differential operators are a deformation quantization of the unshifted cotangent bundle. cite Beraldo

Again, the classical center of R is R. But HH^* is the *derived* center. So even though R is commutative, the derived center is only $\mathbb{E}_2$, and it's even nondegenerate, i.e. the Poisson bracket is nondegenerate.

So far X could have been smooth or singular. If X is smooth, then $T^*X[1]$ is some kind of little nilpotent thickening: it is Spec of an exterior algebra over R. So $T^*[1]$ is set-theoretically the same as X, only it also comes with some nilpotent fuzz. However $T^*X[1]$ becomes much more interesting when X is singular.

15.3.1 Singular support of coherent sheaves

Given $\mathcal{F}$ a coherent sheaf, we can ask how it decomposes over the algebra of local operators HH^* (or, more classically, over $T^*X[1]$). What do we mean by this? By definition $\mathcal{F} \in \mathcal{Z}(\mathrm{pt})$ and, as we have seen, **every structure in the field theory is linear over local operators**, so we can try to spectrally decompose/sheafify over $T^*X[1]$. I.e. sheaves on X actually sheafify over this bigger space. In other words, R-modules are not only linear over R, but also over the center of R, and in the derived world the center is some version of this shifted cotangent bundle.

Let $\mathcal{F} \in \mathcal{Z}(\mathrm{pt})$, i.e. a boundary condition for our two-dimensional field theory. Then we can look at the algebra of operators that act on $\mathcal{F}$, $\mathrm{End}(F)$, which is an algebra over

$$\text{local op's} = \mathcal{Z}\left(S^1\right) = \mathrm{HH}^*(R) \ . \tag{15.50}$$

This is a field-theoretic way of saying that the category $D^b\mathbf{Coh}$ is enriched over $\mathrm{HH}^*(R)$, so in particular $\mathrm{HH}^*(R)$ acts on $\mathrm{End}(\mathcal{F}, \mathcal{F})$.

On the level of cohomology,

$$\mathrm{Sym}\, T_X[1] \circlearrowright \mathrm{Ext}(\mathcal{F}, \mathcal{F}) \ , \tag{15.51}$$

so we get an algebra on $T^*X[1]$. This is a (version of the) *microlocalization* of $\mathcal{F}$. It is an analogue of the *Wigner distribution*.

One way to think about this is in terms of what is known as the *Atiyah class*. This is a version of the theory of Chern classes. Given a vector field $\xi \in \mathbf{Vect}(X)$, we canonically get an extension

$$\xi_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \ , \tag{15.52}$$

i.e.

$$0 \to \mathcal{F} \to \xi_{\mathcal{F}} \to \mathcal{F} \to 0 \ . \tag{15.53}$$

If we take our sheaf and move along the vector field (to first order), we can ask if our sheaf was equivariant or not. This extension is the obstruction to the sheaf being equivariant along the flow given by ξ. So we get a first order deformation of the sheaf by flowing along this vector field. Another way to think about this is that ξ_F is given by 1-jets of sections of $\mathcal{F}$ along ξ. This is how Atiyah wrote a general formula for Chern classes in algebraic geometry. This gives us a map:

$$\xi \mapsto \xi_{\mathcal{F}} \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \ , \tag{15.54}$$

which is the same as a map:

$$TX[1] \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) \ , \tag{15.55}$$

and when we extend this, we get a map:

$$\mathrm{Sym}\, TX[1] = \mathrm{HH}^\bullet \to \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F}) \ , \tag{15.56}$$

which is what we were looking for.

Now let X be singular. In particular, let X be a derived local complete intersection (lci) aka quasismooth. This means X is cut out by equations on a smooth variety: we have Y and Z smooth, and some map $Z \to Y$, such that X is a fiber:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Z \arrow[d] \\ * \arrow[r] & Y \end{tikzcd} \ . \tag{15.57}$$

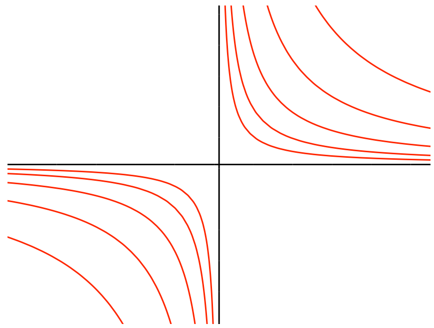


Figure 15.1: The family of subspaces cut out by $xy = t$. The fiber over 0 is the lci space cut out by $xy = 0$, shown in black.

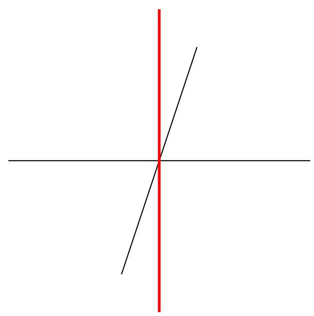


Figure 15.2: The even degree part of the 1-shifted cotangent bundle of the lci space from fig. 15.1 is (set-theoretically) $\mathbb{A}^1$ (shown in red) living over the singular point.

Locally, we just have a map $\mathbb{A}^m \to \mathbb{A}^n$ and $X \hookrightarrow \mathbb{A}^m$, as the fiber over 0:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathbb{A}^m \arrow[d] \\ 0 \arrow[r] & \mathbb{A}^n \end{tikzcd} . \tag{15.58}$$

Note that the map cutting out X is not smooth (if it is smooth, the fiber will be smooth). In this case, we can make this story of singular support very concrete.

Consider the lci space given by

$$X = \{xy = 0\} \ . \tag{15.59}$$

There is a family of subspaces cut out by $xy = t$, and X is the fiber over 0, as in fig. 15.1.

Once we pass from smooth to lci, we get new *even* directions in $T^*X\,[1]$. I.e. we get new non-nilpotent directions, since HH^* has symmetric pieces now. In this example we have an extra copy of $\mathbb{A}^1$ sticking out of the singular point as in fig. 15.2. The idea is that, if X has singular points, then $T^*X\,[1]$ gets new even directions living over these singular points.

This is good for understanding endomorphisms of sheaves. But why is it important to understand automorphisms of sheaves? Well, one of the things one learns about singular varieties, is that $x \in X$ is a smooth point if and only if $\mathcal{O}_x$ is a perfect complex, i.e. it has a finite resolution by vector bundles. But having a finite resolution by vector bundles is equivalent to $\mathrm{Ext}\left(\mathcal{O}_x, \mathcal{O}_x\right)$ being finite dimensional.

But if we take Ext of $\mathcal{O}_0$, where 0 is the singular point of $\{xy = 0\}$, we get:

$$\mathrm{Ext}^{\mathrm{even}}\left(\mathcal{O}_0, \mathcal{O}_0\right) \simeq k\left[t\right] \tag{15.60}$$

where the degree of the variable is $|t| = 2$. So this skyscraper feels very infinite, from the point of view of homological algebra, even though it is just a skyscraper. The point is that these new directions in $T^*X\left[1\right]$, as in fig. 15.2, are exactly build to take this into account. I.e. this Ext exactly looks like the $\mathbb{A}^1$ attached in $T^*X\left[1\right]$:

$$k\left[t\right] = \mathcal{O}\left(\mathbb{A}^1\right) \ . \tag{15.61}$$

In terms of the Arinkin-Gaitsgory theory of coherent singular support, we say that the cite singular support of the skyscraper at 0 is this entire $\mathbb{A}^1$:

$$\mathrm{SingSupp}\left(\mathcal{O}_0\right) = \mathbb{A}^1 \ . \tag{15.62}$$

In general, for any coherent sheaf $\mathcal{F}$, we can define a (conical) subset

$$\mathrm{SingSupp}\left(\mathcal{F}\right) \subset T^*X\left[1\right] \ . \tag{15.63}$$

$\mathcal{F}$ is perfect (e.g. a vector bundle) if and only if

$$\mathrm{SingSupp}\left(\mathcal{F}\right) = X \hookrightarrow T^*X\left[1\right] \ . \tag{15.64}$$

So the singular support gives us a way of differentiating between interesting coherent sheaves, and ones which are just vector bundles.

Example 111. Let C be a compact Riemann surface of genus g, and let $G^\vee$ be an algebraic group. Recall $\mathrm{Loc}_{G^\vee}\left(C\right)$ was the main object of study on the B-side of Geometric Langlands. As it turns out, this space is lci. Recall the definition is:

$$\mathrm{Loc}_{G^\vee}\left(C\right) \coloneqq \left\{\pi_1\left(C\right) \to G^\vee\right\}/G^\vee \ . \tag{15.65}$$

The space of maps $\pi_1\left(C\right) \to G^\vee$ is given by the fiber:

$$\begin{tikzcd}
\left\{\pi_1 \to G^\vee\right\} \arrow[r] \arrow[d] & {G^\vee}^{2g} \arrow[d, "\Pi{[,]}"] \\
1 \arrow[r] & G^\vee \ .
\end{tikzcd} \tag{15.66}$$

Now we can mod out by $G^\vee$ everywhere to get $\mathrm{Loc}_{G^\vee}$. I.e. the space of local systems is cut out of ${G^\vee}^{2g}$ by one equation (up to simultaneous conjugation). This means there is a nice theory of singular support for sheaves on $\mathrm{Loc}_{G^\vee}\left(C\right)$.

So we have discovered that if we consider the $\mathcal{B}$-side of GLC, the 4-dimensional theory $\mathcal{B}_{G^\vee}$ on the surface C (i.e. the 2-dimensional theory $\mathcal{B}_{G^\vee}\left(C \times (-)\right)$) is equivalent to the $\mathcal{B}$-model on $\mathrm{Loc}_{G^\vee}\left(C\right)$.

Recall the observables for the $\mathcal{B}$-model are given by HH^*, which gave rise to $T^*X\left[1\right]$. So the states of the $\mathcal{B}$-model are some kind of geometric quantization of $T^*X\left[1\right]$. In other words, just given $T^*X\left[1\right]$, it is hard to produce the category $D^b\mathbf{Coh}\left(X\right)$: this is the "art".[4]

[4] Ludwig Faddeev once said "quantization is an art, not a science."

Remark 112. Lagrangian field theory in general has to do something with this quantization picture, but everything is much more subtle on the $\mathcal{A}$-side. This is an example of the life lesson: things are much easier on the $\mathcal{B}$-side; the $\mathcal{B}$-side provides the answers to the questions on the $\mathcal{A}$-side. On the A-side we deal with locally constant maps, i.e. topology. On the B-side, the maps are not locally constant, but rather algebraic or holomorphic maps.

Chapter 16

(Derived) defects in Geometric Langlands

16.1 TFT via symplectic geometry

Recall that geometric Satake identifies line operators in the theory $\mathcal{A}_G$ with line operators in the theory $\mathcal{B}_{G^\vee}$. I.e. it identifies: Lecture 35; August 20, 2021

$$\begin{array}{c} \mathcal{A}_G\left(S^2\right) = \mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \\ \updownarrow \\ \mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathrm{Loc}_{G^\vee} S^2\right) \ . \end{array} \tag{16.1}$$

The B-side can be expressed as:

$$\mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathrm{Loc}_{G^\vee} S^2\right) = \mathbf{QC}\left(\mathrm{pt}/G^\vee\right) = \mathbf{Rep}\left(G^\vee\right) \ . \tag{16.2}$$

Note that the product on the A-side $\mathbf{Shv}\left(LG_+\backslash LG/LG_+\right)$ is given by convolution, and the product on the B-side $\mathbf{Rep}\left(G^\vee\right)$ is tensor product.

One way to think about this result is that it says:

$$\mathcal{M}^1_B = \mathcal{M}^1_A = \mathrm{pt}/G^\vee \ . \tag{16.3}$$

We can also ask about local operators in these theories. The idea is that local operators for a theory $\mathcal{Z}$ comprise $\mathcal{Z}\left(S^3\right)$, or equivalently, End of the unit line operator, i.e. End of the trivial representation in $\mathcal{Z}\left(S^2\right) = \mathbf{Rep}\left(G^\vee\right)$. The idea is that taking End of the unit is a version of suspension. But the unit has no endomorphisms:

$$\mathrm{End}_{\mathbf{Rep}(G^\vee)}\left(\mathrm{triv}\right) = k \ . \tag{16.4}$$

This is one reason we haven't talked very much about local operators in geometric Langlands.

Remark 113. We won't discussed surface defects much today, but they are the *ramification data*.

16.1.1 The derived version

The derived version is actually more natural from the point of view of physics. The following is a general yoga in TFT. Let $\mathcal{Z}$ be a 4-dimensional TFT. Then local operators, $\mathcal{Z}\left(S^3\right)$, comprise an $\mathbb{E}_4$ algebra, i.e. something like an odd Poisson algebra. So it is slightly non-commutative. Then

$$\mathcal{M}^0 = \operatorname{Spec} \mathcal{Z}\left(S^3\right) \tag{16.5}$$

is an odd Poisson variety. Similarly, line operators form an $\mathbb{E}_3$-algebra:

$$\mathcal{Z}\left(S^2\right) \ , \tag{16.6}$$

since it is attached to a 2-sphere. Although it is now a category rather than an algebra. So it is a category with a product which is not quite commutative. On the abelian level, $\mathbb{E}_3$ is the same as symmetric monoidal. So there is no place to "feel" this lack of commutativity without passing to the derived setting. $\mathbb{E}_2$ can be detected (as a braiding) without going derived, but not $\mathbb{E}_3$. The relationship between this $\mathbb{E}_4$-algebra and $\mathbb{E}_3$-category is:

$$\mathcal{Z}\left(S^3\right) = \operatorname{End} 1_{\mathcal{Z}(S^2)} \ . \tag{16.7}$$

This is the additive relationship between $\mathbb{E}_n$ and $\mathbb{E}_m$.

Remark 114. We can continue in this manner, and consider e.g. surface defects which form an $\mathbb{E}_2$ 2-category, but we won't do this now.

The 0th moduli space is:

$$\mathcal{M}^0 = \operatorname{Spec} \mathcal{Z}\left(S^3\right) \ , \tag{16.8}$$

and then the (refined) version coming to us from Tannakian reconstruction is:

$$\mathcal{Z}^1 = \text{“Spec”} \mathcal{Z}\left(S^2\right) \ . \tag{16.9}$$

Recall that this Tannakian reconstruction takes a tensor category $\mathcal{C}$, and forms $\operatorname{Spec} \mathcal{C}$, defined by:

$$\left(\operatorname{Spec} \mathcal{C}\right)(R) = \operatorname{Hom}_{\otimes}\left(\mathcal{C}, R\text{-}\mathbf{mod}\right) \ . \tag{16.10}$$

Now R is an $\mathbb{E}_4$-algebra, and $\mathcal{C}$ is an $\mathbb{E}_3$-category so the spectrum is:

$$\left(\operatorname{Spec} \mathcal{C}\right)(R) = \operatorname{Hom}_{\mathbb{E}_3}\left(\mathcal{C}, R\text{-}\mathbf{mod}\right) \ . \tag{16.11}$$

I.e. this makes sense in the derived world. $\mathcal{M}^0$ is an affine $\mathbb{E}_4$-scheme, and $\mathcal{M}^1$ is a (nonaffine) $\mathbb{E}_4$ stack/scheme. The relationship between them is:

$$\mathcal{M}^0 = \operatorname{Spec} \operatorname{End}\left(1_{\mathcal{C}}\right) = \operatorname{Spec} \operatorname{End}\left(\mathcal{O}_{\mathcal{M}^1}\right) = \operatorname{Spec} \Gamma\left(\mathcal{O}_{\mathcal{M}^1}\right) = \operatorname{Aff}\left(\mathcal{M}^1\right) \ . \tag{16.12}$$

16.1.2 Shifted symplectic spaces in Geometric Langlands

We will focus on the $\mathcal{B}$-side, where the 'answers' (coming from the $\mathcal{A}$-side) are. The question is: what is the moduli space of vacua associated to the theory $\mathcal{B}_{G^\vee}$? I.e. we want to identify:

$$\mathcal{M}^1_{\mathcal{B}_{G^\vee}} \to \mathcal{M}^0_{\mathcal{B}_{G^\vee}} \ . \tag{16.13}$$

In the non-derived version this was:

$$\mathrm{pt}/G^\vee \to \mathrm{pt} \ . \tag{16.14}$$

Now we want to do the derived version.

Theorem 16. $\mathcal{M}^1_{\mathcal{B}_{G^\vee}} = \mathfrak{g}^{\vee *}[2]/G^\vee$.

I.e. the structure sheaf is:

$$\mathcal{O}_{\mathcal{M}^1_{\mathcal{B}}} = \operatorname{Sym} \mathfrak{g}^\vee [-2] \tag{16.15}$$

as a representation of $G^\vee$. Putting aside the shift by two, this is something very familiar geometrically: it is the coadjoint representation with the action of $G^\vee$, one of the basic objects of representation theory. One upshot of this is that coadjoint representations have the so-called *Kostant-Kirillov* Poisson bracket, so we expect some Poisson geometry to be around. One way to say this is:

$$\mathcal{M}^1_{\mathcal{B}} = T^*[3](\mathrm{pt}.G^\vee) \ . \tag{16.16}$$

This is 3-shifted symplectic, i.e. a $\mathbb{P}_4$ / $\mathbb{E}_4$ stack. The point is that the tangent space is:

$$T(\mathrm{pt}/G^\vee) = \mathfrak{g}^\vee[1] \ , \tag{16.17}$$

and then

$$T^*(\mathrm{pt}/G^\vee) = \mathfrak{g}^\vee[-1] \ . \tag{16.18}$$

This gets 3-shifted, so the 2-shifted space from the theorem was really shifted by $-1+3$.

16.1.3 Where does this come from?

Consider $\mathrm{Loc}_{G^\vee}(S^2)$. We know that S^2 can be described as two disks glued along a circle. Since disks are contractible, local systems on a disk are really trivial: there is only one local system on the disk with automorphisms given by $G^\vee$. I.e.

$$\mathrm{Loc}_{G^\vee}(D^2) = \mathrm{pt}/G^\vee \ . \tag{16.19}$$

So we have two copies of $\mathrm{pt}/G^\vee$, and we want to glue along local systems on the circle. But we do have interesting local systems on the circle since we can have monodromy. In fact a local system on the circle is completely determined by its monodromy up to conjugation:

$$\mathrm{Loc}_{G^\vee}(S^1) = G^\vee/G^\vee \ . \tag{16.20}$$

Putting this all together we get:

$$\mathrm{Loc}_{G^\vee}(S^2) = (\mathrm{pt}/G^\vee) \times_{G^\vee/G^\vee} (\mathrm{pt}/G^\vee) \ . \tag{16.21}$$

Remark 115. We have implicitly used that local systems on a colimit can be described as a limit of local systems on the pieces. This is tautologically true since we can think of Loc as:

$$\mathrm{Loc}_{G^\vee}(-) = \mathrm{Hom}(-, G^\vee) \ . \tag{16.22}$$

We can rewrite[1] this as:

$$\begin{aligned}\mathrm{Loc}_{G^\vee}(S^2) &= (\mathrm{pt}/G^\vee) \times_{G^\vee/G^\vee} (\mathrm{pt}/G^\vee) && (16.23)\\ &= (\mathrm{pt} \times_{G^\vee} \mathrm{pt})/G^\vee \ . && (16.24)\end{aligned}$$

[1]It *does* require some thought to see why quotients commute with intersections.

I.e. local systems on S^2 can be described as the self intersection of a point in $G^\vee$, modulo $G^\vee$. In the abelian setting, the self-intersection of a point is trivial. But not in the derived setting. This is what is called the *based loop space*,[2] written:

$$\Omega G^\vee = \mathrm{pt} \times_{G^\vee} \mathrm{pt} \ , \tag{16.25}$$

i.e.

$$\mathrm{Loc}_{G^\vee}\left(S^2\right) = \left(\mathrm{pt} \times_{G^\vee} \mathrm{pt}\right)/G^\vee = \Omega G^\vee/G^\vee \ . \tag{16.26}$$

These loops are 'very small': they don't detect the whole topology of $G^\vee$. In fact, they only depend on a small neighborhood of the identity. So if we choose some coordinates, we can identify:

$$\mathrm{Loc}_{G^\vee}\left(S^2\right) = \left(\mathrm{pt} \times_{\mathfrak{g}} \mathrm{pt}\right)/G^\vee \ . \tag{16.27}$$

I.e. we are computing the derived self-intersection of a point in a vector space.

The whole point of the derived world is to correct fiber products, or in particular intersections, when they're not transverse. Let V be a vector space. Then consider the derived self-intersection of $0 \in V$:

$$0 \cap_V 0 = \mathrm{pt} \times_V \mathrm{pt} \ , \tag{16.28}$$

i.e. the based loop space of V at 0. This is the derived intersection, but we omit this from the notation. This self-intersection is the worst 'offender' in terms of transversality: the tangent space of 0 is trivial, so when we ask if the intersection is transverse (i.e. that the tangent spaces span the tangent space of the ambient space at that point) we find that we are missing the entire tangent space $T_0V \simeq V$. So this is as far away from transverse as possible, but the derived structure will exactly keep track of this degeneracy.

As usual in algebraic geometry, we will try to understand this space via functions on it. The principle we will use is that intersection becomes tensor product when we take $\mathcal{O}$:

$$\mathcal{O}\left(\mathrm{pt} \times_V \mathrm{pt}\right) = k \otimes_{\mathcal{O}(V)} k = k \otimes_{\mathrm{Sym}\, V^*} k \ , \tag{16.29}$$

since $\mathcal{O}(V) = \mathrm{Sym}_{V^*}$. This is the derived tensor product, but we omit the notation. This can be calculated using the Koszul resolution to find that this is an exterior algebra:

$$\mathcal{O}\left(\mathrm{pt} \times_V \mathrm{pt}\right) = k \otimes_{\mathcal{O}(V)} k \simeq \wedge^\bullet V^* \ . \tag{16.30}$$

This is a commutative (d)ga:

$$\wedge^\bullet V^* = \mathrm{Sym}\left(V^*[1]\right) \ . \tag{16.31}$$

Now we want to calculate the category of modules over this, i.e. quasi-coherent sheaves on this derived intersection $\mathrm{pt} \times_V \mathrm{pt}$. Whatever this category of modules is, let us first point out some structures it has. It will have a tensor product, since it is modules over a commutative ring, but we aren't interested in this product. Instead, we will be interested in the (associative) convolution structure.

Whenever we have something of the form $X \times_Y X$, coming from a map $X \to Y$, it is a kind of groupoid mapping down to X in two ways. This has a convolution structure, e.g. when X is a finite set, this convolution structure is matrix multiplication. In our case we

[2]It is important to keep track of notation and context. This is based loops in the world of derived algebraic geometry, not in homotopy theory.

are taking $X = \mathrm{pt}$ and $Y = V$. Whatever this convolution structure is, the corresponding operation on the other side of Koszul duality will arise more obviously.

To describe this category $\wedge^\bullet V^*$**-mod**, we will focus on the only objects we really understand: the free module, and the augmentation module. The augmentation module is just k, where all the exterior algebra generators of $\wedge^\bullet V^*$**-mod** just act as zero. This is the unit of convolution.

Another point of view is that this derived intersection is a based loop space at 0:

$$\mathrm{pt} \times_V \mathrm{pt} = \Omega_0 V \ . \tag{16.32}$$

And we know that based loop spaces are groups under concatenation of loops, at least in some homotopical sense. The unit is the skyscraper at identity, i.e. the augmentation module.

To understand this category, our strategy will be as follows. To study a category $\mathcal{C}$, we might consider an object we like, P, and then we might hope that the category can be described as modules over $A = \mathrm{Hom}\,(P, P)$. I.e. there is a functor:

$$\mathcal{C} \xrightarrow{\mathrm{Hom}(P,-)} (\mathrm{Hom}\,(P,P))\,\textbf{-mod} \ , \tag{16.33}$$

and in good situations it will be an equivalence.

In our situation, we will attempt to understand $\wedge^\bullet V^*$**-mod** by calculating $\mathrm{End}\,(k)$. This will lead us to Koszul duality. So we have to calculate:

$$\mathrm{End}_{\wedge^\bullet V^*}(k) \ . \tag{16.34}$$

The interesting this is that $\mathrm{pt} \times_V \mathrm{pt}$ is singular, i.e. it is something funny and derived. As we saw before, if we consider a skyscraper at a singular point, then it has infinite-dimensional endomorphisms. I.e. we can detect that we're at a singular point by noting that $k \in \wedge^\bullet V^*$**-mod** is not perfect. The interpretation of this was that this looks like functions on the singular support, i.e. functions on these extra 'wings' that the space gets attached to the singular points in the cotangent bundle. Recall the example from figs. 15.1 and 15.2, when the node grew a wing at the singular point to account for the huge endomorphisms of the skyscraper. This is not the same example, but it is the same phenomenology. This is why we should expect something like a symmetric algebra to appear:

$$\mathrm{End}_{\wedge^\bullet}(k) \simeq S = \mathrm{Sym}\,(V\,[-2]) = \mathcal{O}\,(V^*\,[2]) \ . \tag{16.35}$$

So our space grew a 'wing': $V^*\,[2]$.

This is a lot of algebra. We will not explain this in the topological setting. We will think of this exterior algebra as:

Good reference: Goresky-Kottwitz-Macpherson

$$\wedge^\bullet V^* = H_*\left(S^1\right) \ . \tag{16.36}$$

Recall $H_*\left(S^1\right)$ is an exterior algebra on one generator. If V is not one-dimensional, imagine a torus instead of just S^1. $\left(H_*\left(S^1\right), *\right)$ is an associative algebra, which one might call the *group algebra* of S^1 in homotopy theory. For a finite group, the homology of the group is literally the group algebra. In general the homology of the group always has an associative multiplication called the *Pontrjagin product*. Then we have that:

$$S = \mathrm{End}_{\wedge^\bullet}(k) = \mathrm{End}_{H_*(S^1)\textbf{-mod}}\left(H^*\,(\mathrm{pt})\right) = \mathrm{Hom}_{S^1}\left(k, H^*\,(\mathrm{pt})\right) \ . \tag{16.37}$$

This is another way of saying equivariant cohomology:

$$S = H_{S^1}(\mathrm{pt}) = H^*\left(BS^1 \simeq \mathbb{CP}^\infty\right) \simeq k[u] \ , \tag{16.38}$$

where the $|u| = 2$.

Koszul duality basically says:

$$\wedge^\bullet V^*\text{-}\mathbf{mod} \xrightarrow[\sim]{\mathrm{Hom}_{\wedge^\bullet V^*}(k,-)} S\text{-}\mathbf{mod} \ . \tag{16.39}$$

There are some standard technical difficulties that we encounter here. First of all, we need to be careful when deciding what kind of modules we are considering. For example we might consider coherent $\wedge^\bullet V^*$-modules, $\mathbf{Coh}_\Lambda$, (i.e. bounded complexes with finite-dimensional cohomology) and then we have an equivalence of categories:

$$\mathbf{Coh}_\Lambda \leftrightarrow \mathbf{Coh}_S \ . \tag{16.40}$$

What is interesting is that S is just a symmetric algebra, so it looks like sheaves on a vector space, which is smooth, so:

$$\mathbf{Coh}_S \simeq \mathbf{Perf}_S \ . \tag{16.41}$$

This is perfectly nice, except:

$$\mathbf{Coh}_\Lambda \neq \mathbf{Perf}_\Lambda \ . \tag{16.42}$$

The issue is that the augmentation module $k \in \mathbf{Coh}_\Lambda$ gets sent to:

$$k \mapsto \mathrm{Hom}_{\wedge^\bullet V^*}(k,k) = S \ . \tag{16.43}$$

But k was not perfect/compact, and $S \in \mathbf{Coh}_S$ is. Another version of this phenomenon is as follows. We have: $k\left[u,u^{-1}\right] \in \mathbf{Perf}_S$, which maps (under the adjoint functor) to $0 \in \mathbf{Perf}_\Lambda$. So there is some tension here, which is resolved by ind-coherent sheaves. This is written **IndCoh** or $\mathbf{QC}^!$. The idea is that when we look at singular spaces, we put an exclamation mark to warn you that you have to be careful(!). But the upshot is that:

$$\mathbf{QC}^!(\mathrm{pt} \times_V \mathrm{pt}) \simeq S\text{-}\mathbf{mod} \ . \tag{16.44}$$

We should think about this as a correction/enlargement of Λ-**mod**:

$$\mathbf{QC}^!(\mathrm{pt} \times_V \mathrm{pt}) \supset \Lambda\text{-}\mathbf{mod} = \mathbf{QC}(\mathrm{pt} \times_V \mathrm{pt}) \ . \tag{16.45}$$

The idea is that k is not compact (poorly behaved) in the coherent world, but when we pass to the ind-coherent world, k becomes a compact object (well behaved).

In any case, Koszul duality is a statement of the form:

$$(\Lambda\text{-}\mathbf{mod}, *) \xrightarrow{\sim} (S\text{-}\mathbf{mod}, \otimes) \ . \tag{16.46}$$

We can think of this as a version of Cartier/Pontrjagin duality. This tells us that the convolution structure on $\wedge^\bullet V^*$-**mod** corresponds to a naturally occurring tensor product structure after passing through Koszul duality.

Recall that, in the context of derived Satake, we are interested in the space:

$$\mathrm{Loc}_{G^\vee}\left(S^2\right) = \left(\mathrm{pt} \times_{\mathfrak{g}^\vee} \mathrm{pt}\right)/G^\vee \ . \tag{16.47}$$

The category we are interested in is:

$$\mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}^!\left(\mathrm{Loc}_{G^\vee}\left(S^2\right)\right) = \mathbf{QC}^!\left(\mathrm{pt}\times_{\mathfrak{g}^\vee}\mathrm{pt}\right)^{G^\vee} \ . \tag{16.48}$$

We use $\mathbf{QC}^!$ to remind us we are dealing with a singular space, and so we're treating skyscrapers at the singular points as 'valued members of society'. Recall the abelian version considered $\mathbf{QC}\left(\mathrm{pt}\right)^{G^\vee}$ (or $\mathbf{QC}^!$, but there isn't a difference since pt isn't singular), but now in the derived version we see that we're actually supposed to consider sheaves on the derived self-intersection of the point rather than just the point itself. Back to the calculation, we have:

$$\mathcal{B}_{G^\vee}\left(S^2\right) = \left(\mathrm{Sym}\left(\mathfrak{g}^\vee\left[-2\right]\right)\right)^{G^\vee} \ , \tag{16.49}$$

with tensor product. Dualizing, we see that this is ($G^\vee$-equivariant) sheaves on:

$$\left(\mathfrak{g}^{\vee *}\left[2\right]\right)/G^\vee = T^*\left[3\right]\left(\mathrm{pt}/G^\vee\right) \ . \tag{16.50}$$

16.2 Local operators

Recall we only had trivial local operators in the abelian story. Let's revisit this. Recall that local operators are given by endomorphisms of the unit line operator. So here this is the affinization of $T^*\left[3\right]\left(\mathrm{pt}/G^\vee\right)$. I.e. it is:

$$\left(\mathrm{Sym}\left(\mathfrak{g}^\vee\left[-2\right]\right)\right)^{G^\vee} = \mathrm{Sym}\left(\mathfrak{h}^\vee\left[-2\right]\right)^W = \mathcal{O}\left(\mathfrak{h}^\vee\left[2\right]/W\right) \tag{16.51}$$

where $\mathfrak{h}$ is a Cartan subalgebra. So it is a graded version of the usual space of invariant polynomials on a Lie algebra. In other words, we have:

$$\mathcal{M}^1 = \mathfrak{g}^{\vee *}\left[2\right]/G^\vee \tag{16.52}$$

and then we also have $\mathcal{M}^0$, which is usually called the Coulomb branch (of $\mathcal{N}=4$ SYM), which is:

$$\mathcal{M}^0 = \mathfrak{h}^{\vee *}\left[2\right]/W \ , \tag{16.53}$$

and then the map between them is given by the characteristic polynomial map:

$$\mathcal{M}^1 = \mathfrak{g}^{\vee *}\left[2\right]/G^\vee \to \mathcal{M}^0 = \mathfrak{h}^{\vee *}\left[2\right]/W \ . \tag{16.54}$$

So we do find local operators: they look exactly like the coefficients of the characteristic polynomial. They're just all in nonzero degrees, which is why we didn't see them before.

16.2.1 $\mathcal{A}$-side

Now that we have some nontrivial local operators, we can ask some questions: where do we find them in practice, and what do they detect/do? This is easiest to see on the automorphic side. Consider the Cartan modulo the Weyl group:

$$\mathfrak{h}^{\vee *}\left[2\right]/W = \mathfrak{h}\left[2\right]/W \ . \tag{16.55}$$

It is often convenient to just write this as $\mathfrak{h}/W$ and just keep track of the grading with an action of $\mathbb{C}^\times$. This space appears in a very natural way. Recall that the automorphic Satake category is:

$$\mathcal{A}_G\left(S^2\right) = \mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \ , \tag{16.56}$$

i.e. LG_+-equivariant sheaves on the affine grassmannian. This is some kind of derived beast, but we just need to care about the unit with respect to convolution, which is the skyscraper at the identity coset: we have $\mathrm{pt} \in \mathrm{Gr} = LG/LG_+$, and we're looking at sheaves equivariant with respect to LG_+, so the unit is the skyscraper $\underline{k}_{\mathrm{pt}/LG_+}$. But homotopically this LG_+ action is the same as a G-action:

$$\underline{k}_{\mathrm{pt}/LG_+} \simeq \underline{k}_{\mathrm{pt}/G} \ . \tag{16.57}$$

In particular this means that local operators on the $\mathcal{A}$-side form:

$$\mathcal{A}_G\left(S^3\right) = \mathrm{End}\,(1) = H^*\,(BG) = H^*_G\,(\mathrm{pt}) \ . \tag{16.58}$$

As in topology,

$$H^*_G\,(\mathrm{pt}) = \mathrm{Sym}\,\mathfrak{h}^*\,[2]^W \simeq (\mathrm{Sym}\,\mathfrak{g}^*\,[2])^G \ . \tag{16.59}$$

This is a long-winded way of saying that local operators on the A-side are just $H^*_G\,(\mathrm{pt})$, which is just the coefficient ring for G-equivariant cohomology. This is something that is familiar from e.g. Donaldson theory: the natural coefficient ring is given by equivariant cohomology.

Recall that the point of local operators is that everything is linear over this on the A-side. I.e. things like $\mathbf{Shv}\,(\mathrm{Bun}_G\,(C))$ can be written as a quotient by G:

$$\mathbf{Shv}\,(\mathrm{Bun}_G\,(C)) = \mathbf{Shv}\,(\mathrm{Bun}_G\,(C,x)\,/G) \tag{16.60}$$

meaning it is linear over $H^*\,(BG)$.

16.2.2 B-side

The derived enhancement of local and line operators on the B-side will measure singular support. This is what Arinkin-Gaitsgory were describing.. Let C be a Riemann surface (or just a topological surface). The states on C form: cite

$$\mathcal{B}_{G^\vee}\,(C) = \mathbf{QC}^!\,(\mathrm{Loc}_{G^\vee}\,(C)) \ . \tag{16.61}$$

There are different models of this category of sheaves. E.g. we used to always write $\mathbf{QC}$, quasi-coherent sheaves, but now we have realized that we need to be more careful, and use some kind of ind-coherent sheaves, so $\mathbf{QC}^!$, since this is a singular space. As we saw last time, sheaves on a singular space have a notion of singular support. In particular, this space $\mathrm{Loc}_{G^\vee}\,(C)$ arises as a fiber product:

$$\begin{tikzcd} \mathrm{Loc}_{G^\vee} \arrow[r] \arrow[d] & \left(G^\vee\right)^{2g}/G^\vee \arrow[d, "\Pi{[,]}"] \\ 1/G^\vee \arrow[r] & G^\vee/G^\vee \end{tikzcd} \ . \tag{16.62}$$

This will have a symmetry coming from based loops:

$$\Omega_1 G^\vee/G^\vee \circlearrowright \mathrm{Loc}_{G^\vee} \ . \tag{16.63}$$

The upshot is that even though $\mathrm{Loc}_{G^\vee}\,(C)$ is not smooth, it is very close to being smooth: it is the fiber of a map between two smooth spaces. This space of "wings", the place where singular support lives, $T^*[1]\,\mathrm{Loc}_{G^\vee}$, really looks like: a local system E along with a section of $\mathfrak{g}^{\vee *}[2]$ twisted by E. Whatever this thing is, we can just worry about the invariant polynomial part:

$$T^*[1]\,\mathrm{Loc}_{G^\vee} \to \mathfrak{h}^{\vee *}[2]/W \ . \tag{16.64}$$

This is all just to say, that local operators, which we said are $\mathcal{O}\left(\mathfrak{h}^{\vee *}[2]\right)^W$, act on the category $\mathbf{QC}^!(\mathrm{Loc})$ by measuring (the characteristic polynomial of) singular support.

Arinkin-Gaitsgory proposed that we restrict to the 0-fiber of this map, i.e. ind-coherent sheaves with Nilpotent singular support, written:

$$\mathbf{QC}^!_{\mathcal{N}}\left(\mathrm{Loc}_{G^\vee}\right) \ . \tag{16.65}$$

They propose this because it matches the $\mathcal{A}$-side. The "obvious" version of $\mathbf{Shv}(\mathrm{Bun}_G)$ is torsion wrt $H^*_G(\mathrm{pt})$. I.e. it lives at

$$0 \in \mathfrak{h}[2]/W = \mathcal{M}^0 \ . \tag{16.66}$$

So to match this with something on the spectral side:

$$\mathbf{Shv}(\mathrm{Bun}_G) \overset{?}{\simeq} \mathbf{QC}^!_{\mathcal{N}}\left(\mathrm{Loc}_{G^\vee}\right) \tag{16.67}$$

then it should live at 0 too.

Theorem 17 (Beraldo)**.** *The observables on a Riemann surface C, constructed from line operators, form:*

$$\int_C \mathcal{B}_{G^\vee}\left(S^2\right) \simeq \mathbf{DQ}\left(T^*[1]\,\mathrm{Loc}_{G^\vee}\right) \ ,$$

where $\mathbf{DQ}$ *denotes "deformation quantization modules", i.e. sheaves on the quantization of this space.*

This is saying that if we use line operators as our observables, then we can completely detect singular support of coherent sheaves: line operators do no more and no less than detect singular support. We can compare this to what we had before:

$$\int_C \mathbf{Rep}\left(G^\vee\right) = \mathbf{QC}\left(\mathrm{Loc}_{G^\vee}(C)\right) \ . \tag{16.68}$$

These are observables from "abelian" line operators on C, so it naturally acted on:

$$\mathcal{A}_G(C) = \mathbf{Shv}\left(\mathrm{Bun}_G(C)\right) \ . \tag{16.69}$$

So when we passed from line operators to derived line operators, we gained singular support.

Should have said before: derived geometric Satake says:

$$\mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \simeq \mathbf{QC}^!\left(\mathrm{Loc}_{G^\vee}\left(S^2\right)\right) = \left(\mathrm{Sym}\left(\mathfrak{g}^\vee[-2]\right)\textbf{-mod}\right)^{G^\vee} \ . \tag{16.70}$$

This is a theorem of Bezrukavnikov-Finkelberg. I.e. derived geometric Satake is saying that cite
derived line operators are the same on the A and B-sides.

So the takeaway is that, since $\mathrm{Loc}_{G^\vee} C$ is singular, the derived version of line operators spectrally composes over $T^*[1]\,\mathrm{Loc}_{G^\vee}$.

Remark 116. The smooth points are what are called *cuspidal* Langlands parameters. So this whole theory of singular support tells you how to understand Langlands parameters which are not cuspidal.

16.3 Boundary conditions (codimension one defects)

So far we have talked about point (local) operators, line operators, and surface operators. Now we will talk about codimension 1 operators/defects. We can consider two different kinds of codimension 1 defects. Unlike the earlier notions of defects, codimension 1 defects naturally separate the manifold into two parts. You can talk about a boundary, or an embedded codimension 1 submanifold. In physics these go by the name of *domain walls*. Because the space is decomposed into two parts, a domain wall in general, is just something which separates two different field theories $\mathcal{Z}$ and $\mathcal{Z}'$. One of the most natural notions of a map of field theories is a domain wall, i.e. should think of a domain wall separating two theories as being a map from one of the theories to the other. Formally speaking, we are extending the theory defined on ordinary bordisms, to be defined on bipartite manifolds. I.e. a map from $\mathcal{Z}$ to $\mathcal{Z}'$ is defined by extending both to bipartite manifolds, where we have $\mathcal{Z}$ on one side and $\mathcal{Z}'$ on the other side.

16.3.1 Langlands functoriality

We won't try to relate the theories on the $\mathcal{A}$-side and $\mathcal{B}$-side, although we could think about it this way, i.e. that the Langlands correspondence itself is given by some magical domain wall. Instead, we will do the following. When we talk about Langlands functoriality, they're saying that we might try to relate $\mathcal{B}_{G^\vee}$ to $\mathcal{B}_{H^\vee}$ for two different groups $G^\vee$ and $H^\vee$. The most obvious thing to do, and indeed this is what people usually mean by Langlands functoriality, is to take a subgroup, $H^\vee \subset G^\vee$, and then we want to build a map:

$$\mathcal{B}_{H^\vee} \to \mathcal{B}_{G^\vee} \ . \tag{16.71}$$

In particular, if we look at local systems on a Riemann surface C, then we get an actual map:

$$\mathrm{Loc}_{H^\vee}(C) \to \mathrm{Loc}_{G^\vee}(C) \ . \tag{16.72}$$

Which induces a pushforward functor:

$$\mathbf{QC}^!\left(\mathrm{Loc}_{H^\vee}(C)\right) \to \mathbf{QC}^!\left(\mathrm{Loc}_{G^\vee}(C)\right) \tag{16.73}$$

We didn't use that C was a Riemann surface. Indeed this extends to the entire field theory. And we didn't even use that it was a subgroup, so we could have just started with a group homomorphism $H^\vee \to G^\vee$. Then what Langlands functoriality asks is: what does this correspond to on the $\mathcal{A}$-side:

$$\begin{tikzcd} \mathcal{B}_{H^\vee} \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{B}_{G^\vee} \arrow[d, phantom, "\wr"] \\ \mathcal{A}_H \arrow[r, dashed, "?"] & \mathcal{A}_G \end{tikzcd} \tag{16.74}$$

The issue is that $H^\vee \to G^\vee$ does *NOT* give any kind of map $H \to G$.

This is actually much closer to how Langlands originally formulated the program. I.e. that there is some Langlands functoriality principle, which imprecisely says: for any map $H^\vee \to G^\vee$, there should exist an operation going from automorphism forms for H to automorphism forms for G.

Another key example of this kind of functoriality is parabolic induction/the theory of Eisenstein series. In this context we gave the group $G^\vee$, and then the maximal torus $T^\vee$.

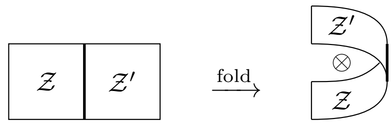


Figure 16.1: By "folding" along the domain wall, we can think of a domain wall between $\mathcal{Z}$ and $\mathcal{Z}'$ as a tensor product $\mathcal{Z} \otimes \mathcal{Z}'$.

We could think of this as a subgroup, but instead we will think of the Borel as interpolating between these two:

$$\begin{tikzcd} & B^\vee \arrow[dl, hook] \arrow[dr, two heads] & \\ G^\vee & & T^\vee \end{tikzcd} . \tag{16.75}$$

This gives us a domain wall between $\mathcal{B}_{G^\vee}$ and $\mathcal{B}_{T^\vee}$. For example, on a Riemann surface C, we get a correspondence:

$$\begin{tikzcd} & \mathrm{Loc}_{B^\vee}(C) \arrow[dl] \arrow[dr] & \\ \mathrm{Loc}_{G^\vee}(C) & & \mathrm{Loc}_{T^\vee}(C) \end{tikzcd} , \tag{16.76}$$

$\mathrm{Loc}_{B^\vee}(C)$ consists of $G^\vee$-local systems with a flag. Then there's a natural operation, to get from $\mathrm{Loc}_{G^\vee}$ to $\mathrm{Loc}_{T^\vee}$: pullback and pushforward. In this case we actually know what to do on the other side: we again think that:

$$G \supset B \twoheadrightarrow T , \tag{16.77}$$

and then we get a correspondence:

$$\begin{tikzcd} & \mathrm{Bun}_B \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_T \end{tikzcd} , \tag{16.78}$$

so we can again pullback and pushforward.

16.3.2 Domain walls versus boundary conditions

Now we will change perspective a bit. A bimodule ${}_A M_B$ is the same a module over $A \otimes B^{\mathrm{op}}$. In linear algebra this is just saying that a map $f \colon V \to W$ is the same as an element $f \in V^* \otimes W$. So we're taking a bimodule ${}_A M_B$, which we can think of as a functor

$${}_A M_B \colon A\textbf{-mod} \to B\textbf{-mod} , \tag{16.79}$$

and we're just alternatively thinking of it as an element of $A \otimes B^{\mathrm{op}}\textbf{-mod}$.

Visually speaking, instead of thinking of a bimodule/domain wall, we will fold our paper as in fig. 16.1. A boundary condition is a domain wall to the trivial theory. So rather than a domain wall/map from $\mathcal{Z}$ to $\mathcal{Z}'$, we're thinking of a codimension 1 defect as a boundary condition for the single theory $\mathcal{Z} \otimes \mathcal{Z}'$. The point is that domain walls, or functoriality, is a special case of boundary conditions.

For example, in this Eisenstein series example, instead of thinking of a domain wall given by B,

$$G \supset B \twoheadrightarrow T \ , \tag{16.80}$$

we can fold it around, and think of B as a boundary condition for the $G \times T$ theory. Or we could think about this geometrically. If we have:

$$U \hookrightarrow B \twoheadrightarrow T \ , \tag{16.81}$$

then G/U is a space with two actions:

$$G \circlearrowright G/U \circlearrowleft T \ . \tag{16.82}$$

Then if we mod out by the groups, we get:

$$\begin{tikzcd} & \mathrm{pt}/B \arrow[dl] \arrow[dr] & \\ \mathrm{pt}/G & & \mathrm{pt}/T \end{tikzcd} \ . \tag{16.83}$$

The upshot of this is that we can just focus on boundary conditions without worrying about domain walls.

16.3.3 Hamiltonian actions

The punchline will be the following.

Claim 3. A boundary condition for $\mathcal{B}_{G^\vee}$ is roughly the same as/gives rise to a (shifted) Hamiltonian $G^\vee$-space.

This is a version of the theory of Coulomb branches (Braverman-Finkelberg-Nakajima), which appeared earlier in the context of physics, e.g. Gaiotto-Witten

cite BFN and GW

Example 112. Let G be a group, and X be a G-space. Now consider T^*X. This is symplectic, and it gets an action of G from X, but it also has a moment map:

$$T^*X \to \mathfrak{g}^* \ , \tag{16.84}$$

which is G-equivariant. This moment map gives Hamiltonians for the $\mathbf{Lie}\,(G) = \mathfrak{g}$-action. The idea is that this map $T^*X \to \mathfrak{g}^*$ gives us a map on functions:

$$H \colon \mathcal{O}\left(\mathfrak{g}^*\right) = \mathrm{Sym}\,\mathfrak{g} \to \mathcal{O}\left(T^*X\right) \ , \tag{16.85}$$

which is the Hamiltonian operator.

So for $x \in \mathfrak{g}$, we can look at the corresponding Hamiltonian function, $H_x \in \mathcal{O}\left(T^*X\right)$, and then saying that this is Hamiltonian for the action is saying that Poisson bracket with H_x is the same as the action of x:

$$\{H_x, -\} = \text{ action of } x \text{ on } T^*X \ . \tag{16.86}$$

So we have a map $\mathfrak{g} \to \mathcal{O}(T^*M)$ lifting the map $\mathfrak{g}$ to vector fields:

$$\begin{tikzcd} & \operatorname{Sym}\mathfrak{g} \arrow[dr] & \\ \mathfrak{g} \arrow[ur, hook] \arrow[rr] \arrow[drr, "\text{action}"'] & & \mathcal{O}(T^*M) \arrow[d, "\{,\}"] \\ & & \mathbf{Vect}(T^*M) \end{tikzcd} \quad . \tag{16.87}$$

Now if we're given an arbitrary symplectic space M with a G-action (i.e. not necessarily of the form T^*X), we might ask for the same data:

$$\begin{tikzcd} M \arrow[rr] & & \mathfrak{g}^* \\ \circlearrowright & & \circlearrowright \\ & G & \end{tikzcd} \quad . \tag{16.88}$$

I.e. a map to $\mathfrak{g}^*$, equivariant for the group, which generates the Lie-algebra action of G, all compatible with the symplectic structure.

We already see something that should remind us of geometric Satake. We can rephrase this data as a map of stacks:

$$M/G \xrightarrow{\mu} \mathfrak{g}^*/G \ . \tag{16.89}$$

Up to issues of grading (e.g. maybe we would expect $\mathfrak{g}^*[2]/G$) the target of the moment map looks like the moduli space of vacua for the B-theory:

$$M/G \xrightarrow{\mu} \mathfrak{g}^*[2]/G = \mathcal{M}^1_{\mathcal{B}} \ . \tag{16.90}$$

So the place where Hamiltonian geometry comes in is that a map to to $\mathcal{M}^1_{\mathcal{B}}$ will be very close to a Hamiltonian G-action. So we would like to extract data that looks like a Hamiltonian action from a boundary condition.

Let's do some more field theoretic defectology. The idea is to do the same kind of spectral nonsense. We summarize this as follows. Given a boundary condition $\mathcal{T}$ for a TFT $\mathcal{Z}$, we want to extract an algebra of operators and take Spec. This will be the space M, and in our case it will be a Hamiltonian space. So the claim is that this space lives over the moduli space of vacua for the theory $\mathcal{Z}$. I.e. we already took a theory $\mathcal{Z}$ and made algebraic geometry out of it, and now we want to make algebraic geometry in tandem for a theory $\mathcal{Z}$ along with a boundary condition $\mathcal{T}$.

Recall we did something similar for 2-dimensional theories. If we have a category $\mathcal{C}$, which we're thinking of as corresponding to a two-dimensional TFT, then we can consider an object $\mathcal{F} \in \mathcal{C}$. Then we can look at the algebra of endomorphisms of $\mathcal{F}$, and this received a map from the local operators for the two-dimensional theory $\mathcal{Z}$:

$$\mathrm{HH}(\mathcal{C}) = \mathcal{Z}\left(S^1\right) \to \mathrm{End}(F) \ . \tag{16.91}$$

So we have an algebra over local operators, i.e. it sheafifies over Spec of the local operators, and this is where singular support came from.

To repeat this in the language of algebra: we have an algebra A acting on a module M, and from this we get a map:

$$\mathrm{center}(A) \to \mathrm{End}(M) \ , \tag{16.92}$$

which is the kind of thing we're going to get. In this setting $\mathrm{End}(M)$ is just an associative ($\mathbb{E}_1$) algebra, so we don't really want to take Spec of it since it's not commutative enough. But in 4-dimensions everything is more commutative. So we can take Spec and see geometry.

Let $\mathcal{Z}$ be a 4-dimensional TFT, and let $\mathcal{T}$ be a boundary condition. To warm up, let's first assume that $\mathcal{Z}$ is trivial. In this case $\mathcal{T}$ is the same as an ordinary 3-dimensional TFT. Then local operators for $\mathcal{T}$, $\mathcal{T}\left(S^2\right)$, forms an $\mathbb{E}_3$-algebra. On the level of cohomology it just looks like a (graded, even) Poisson algebra. So we can define the moduli space of vacua for this theory $\mathcal{T}$ to be:

$$\mathcal{M}^0_{\mathcal{T}} = \mathrm{Spec}\left(\mathcal{T}\left(S^2\right)\right) \ , \tag{16.93}$$

which we can think of as an (affine) Poisson variety. If we take an ordinary Poisson variety with a $\mathbb{C}^\times$ action, then its ring of functions is graded, and we can declare that to be the cohomological grading. I.e. take the $\mathbb{C}^\times$-action for which $\{,\}$ has degree -2. So this is a $\mathbb{P}_3$-variety.

Example 113. T^*X always has a rescaling action by $\mathbb{C}^\times$. This has degree 2, so its inverse, the Poisson bracket, has degree -2. I.e. we can make this into a $\mathbb{P}_3$-variety.

The moduli space $\mathcal{M}^0_{\mathcal{T}}$ will always be affine, because we only looked at local operators. Instead we could look at line operators, $\mathcal{T}\left(S^1\right)$, which is an $\mathbb{E}_2$-category (braided). The resulting moduli space will be an $\mathbb{E}_3$-stack which is not necessarily affine. We're doing a kind of reconstruction (of M from the theory $\mathcal{T}$), but it's maybe easier to think of it the other way. If we start with M graded symplectic (i.e. it has a $\mathbb{C}^\times$-action where the symplectic form has weight 2), then we can construct a 3-dimensional TFT called *Rozansky-Witten theory.* The local operators for this theory form:

$$\mathcal{T}_M\left(S^2\right) = \left(\mathcal{O}\left(M\right), \{,\}\right) \ , \tag{16.94}$$

thought of as a $\mathbb{P}_3$-algebra, and the line operators are:

$$\mathcal{T}_M\left(S^1\right) = \mathbf{QC}\left(M\right) \ . \tag{16.95}$$

But the tensor product on the line operators actually has a braided deformation, so it is actually $\mathbb{E}_2$. This is really only well-understood in the case when $M = T^*X$, in which case we're really using Koszul duality:

$$\mathbf{QC}\left(T^*X\right) \leftrightarrow \mathbf{QC}^!\left(\mathcal{L}X\right) \ , \tag{16.96}$$

and $\mathbf{QC}^!\left(\mathcal{L}X\right)$ has a natural $\mathbb{E}_2$-structure.

So we start with $\mathcal{T}$, produce:

$$\mathcal{M}^1 \to \mathcal{M}^0 \ , \tag{16.97}$$

and then hope that $\mathcal{T}$ is $\mathbf{RW}$ on $\mathcal{M}^0$, or more likely $\mathbf{RW}$ on $\mathcal{M}^1$.

Remark 117. Physicists might call $\mathbf{RW}$ on one of these moduli spaces as a low-energy effective field theory for $\mathcal{T}$.

Warning 3. This reconstruction fails miserably sometimes. For example, consider Chern-Simons. There are no local operators, so the affine version would be a point. There are interesting line operators, given by this modular tensor category, but if we try to take its spectrum it is also just a point. There is nothing derived about this modular tensor category, so it doesn't talk to $\mathbb{E}_3$-algebras. However the BG is 2-shifted symplectic, thanks to the killing form, and Chern-Simons is morally $\mathbf{RW}$ of BG, yet we're not seeing this from either $\mathcal{M}^0$ or $\mathcal{M}^1$.

Now let our bulk theory $\mathcal{Z}$ be nontrivial, still with a boundary condition $\mathcal{T}$. The point will be that there is a relative version of everything we just said. Namely, local operators for T, $T\left(S^2\right)$, is not a naked $\mathbb{E}_3$-algebra like before, but rather an $\mathbb{E}_3$-algebra in the category $\mathcal{Z}\left(S^2\right)$. One way to say this is that we can take local operators in the bulk and collide them with the boundary, making $T\left(S^2\right)$ an algebra over $\mathcal{Z}\left(S^3\right)$. This is analogous to the situation from earlier: if we have a module $A \circlearrowright M$, then $\mathrm{End}\,(M)$ is an algebra over the center of A. I.e. it is an $(\mathbb{E}_1)$ algebra object in the category $\mathrm{center}\,(A)$**-mod**. It makes sense to talk about an algebra object of this category, because the center of A is $\mathbb{E}_2$, so the category of modules over it is $\mathbb{E}_1$ itself.

The same picture works in the three-dimensional theory: local operators on the boundary, $\mathcal{T}\left(S^2\right)$, is an $\mathbb{E}_3$-algebra over the $\mathbb{E}_4$-algebra $\mathcal{Z}\left(S^3\right)$, i.e. an $\mathbb{E}_3$-algebra object in the $\mathbb{E}_3$-category $\mathcal{Z}\left(S^3\right)$**-mod**. Now we're in this more commutative setting and we can take Spec. This means that we have a map:

$$\begin{tikzcd} \mathrm{Spec}\, T\left(S^2\right) \arrow[d] \\ \mathcal{M}^0 = \mathrm{Spec}\, \mathcal{Z}\left(S^3\right) \end{tikzcd} . \tag{16.98}$$

This map is an example of what is called *Lagrangian* (it has some compatibility with the Poisson structures). This category $\mathcal{Z}\left(S^3\right)$**-mod** is really an approximation for $\mathcal{Z}\left(S^2\right)$. A better way to think about this is that $T\left(S^2\right)$ is an $\mathbb{E}_3$-algebra object in the $\mathbb{E}_3$-category $\mathcal{Z}\left(S^2\right)$.

If we're in the context of Langlands, i.e. our bulk theory $\mathcal{Z}$ is actually $\mathcal{B}_G^\vee$ we see that for a boundary theory $\mathcal{T}$ of $\mathcal{B}_{G^\vee}$ we get an algebra over $\mathfrak{g}^{\vee *}[2]/G^\vee$, since this category $\mathcal{Z}\left(S^2\right)$ is sheaves on the coadjoint representation $\mathfrak{g}^{\vee *}[2]/G^\vee$. And so when we take Spec, we get an affine morphism:

$$M/G^\vee \to \mathfrak{g}^{\vee *}[2]/G^\vee , \tag{16.99}$$

which is compatible with Poisson structures. Which is the same thing as giving an affine Hamiltonian $G^\vee$-variety (up to this grading shear) $G^\vee \circlearrowright M$. Then if we want to see something non-affine, we can look at $\mathcal{T}\left(S^1\right)$, which is an $\mathbb{E}_2$-algebra over $\mathcal{Z}\left(S^2\right)$. And then we get something like

$$\mathbf{QC}\,(M/G^\vee) \tag{16.100}$$

acted on by

$$\mathbf{QC}\left(\mathfrak{g}^{\vee *}[2]/G^\vee\right) \tag{16.101}$$

via the moment map. So a boundary condition for $\mathcal{B}_{G^\vee}$ has a Hamiltonian-$G^\vee$ space as a "shadow". Conversely, if $G^\vee \circlearrowright M$ is a Hamiltonian action on a holomorphic-symplectic variety, then $\mathbf{RW}_M$ defines a boundary condition for $\mathcal{B}_{G^\vee}$.

Chapter 17

Boundary conditions in geometric Langlands

Specifically, we will talk about the applications of boundary conditions in geometric Langlands to number theory.

17.1 Summary of boundary conditions

The rough idea, is that a boundary condition for an n-dimensional gauge theory (with gauge group G), should be roughly the same as an $(n-1)$-dimensional theory with G-symmetry. First we will do some examples as a warmup for what we're really interested in. Lecture 36; August 24, 2021

Example 114. If G is a compact Lie group, then there is a two-dimensional QFT, topological Yang-Mills, $\mathrm{YM}_G^{\mathrm{top}}$. The boundary conditions are one-dimensional theories, now with G-symmetry, so the category of boundary conditions is $\mathbf{Rep}(G)$.

Example 115. Two-dimensional A-type gauge theory is also known (by physicists) as $\mathcal{N} = (2,2)$ super Yang-Mills for the group G in the A-twist. We boundary conditions for this theory as follows. Suppose $G \circlearrowright X$. The boundary theories will be some kind of quantum mechanics, but instead of ordinary QM on X, we can do supersymmetric QM on X by considering the de Rham complex $H^\bullet_{\mathrm{dR}}(X)$ as our space of states. Since $G \circlearrowright X$, we can ask what kind of symmetries the de Rham complex gets. It does act, but in such a way that the $\mathfrak{g} = \mathbf{Lie}(G)$ action is trivialized. This is the content of the famous Cartan homotopy identity. So classically, you only see the group of connected components acting, but a richer thing to say is that in fact you get an action:

$$H_*(G) \circlearrowright H^*(X) \ . \tag{17.1}$$

These are what one might call *locally constant representations of* G, rather than ordinary representations as in the previous example.

This is very similar to what we will see in the four-dimensional A-model. In particular, the local operators in this theory look like

$$H_*(G) \otimes H^*(BG) \ . \tag{17.2}$$

In geometric Langlands, we are studying boundary conditions for this physical gauge theory, called 4-dimensional $\mathcal{N} = 4$ SYM (Gaiotto-Witten) which is a G-gauge theory. A cite boundary condition is some kind of 3-dimensional theory with G-symmetry. Because we're talking about these supersymmetric bulk theories, we also need boundary theories which are suitably supersymmetric. In physics, if we take a manifold M which is not just Riemannian (like we needed in SUSY QM), not just Kähler, but hyperkähler, then this is exactly the condition we need to construct a 3-dimensional σ-model into M. This ends up being an $\mathcal{N} = 4$ SUSY theory (i.e. it has 8 supercharges). We should think of this as a version of 2-shifted quantization: rather than a vector space, i.e. a 1-dimensional field theory, we're constructing a three-dimensional QFT. Because we have sufficient supersymmetry, we can make this topological in a few ways. One is the "Coulomb"/$\mathcal{B}$-twist, which gives us the Rozansky-Witten TFT on M. There is also the "Higgs"/$\mathcal{A}$-twist, which is the lesser-known 3-dimensional $\mathcal{A}$-model. We won't really discuss the $\mathcal{A}$-model. A hyperkähler manifold has a lot of structure, but in particular it is holomorphic symplectic. For the Rozansky-Witten theory we just need M to be holomorphic symplectic, i.e. we don't actually need the Riemannian metric.

Suppose we have a compact group G_c acting by hyperkähler isometries $G_c \circlearrowright M$. (Or we could think of the complex group $G_{\mathbb{C}} \circlearrowright M$ by holomorphic symplectic symmetries.) From this data, we get a boundary condition for this four-dimensional $\mathcal{N} = 4$ Yang-Mills theory. And indeed the types match: the $\mathcal{A}$-type version gives a boundary condition for the four-dimensional $\mathcal{A}$-theory, $\mathcal{A}_G$, and the $\mathcal{B}$-type gives a boundary condition for the four-dimensional $\mathcal{B}$-theory, $\mathcal{B}_G$. This will be our source of boundary theories.

What Gaiotto and Witten tell us, is that if you have electromagnetic/S-duality, then this is a symmetry which identifies $\mathcal{N} = 4$ SYM for G with $\mathcal{N} = 4$ SYM for $G^\vee$. Since the theories themselves are identified, the boundary conditions are identified as well. So we have a correspondence between boundary conditions for $\mathcal{A}_G$ and boundary conditions for $\mathcal{B}_{G^\vee}$. Among the boundary conditions for $\mathcal{A}_G$, we can consider the "geometric ones", which come from Hamiltonian G-spaces $G \circlearrowright M$. Then we might hope that these match with the Rozansky-Witten theories coming from Hamiltonian $G^\vee$-spaces $G^\vee \circlearrowright M^\vee$. We could even be more ambitious, and hope that we can actually match the ones which are cotangent bundles on the $\mathcal{A}$-side:

$$G \circlearrowright X \rightsquigarrow M = T^*X \qquad \overset{?}{\leftrightsquigarrow} \qquad G^\vee \circlearrowright X^\vee \rightsquigarrow M^\vee = T^*X^\vee \ . \tag{17.3}$$

17.2 Relative spectrum

Recall from last time, that we can do a little better: given a boundary condition, we can actually construct a Hamiltonian G or $G^\vee$-space. This is a 'relative spectrum' construction.

Let $\mathcal{Z}$ be a four-dimensional TFT with boundary theory/condition $\mathcal{T}$. What kind of algebraic structure does this give us? We can think of $\mathcal{T}$ as a map from the trivial theory to $\mathcal{Z}$:

$$\mathcal{T} \in \mathrm{Hom}\,(1, \mathcal{Z}) \ . \tag{17.4}$$

This means that for N of dim ≤ 3, we can look at $\mathcal{Z}\,(N)$, and we get that

$$\mathcal{T}\,(N) \in \mathrm{Hom}\,(1, \mathcal{Z}\,(N)) \ . \tag{17.5}$$

We can really think of this as defining an element

$$\mathcal{T}\,(N) \in \mathcal{Z}\,(N) \ . \tag{17.6}$$

Remark 118. We can sometimes think of this as a functional

$$\mathcal{T}(N) \in \operatorname{Hom}(\mathcal{Z}(N), 1) \ , \tag{17.7}$$

but if we don't have enough duality these might not quite be the same.

So a boundary theory picks out an element of whatever the theory assigns to a bordism, in a way which is compatible with all of the structure. For example, we get a particular object $\mathcal{T}\left(S^2\right) \in \mathcal{Z}\left(S^2\right)$, of the category of line operators from the bordism $S^2 \times I$, with one end of the interval labelled with $\mathcal{T}$. It is compatible with all of the structure on $\mathcal{Z}\left(S^2\right)$, i.e. it is an $\mathbb{E}_3$-algebra object of the category of line operators.

Another way to think about this picture, is the following. To describe local operators on the boundary, like all defects, we look at the link of that singularity. The link is actually just a 3-ball, with boundary S^2. The relation between this picture and the bordism $S^2 \times I$ above is simple: remove the singularity from B^3 to get $B^3 \setminus \mathrm{pt} \simeq I \times S^2$. The upshot is that $\mathcal{T}\left(S^2\right) \in \mathcal{Z}\left(S^2\right)$ is a relative refinement of the $\mathbb{E}_3$-algebra of local operators on the boundary $\mathcal{T}$: we can obtain the ordinary $\mathbb{E}_3$-algebra of local operators by filling in the ball, i.e. pairing with the unit line operator.

We can also look at line operators on $\mathcal{T}$. When we excise a tubular neighborhood of some lines, we get that line operators for $\mathcal{T}$ carry an action of line operators for $\mathcal{Z}$. For us, i.e. when $\mathcal{Z} = \mathcal{B}_{G^\vee}$, line operators are:

$$\mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \ , \tag{17.8}$$

where we are omitting the shift by 2. Now if we have a boundary condition $\mathcal{T}$, then $\mathcal{T}\left(S^2\right)$ is an $\mathbb{E}_3$ (graded Poisson) algebra in $\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right)$. So when we take its spectrum, we get an affine variety over the coadjoint representation:

$$M^\vee/G^\vee \to \mathfrak{g}^{\vee *}/G^\vee \ , \tag{17.9}$$

which is the moment map for $M^\vee$ as a Hamiltonian $G^\vee$-space. So this is what we mean by a relative spectrum: it lives over $\mathfrak{g}^{\vee *}/G^\vee$. If we just took the non-relative Spec of $\mathcal{T}\left(S^2\right)$, then we just get:

$$\operatorname{Spec} \mathcal{O}\left(M^\vee\right)^{G^\vee} \simeq M^\vee /\!\!/ G^\vee \ , \tag{17.10}$$

which is called the affine GIT quotient. So this just an affine Poisson variety. But when we do the relative version, we get an actual Hamiltonian $G^\vee$-space from line operators. Or, if you prefer, given such a map $M^\vee/G^\vee \to \mathfrak{g}^{\vee *}/G^\vee$, we can pass to categories of sheaves, and ($G^\vee$-equivariant) sheaves on $M^\vee$ are acted on by sheaves on $\mathfrak{g}^{\vee *}/G^\vee$ by pullback:

$$\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \circlearrowright \mathbf{QC}\left(M^\vee\right)^{G^\vee} \ . \tag{17.11}$$

I.e. line operators for $\mathcal{T}$ are acted on by line operators for $\mathcal{B}_{G^\vee}$.

Notice that all we needed to know about the boundary condition was its local operators, which are the same for the $\mathcal{A}$ and $\mathcal{B}$-theories by geometric Satake. So we can be ambivalent about whether we started with a boundary condition on the $\mathcal{A}$-side or the $\mathcal{B}$-side.

So the idea is to start with a boundary condition for the theory $\mathcal{A}_G$, i.e. a Hamiltonian G-space, and then we can build some kind of spectrum out of it, which is this Hamiltonian space for $G^\vee$. (This is very similar to the Coulomb branch construction of Braverman-Finkelberg-Nakajima, only we're in four dimensions rather than three.) And then we might hope that this new space $M^\vee$ is the actual dual of our space M, i.e. that this construction realizes the duality between boundary conditions for $\mathcal{A}_G$ and boundary conditions for $\mathcal{B}_G^\vee$.

17.3 L-functions ($\mathcal{B}$-side)

Now we want to interpret boundary conditions on both sides and relate them to objects of interest in number theory. Namely, the boundary conditions on the $\mathcal{B}$-side will be related to L-functions, and the boundary conditions on the $\mathcal{A}$-side will be related to periods. First we will discuss how to interpret boundary conditions on the $\mathcal{B}$-side, and what they have to do with L-functions.

Recall we discussed this a little bit already, e.g. in section 13.4. The basic mechanism is as follows. We have a vector space V with an endomorphism $F \circlearrowright V$. The local version of an L-function is a "super" version of the characteristic polynomial. We can write the usual characteristic polynomial as:

$$\det(1 - tF) = \mathrm{Tr}_{\mathrm{gr}}\left(F, \wedge^\bullet V\right) \ . \tag{17.12}$$

Similarly:

$$\frac{1}{\det(1 - tF)} = \mathrm{Tr}_{\mathrm{gr}}\left(F, \mathrm{Sym}^\bullet V\right) \ . \tag{17.13}$$

In fact we should really think of these simultaneously: if V is a super vector-space (so we have an even part and an odd part), then the exterior algebra is just the symmetric algebra of the odd part. For example, if $V^\bullet$ is a chain complex (the differential won't matter, so we can just think of this as a graded vector space) then we can define a similar L-factor by doing a combination of both of these formulas:

$$\begin{aligned} L(V, F, t) &= \prod \det\left((1 - tF)|_{H^k(V)} \right)^{(-1)^{k+1}} &(17.14)\\ &= \prod \mathrm{Tr}_{\mathrm{gr}}\left(F, \mathrm{Sym}\, H^k(V)\right) &(17.15)\\ &= \mathrm{Tr}_{\mathrm{gr}}\left(F, \mathrm{Sym}^\bullet V\right) \ . &(17.16)\end{aligned}$$

where we have omitted some parity correction on $H^k(V)$ from the notation, and $\mathrm{Sym}^\bullet$ is the graded symmetric algebra.

So we should think of the L-factor as some version of the characteristic polynomial. Unlike the usual characteristic polynomial, which is finite, this is not in general a finite formula. I.e. if V has an even part we will get a power series in t. There are two main questions one might ask about L-functions: there are analytic questions about the dependence on t (e.g. the Riemann hypothesis), however we are interesting in the L-values instead:

$$L(V, F) = L(V, F, t = 1) \ . \tag{17.17}$$

For example, if $F = \mathrm{id}$, then $L(V, F) = \dim(\mathrm{Sym}(V))$. For general F we have $L(V, F) = \mathrm{Tr}\left(F|_{\mathrm{Sym}(V)} \right)$. Since $\mathrm{Sym}(V)$ is usually infinite-dimensional, this number won't make sense in general. The point is that this number needs some kind of renormalization. One punchline is that this ∞ is the same kind of ∞ that appears in quantum field theory: in QFT you often write expressions for partition functions which end up being infinite. And now you need some regularization of renormalization procedure. In our setting the L-values will literally appear as a partition function and in fact the techniques to renormalise will often be the same as in physics.

This is kind of the function field notation, but we can write this L-value in the more classical number-theoretic notation, where there isn't a single variable t. We really have to think of $t = q^{-s}$, where q is a prime power (order of residue field in a local field situation). The idea is that t is the multiplicative variable, and then the additive variable s is the 'logarithm' of t. But if we have different primes, we have to think about them differently. We talked a bit about this earlier, but the idea is that a number field kind of looks like a 3-manifold. It's not quite fibered over the circle: it has a 'circle' corresponding to every prime, and each one has a 'length', which is basically $\log p$. E.g. you will see expressions like

$$\frac{1}{\det\left(1 - p^{-s}F\right)} \ , \tag{17.18}$$

so p^{-s} is playing the role of t. Since we're consider the value of the L-function at $t = 1$, this corresponds to $s = 0$. Changing the point we're evaluating, e.g. passing from $s = 0$ to $s = n \in \mathbb{Z}$, corresponds to changing the grading on the vector space V. In particular:

$$L\left(V, F, t = q^{-k/2}\right) = L\left(V, F, s = k/2\right) = L\left(V\left\langle k\right\rangle, F, s = 0\right) \ , \tag{17.19}$$

where $V\langle k\rangle$ means that $F \circlearrowright V\langle k\rangle$ by $q^{-k/2}F$. This isn't going to be very important for us, it's just to say that the precise point of evaluation isn't that important: the value of an L-function at $s \in \mathbb{Z}$ is the same as another L-function at $s = 0$. So we will talk about *the* L-value of $F \circlearrowright V$. Again, this completely avoids questions about analytics properties of L-functions in s.

Example 116. In number theory the first L-function you meet is the Riemann ζ-function:

$$\zeta\left(s\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ . \tag{17.20}$$

We can also think of this via the Euler product:

$$\zeta\left(s\right) = \prod_p \frac{1}{1 - p^{-s}} \ . \tag{17.21}$$

So we started with an L-factor for all p, and then we take the product to get the L-function. If you like, the operator F in this case is the identity. I.e. this L-factor is what we get when we consider the trivial 1-dimensional vector space with the identity operator acting on it. Then we might be interested in the L-value:

$$\zeta\left(0\right) = \sum 1 = \#\mathbb{N} = -\frac{1}{2} \ . \tag{17.22}$$

Let's set this up in the setting we need Let $G^\vee$ be a reductive group, and instead of an endomorphism, consider some element $F \in G^\vee$, and a representation:

$$V \colon G^\vee \to \mathrm{GL}\left(V\right) \ . \tag{17.23}$$

Then want to give some version of a characteristic polynomial in this setting. To this F and V we can attach the L-factor

$$L\left(V \circlearrowleft F\right) \ . \tag{17.24}$$

In particular, if we have a Galois representation:

$$\mathrm{Gal}\left(\overline{\mathbb{Q}_p}, \mathbb{Q}_p\right) \to G^\vee \ , \tag{17.25}$$

we can ask if this representation is unramified, i.e. that it factors through the Frobenius:

$$\begin{tikzcd} \mathrm{Gal}\left(\overline{\mathbb{Q}_p}/\mathbb{Q}_p\right) \arrow[rr] \arrow[dr] & & G^\vee \arrow[r, "V"] & \mathrm{GL} \\ & \langle F \rangle \arrow[ur, dashed] & & \end{tikzcd} . \tag{17.26}$$

Then we can attach an L-factor this.

So these are the local factors, and we can define global L-functions for Galois representations as follows. So suppose we have representations:

$$\mathrm{Gal}\,(F) \xrightarrow{\rho} G^\vee \xrightarrow{V} \mathrm{GL}\,(V) \ . \tag{17.27}$$

Then we can define:

$$L\,(\rho, V, s) = \prod_p L_p\,(V \circlearrowleft \rho\,(F_p)) \tag{17.28}$$

where F_p is the Frobenius at p. A priori, this is not well-defined where the representation is ramified. At all but finitely many places this representation factors through the Frobenius, so we can just restrict to the places where it is unramified, or if p is ramified, we can still define an L-factor at p as follows. The Frobenius at p no longer acts on V, but it does act on the invariants of the inertia:

$$V^{\mathrm{Inertia}} \circlearrowleft \rho\,(F_p) \ . \tag{17.29}$$

The inertia is the kernel

$$\mathrm{Inertia} \to \mathrm{Gal}\,(F_p) \to \langle F \rangle \ . \tag{17.30}$$

This L-function is something like the trace of this big tensor product of Frobenius operators acting on this big tensor product of symmetric products of V, but we have to take the inertia invariants:

$$L\,(\rho, V, s) = \mathrm{tr}\left(\bigotimes_p F_p \circlearrowright \bigotimes_p \mathrm{Sym}\, V^{I_p}\right) \ . \tag{17.31}$$

Again, just as with ζ, this does not converge at $s = 0$, but there will be some domain where it does converge, and we can ask if we can analytically continue this and evaluate at $s = 0$.

If C is a curve over $\mathbb{F}_q$, then we have the unramified part of the Galois group of C:

$$\mathrm{Gal}^{\mathrm{unram}}\,(C) = \pi_1^{\mathrm{arith}}\,(C) \ , \tag{17.32}$$

which maps to a copy of $\mathbb{Z}$ generated by the Frobenius, and the kernel is the geometric π_1:

$$\pi_1^{\mathrm{geom}}\,(C) \to \pi_1^{\mathrm{arith}}\,(C) \to \langle F \rangle \ , \tag{17.33}$$

where

$$\pi_1^{\mathrm{geom}}\,(C) = \pi_1\left(\overline{C} = C \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}\right) \ . \tag{17.34}$$

So the picture is that if we have a curve over a finite field, and we think of this as a three manifold fibered over a circle, then we have a map to π_1 of the circle, and the kernel is π_1 of the fiber. This geometric π_1 is playing the role of the inertia from above. So we can define the L-function in this case as the trace of this single Frobenius acting on π_1^{geom}-invariants.

17.3.1 Global function field

Now that we have discussed a little about L-function in number theory, we will say this all slower in the function field setting. We would like to interpret this L-function in a more geometric way.

Consider a curve $C/\mathbb{F}_q$, and let V be a representation of reductive group $G^\vee$. We want to study Galois representations into a $G^\vee$. We will also take V to be self-dual, and in fact symplectic. This will relate to the general theory of L-functions for arbitrary representations: if you're interested in a general representation W, then $T^*W = W \oplus W^*$ will be symplectic, and then

$$L\left(T^*W\right) = L\left(W\right) \cdot L\left(W^*\right) \ . \tag{17.35}$$

Now we will fix a Galois representation. We're in geometry now, so this is a representation of π_1, i.e. we will fix some local system $\rho \in \mathrm{Loc}_{G^\vee}\left(C\right)$. Recall that local systems in this context are:

$$\mathrm{Loc}_{G^\vee}\left(C\right) = \left\{\pi_1^{\mathrm{arith}} \to G^\vee\right\}/G^\vee \ . \tag{17.36}$$

This is what one might call an arithmetic local system. In the case of a curve over a finite field, this is the same as a local system on $C/\overline{\mathbb{F}_q}$ fixed by the Frobenius F. Note that we are assuming we are unramified everywhere. One can try to do a lot of these things in the ramified setting, but this is harder.

Now we can talk about the L-value $L\left(\rho, V\right)$. We have $\pi_1\left(C\right) \to G^\vee \circlearrowright V$, so we can take:

$$L\left(\rho, V\right) = L\left(\rho, V, s=0\right) \tag{17.37}$$

$$= L\left(V^{\pi_1^{\mathrm{geom}}} \circlearrowleft F\right) \ . \tag{17.38}$$

Now we want to interpret this more geometrically. Inside of this full arithmetic π_1, we have the geometric part, which acts on V

$$\rho\colon \pi_1^{\mathrm{geom}} \to \mathrm{End}\left(V\right) \ . \tag{17.39}$$

In other words, V defines a local system of vector spaces, V_ρ, associated to ρ as a $G^\vee$-local system and the representation V. Then this vector space from above is really just cohomology with coefficients in this local system:

$$L\left(\rho, V\right) = L\left(H^*\left(C, V_\rho\right) \circlearrowleft F\right) \ . \tag{17.40}$$

So this is what our global L-function is.

We can say this in several ways. We have a tautological sheaf on $\mathrm{Loc}_{G^\vee}$, given by:

$$\rho \mapsto H^*\left(C, V_\rho\right) \ . \tag{17.41}$$

I.e. cohomology of C with coefficients in the V-twist of the universal local system. I.e. we have a sheaf $\{V_\rho\}$ on $C \times \mathrm{Loc}_{G^\vee}$ (which is quasi-coherent in the Loc direction and locally constant (local system) in the C direction), and then we can push it along the map:

$$\begin{tikzcd} C \times \mathrm{Loc}_{G^\vee} \arrow[d, "\pi"] \\ \mathrm{Loc}_{G^\vee} \end{tikzcd} \ . \tag{17.42}$$

So we have this space and Frobenius automorphism:

$$\mathrm{Loc}_{G^\vee}\left(\overline{C}\right) \circlearrowleft F \ , \tag{17.43}$$

where the fixed points are arithmetic local systems:

$$\mathrm{Loc}_{G^\vee}^{\mathrm{arith}} = \mathrm{Loc}_{G^\vee}\left(C\right) \ . \tag{17.44}$$

At a fixed point ρ,

$$F \circlearrowright \text{fiber } = H^*\left(C, V_\rho\right) \tag{17.45}$$

and we are taking L the trace to get L. So we get a function $\rho \mapsto L\left(\rho, V\right)$ on the space of arithmetic local systems, $\mathrm{Loc}_{G^\vee}^{\mathrm{arith}}$, given any representation of the group.

So we attached $\rho \mapsto \mathrm{Tr}\left(F \text{ on } \mathrm{Sym}\, H^\bullet\left(C, V_\rho\right)\right)$. We want to really take this symmetric algebra seriously. In particular, it is a commutative ring, and we want to view $\mathrm{Sym}\, H^\bullet\left(C, V_\rho\right)$ as functions on some derived scheme. The point is that this scheme has a nice interpretation. For example, if ρ is trivial we have:

$$\mathrm{Sym}\, H^\bullet\left(C, V\right) = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}\left(C, V\right)\right) \ . \tag{17.46}$$

For general ρ, we have V_ρ living over C, and we can look at locally constant sections:

$$\begin{tikzcd} V_\rho \arrow[d, shift right] \\ C \arrow[u, bend right] \end{tikzcd} \ . \tag{17.47}$$

Then we have that:

$$\mathrm{Sym}\, H^*\left(C, V_\rho\right) = \mathcal{O}\left(\text{locally constant sections of } V_\rho \to C\right) \ . \tag{17.48}$$

Now let's write this in a more unified way. At some point we might get sloppy about V versus V^* (e.g. $\mathrm{Sym}\, V$ versus $\mathrm{Sym}\, V^*$) but we took V to be symplectic, so it is self-dual. When we assemble these together for all ρ, we get a space:

$$\begin{tikzcd} \mathrm{Loc}_{G^\vee}^{V} \arrow[r, equal] \arrow[d] & \left\{\rho + \text{flat section of } V_\rho\right\} = \mathrm{Map}_{\mathrm{lc}}\left(C, V/G^\vee\right) \\ \rho \in \mathrm{Loc}_{G^\vee} \arrow[r, equal] & \mathrm{Map}_{\mathrm{lc}}\left(C, \mathrm{pt}/G^\vee\right) \end{tikzcd} \ . \tag{17.49}$$

Now we define:

$$\mathcal{A}_V \in \mathbf{QC}^!\left(\mathrm{Loc}_{G^\vee}\right) \tag{17.50}$$

by:

$$\mathcal{A}_V \coloneqq \pi_* \mathcal{O} \ . \tag{17.51}$$

I.e. it sends

$$\rho \mapsto \mathcal{O}\left(\text{loc. const. sections of } V_\rho\right) = \mathrm{Sym}\, H^\bullet\left(C, V_\rho\right) \ . \tag{17.52}$$

The point of all this is that we have incorporated this symmetric algebra into the geometric story. In other words, we can write the L-function down without writing down the symmetric algebra:

$$L\left(V, \rho\right) = \mathrm{Tr} \text{ of } F \text{ on the fiber of } \mathcal{A}_V \text{ at } \rho \ . \tag{17.53}$$

This is the "L-observables" sheaf. It is a sheaf of commutative algebras.

Remark 119. Note that functions on $\mathrm{Loc}_{G^\vee}(C)$ is exactly $\mathcal{B}_{G^\vee}(C)$, where we are thinking of C as a version of a three-manifold. In fact, we have just seen that it is also coming from a sheaf on the geometric object:

$$\mathcal{A}_V \in \mathbf{QC}^! \left(\mathrm{Loc}_{G^\vee} \overline{C}\right) = \mathcal{B}_{G^\vee}\left(\overline{C}\right) \ . \tag{17.54}$$

For this whole story, we don't actually need V to be a vector space. We can replace V by some variety $M^\vee \circlearrowleft G^\vee$. Really it's important that we're replacing the symplectic representation V by a Hamiltonian $G^\vee$-space. Now we can define an analogue of an L-function for $M^\vee$. We have a space:

$$\begin{tikzcd} \mathrm{Loc}_{G^\vee}^{M^\vee} \arrow[r, equal] \arrow[d, "\pi"] & \mathrm{Map}_{\mathrm{lc}}\left(C, M^\vee/G^\vee\right) \\ \mathrm{Loc}_{G^\vee} & \end{tikzcd} , \tag{17.55}$$

and then we can define the sheaf of L-observables to be the pushforward of the structure sheaf:

$$\mathcal{A}_{M^\vee} = \pi_* \mathcal{O} \ . \tag{17.56}$$

Then we can take the trace of the Frobenius on the fiber at

$$\rho \in \left(\mathrm{Loc}_{G^\vee}\right)^{\mathrm{Fr}} = \mathrm{Loc}_{G^\vee}^{\mathrm{arith}} \ , \tag{17.57}$$

i.e. an arithmetic local system, then we get a generalized L-function. If $\pi_1 \to G^\vee \circlearrowright M^\vee$ has discrete fixed points, then we get the sum of L-functions of the tangent space.

Remark 120. We started with the global story, and it hasn't been clear how the symplectic structure played any role at all. This will come when we relate this to the local story.

17.3.2 Local story

Under this $\mathrm{Tr}(F)$ decategorification, Euler products is the shadow of factorization homology. Suppose $G^\vee \circlearrowright V$ (or more generally $G^\vee \circlearrowright M^\vee$). Then we have that $\mathcal{O}(M^\vee)$ (or $\mathrm{Sym}\, V^*$) are representations of $G^\vee$. From these we can define the local L-factor:

$$L^{\mathrm{local}}\left(M^\vee\right) = \left\{g \mapsto \mathrm{Tr}\left(g, \mathcal{O}\left(M^\vee\right)\right)\right\} \ . \tag{17.58}$$

Again, these functions will have a lot of poles. For example if we take the identity element, we're trying to calculate the dimension of a symmetric algebra.

But one thing that always makes sense, is the categorified version. $\mathcal{O}(M^\vee)$ is a commutative algebra in $\mathbf{Rep}(G^\vee)$. If $M^\vee$ is Hamiltonian, then we have a moment map:

$$M^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *} \ . \tag{17.59}$$

So we can pushforward,

$$\mu_* \mathcal{O}_{\mathcal{M}^\vee} = \int \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) = \mathbf{QC}\left(\mathfrak{g}^{\vee *}\right)^{G^\vee} \ . \tag{17.60}$$

So we're taking functions on $G^\vee$, and spreading it over the coadjoint representation compatibly with the group action.

Now the punchline is that the Euler product expansion of L-functions corresponds to factorization homology in this categorified world. This is a key idea which is not really written down. It is mentioned in Gaitsgory-Lurie: Weil's conjecture on Tamagawa numbers. Recall that the Euler product says that the L-function can be expressed in terms of local cite L-factors:

$$L(\rho, V) = \prod_{x \in C} L_x(\rho, V) \tag{17.61}$$

$$= \prod \mathrm{Tr}\,(\mathrm{Sym}\, V) \tag{17.62}$$

$$= \mathrm{Tr}\left(\bigotimes \mathrm{Sym}\, V\right) . \tag{17.63}$$

So the L-function is the trace of a giant tensor product. The general idea is that factorization homology is also some version of a giant tensor product. If R is a commutative ring then, by definition, the factorization homology receives a map from:

$$\int_C R \leftarrow \bigotimes_{x \in C} R . \tag{17.64}$$

In fact it is the quotient by the relation: whenever you see two nearby points, so you have a copy of R at both, then you have a map $R \otimes R \to R$. This is easiest to formulate in the topological/Betti setting. This can be formalized in the setting of a curve over an arbitrary field using the notion of a Ran space. We should think of this as some kid of improved version of the tensor product: you have this huge tensor product over all points of the curve, and you quotient out by some relations to make it into something reasonable:

$$\int_C R = \mathcal{O}\left(\mathrm{Map}_{\mathrm{lc}}\left(C, \mathrm{Spec}\left(R\right)\right)\right) . \tag{17.65}$$

So if each of these rings have a Frobenius operator, we might want to calculate the trace on this infinite symmetric product, but in general this won't converge. But you can try to compare it with the trace of Frobenius on this nicer object, to get something which is regularized.

In our setting we had this local category:

$$\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) = \mathcal{B}_{G^\vee}\left(S^2\right) , \tag{17.66}$$

and then we can try to calculate the factorization homology of this over a curve, which turns out to be:

$$\int_C \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) = \mathbf{QC}\left(\mathrm{Map}_{\mathrm{lc}}\left(C, \mathfrak{g}^{\vee *}/G^\vee\right)\right) \tag{17.67}$$

$$= \mathbf{QC}\left(T^*[1]\,\mathrm{Loc}_{G^\vee}\right) . \tag{17.68}$$

On the other hand, we had this local object

$$\mathcal{O}\left(M^\vee/G^\vee\right) \in \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) , \tag{17.69}$$

which gave us our local L-factor. So we have:

$$\mathbf{QC}\left(T^*[1]\,\mathrm{Loc}_{G^\vee}\right) = \int_C \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \tag{17.70}$$

and inside of this we have:

$$\mathbf{QC}\left(T^*[1]\operatorname{Loc}_{G^\vee}\right) \ni \mathcal{O}\left(\operatorname{Loc}_{G^\vee}^{M^\vee} = \operatorname{Map}_{\mathrm{lc}}\left(C, M^\vee/G^\vee\right)\right) = \int_C \mathcal{O}\left(M^\vee/G^\vee\right) \ . \tag{17.71}$$

So we have

$$\begin{tikzcd}
\operatorname{Loc}_{G^\vee}^{M^\vee} \arrow[r, "\mu"] \arrow[d] & T^*[1]\operatorname{Loc}_{G^\vee} \arrow[r, equal] \arrow[dl] & \operatorname{Map}\left(C, \mathfrak{g}^{\vee *}/G^\vee\right) \arrow[dl] \\
\operatorname{Loc}_{G^\vee} \arrow[r, equal] & \operatorname{Map}\left(C, \mathrm{pt}/G^\vee\right) &
\end{tikzcd} \ . \tag{17.72}$$

So there is a sense in which factorization homology descends to the Euler product for the L-function associated to $M^\vee$.

17.3.3 Summary

To summarize, boundary conditions/theories on the $\mathcal{B}$-side are a way of organizing L-functions for Galois representations. So for a linear representation $G^\vee \circlearrowright W$ we can double it to get a symplectic vector space $V = W \oplus W^*$, and (more geometrically) for $G^\vee \circlearrowright X^\vee$, we get a Hamiltonian $G^\vee$-space

$$T^*X^\vee = M^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *} \ . \tag{17.73}$$

From this we get a global L-function:

$$\{L(\rho, V)\} \in \mathcal{O}\left(\operatorname{Loc}_{G^\vee}(C)\right) = \mathcal{B}_{G^\vee}(C) \ . \tag{17.74}$$

On the other hand, the local data for the L-function was

$$\mathcal{O}\left(M^\vee/G^\vee\right) \in \mathbf{Alg}\left(\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right)\right) \ , \tag{17.75}$$

i.e. a local operator on the boundary condition. For example, $\operatorname{Sym} V$ gives such an object. The passage from local to global is then given by factorization homology: the observables for the boundary condition $M^\vee$ on C is given by:

$$\int_C \mathcal{O}\left(M^\vee\right) = \mathcal{A}_{M^\vee} \tag{17.76}$$

which categorifies the global L-function of $M^\vee$.

So the L-function attached to a symplectic representation is given by the trace of Frobenius on the observables for a boundary condition.

Remark 121. The deformation quantization of $M^\vee$ is an algebra $\mathcal{A}_M^\vee$, which really lives over $\operatorname{Loc}_{G^\vee}$, such that

$$\operatorname{Tr}(F) = L\left(M^\vee\right) \ . \tag{17.77}$$

Geometric quantization produces the $\mathcal{L}$-sheaf

$$\mathcal{L}_{M^\vee} \in \mathbf{QC}^!\left(\operatorname{Loc}_{G^\vee}\right) \ , \tag{17.78}$$

which carries an action of

$$\mathcal{A}_{M^\vee} = \int_C \mathcal{O}\left(M^\vee\right) \tag{17.79}$$

and

$$\operatorname{Tr}\left(F, \mathcal{L}_{M^\vee}\right) = L\text{-function of } X^\vee \text{ or } W \ . \tag{17.80}$$

Example 117. For ρ trivial, this is observables vs. states for $\mathbf{RW}_{M^\vee}$. Local operators are something like $\mathcal{O}(M^\vee)$, and states attached to a Riemann surface of genus G is not so easy to write down, since they're a geometric quantization, but we do know that there is an action:

$$\mathbf{RW}_{M^\vee}(C) \circlearrowleft \int_C \mathcal{O}(M^\vee) = \mathcal{O}(\mathrm{Map}_{\mathrm{lc}}(C, M^\vee)) \ . \tag{17.81}$$

$\mathcal{O}(\mathrm{Map}_{\mathrm{lc}}(C, M^\vee))$ is something like a Clifford algebra, and $\mathbf{RW}_{M^\vee}(C)$ is something like a spin module.

All of this is coming from [Ben-Zvi-Sakellaridis-Venkatesh] The point is: the theory of L-functions of Galois representations fits into the framework of boundary conditions. This isn't super deep: this side is the answer. The $\mathcal{A}$-side asks the questions. We're just saying that L-functions of Galois representations naturally arise from boundary conditions on the B-side. cite

In fact an L-function of a local system is a partition function of $\mathcal{B}_{G^\vee}$ on the "4-manifold "$C \times I$, where we put ρ and V at either end. By this we mean $\delta_\rho \in \mathcal{O}(\mathrm{Loc}_{G^\vee}(C)) = \mathcal{B}_{G^\vee}(C)$. A boundary condition V defines a state in the theory $\mathcal{B}_{G^\vee}(-)$, for any manifold plugged into $(-)$. So in particular on the curve C.

Lecture 37; August 27, 2021

17.4 Recap about L-functions

Recall we have a representation $G^\vee \circlearrowright W$, which we will use to attach L-functions to things of the form:

$$\rho\colon \ \mathrm{Gal} = \pi_1(C) \to G^\vee \to \mathrm{GL}(W) \ . \tag{17.82}$$

These are given as Euler products of local factors. Locally we can think that we have a copy of $\mathbb{Z}$ (generated by the Frobenius), i.e. we have the short exact sequence:

$$\mathbb{Z}F \to G^\vee \to \mathrm{GL}(W) \tag{17.83}$$

since $[F] \in G^\vee / G^\vee$. Then we take $\mathrm{Tr}(F, \mathrm{Sym}(W))$.

We are going to double this representation W to get a *symplectic representation* of $G^\vee$:

$$V = W \oplus W^* = T^* W \ . \tag{17.84}$$

Before doubling, we could have generalized, and replaced this representation by any variety with a $G^\vee$-action:

$$G^\vee \circlearrowright X^\vee \ . \tag{17.85}$$

We would correspondingly replace $\mathrm{Sym}(W^*)$ with $\mathcal{O}(X^\vee)$.

The common generalization is that that we start with any symplectic variety $M^\vee$, which might be $M^\vee = T^* X^\vee$. This variety is a Hamiltonian $G^\vee$-variety, i.e. there is a map of stacks called the moment map

$$M^\vee / G^\vee \xrightarrow{\mu} \mathfrak{g}^{\vee *} / G^\vee \ , \tag{17.86}$$

which is compatible with the symplectic structure.

These Hamiltonian spaces arise from, and give rise to, boundary conditions for the B-model. Specifically, for $M^\vee$ holomorphic symplectic we can construct a three-dimensional

TFT $\mathbf{RW}_{M^\vee}$, called Rozansky-Witten theory with target $M^\vee$. But if $G^\vee \circlearrowright M^\vee$, then $\mathbf{RW}_{M^\vee}$ becomes a boundary theory for $\mathcal{B}_{G^\vee}$.

Recall the L-function of the symplectic representation V is:

$$\mathrm{Tr}\left(F, \mathrm{Sym}\left(V\right)\right) \ , \tag{17.87}$$

where we're thinking that

$$\mathrm{Sym}\left(V\right) = \mathcal{O}\left(M^\vee\right) \ . \tag{17.88}$$

I.e. given an element of the group up to conjugacy, $F \in G^\vee/G^\vee$, then this acts on $\mathcal{O}\left(M^\vee\right)$ (since it acts on $M^\vee$), and we can take the trace.

We want to think of $\mathcal{O}\left(M^\vee\right)$ in a relative way. I.e. the data of $\mathcal{O}\left(M^\vee\right)$, the action $G^\vee \circlearrowright \mathcal{O}\left(M^\vee\right)$, and μ corresponds to an algebra in:

$$\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) = \mathbf{QC}\left(\mathfrak{g}^{\vee *}\right)^{G^\vee} \ , \tag{17.89}$$

given by pushing $\mathcal{O}$ forward along the moment map. This relates to the story of boundary conditions $\mathcal{T}$ for the theory $\mathcal{B}_{G^\vee}$ as follows. Recall $\mathcal{T}$ produces an $\mathbb{E}_3$-algebra in line operators $\mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right)$ and a grading.

Remark 122. Secretly we are quantizing the $\mathbb{P}_3$-variety $M^\vee$ to an $\mathbb{E}_3$-algebra. One can ignore this by taking cohomology.

More categorically, given the moment map $M^\vee/G^\vee \to \mathfrak{g}^{\vee *}/G^\vee$ we get an action:

$$\mathbf{QC}\left(M^\vee/G^\vee\right) \circlearrowleft \mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \ . \tag{17.90}$$

The Euler product says that if $\rho \in \mathrm{Loc}_{G^\vee}\left(C\right)$, then the local L-function was given by the product of the local factors:

$$L\left(\rho, M^\vee\right) = \prod_{x\in C} \text{local factors} \ , \tag{17.91}$$

which needs to be regularized. This can be categorified to: something like $\mathrm{Tr}\left(F, \bigotimes_{x\in C}\mathcal{O}\left(M^\vee\right)\right)$, which we can formally access via the quotient:

$$\bigotimes_{x\in C}\mathcal{A} \twoheadrightarrow \int_C \mathcal{A}_{M^\vee} \ . \tag{17.92}$$

This is the algebra of observables for T on $\overline{C}$, i.e.

$$L\left(\rho, M^\vee\right) = \mathrm{Tr}\left(F, \int_C \mathcal{A}_{M^\vee}\right) \ . \tag{17.93}$$

So we found the observables. What about states? The states on a Riemann surface will be objects:

$$\mathcal{L}_{M^\vee, C} \in \mathcal{B}_{G^\vee}\left(C\right) = \mathbf{QC}^!\left(\mathrm{Loc}_{G^\vee}\left(C\right)\right) \ , \tag{17.94}$$

i.e. a module for observables. This is called the L-sheaf, and in fact:

$$\mathrm{Tr}\left(F, \mathcal{L}_{M^\vee, C}\right) = L\text{-function of } X^\vee \ , \tag{17.95}$$

or W. I.e. this returns the L-function of the original representation, not the double.

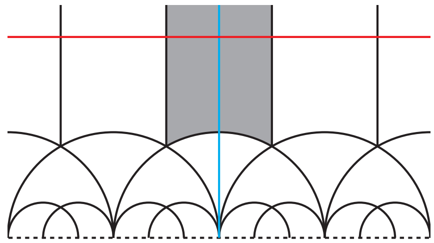


Figure 17.1: Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray. One can define a "period" as taking a modular form and integrating it, e.g. on the red or blue line.

Example 118. If $M^\vee = T^*X^\vee$, then

$$\mathrm{Loc}_{G^\vee}^{X^\vee} = \{\rho \text{ and flat section of associated } X^\vee\text{-bundle}\} \ . \tag{17.96}$$

This has a map

$$\pi\colon \mathrm{Loc}_{G^\vee}^{X^\vee} \to \mathrm{Loc}_{G^\vee} \ , \tag{17.97}$$

and then

$$\mathcal{L} \coloneqq \pi_*\omega \ . \tag{17.98}$$

We have ω instead of $\mathcal{O}$ because of the same reason that we need **IndCoh** instead of **QC**.

17.5 Periods of automorphic forms ($\mathcal{A}$-side)

How does this Hamiltonian symplectic geometry arise from the automorphic side? We want to present this from a classical number theory point of view.

A basic question in the theory of L-functions is: how do we know we can analytically continue? This is not something we have direct access to on the Galois side. We talked a bit about an L-function of an automorphic form. There is a straightforward idea of how to attach an L-function to an automorphic form. The idea is that if f is an automorphic form, then it is unramified at all but finitely many places. So we have Satake parameters at all $p \notin S$. This just means the Hecke eigenvalues. The whole point of the Satake correspondence is that the Hecke eigenvalues correspond to $[F] \in G^\vee/G^\vee$. This is where the dual group came from. And so we know how to attach L-factors: for $V \in \mathbf{Rep}(G^\vee)$ we can define

$$L(f, V, s) = \prod_{p\notin S}{}' \frac{1}{\det\left(1 - p^{-s}F_p\right)} \ . \tag{17.99}$$

Let's ask the question again, in a different way: what are natural functionals on automorphic forms? This is a special case of the question: what are boundary conditions for $\mathcal{A}_G$? The point being that a boundary condition will certainly give a functional on $\mathcal{A}_G(C)$, which is the space of automorphic forms. Let's go to the picture of a modular form, so $G = \mathrm{PSL}_2(\mathbb{Z})$. Given particular cycles, we can integrate the modular form to get a number. See fig. 17.1. The red line is a horocycle, which will give rise to the Eisenstein period. The blue line will give rise to the Hecke period. Any modular form φ on $\mathrm{SL}_2\mathbb{Z}$ has a q-expansion:

$$\varphi = \sum a_n q^n \tag{17.100}$$

i.e. it is the Laurent expansion at ∞. So we could send φ to the number:

$$a_0 = \int_{\text{horocycle}} \varphi \, dt \ . \tag{17.101}$$

This is called the *Eisenstein period*. (It tells the difference between cusp forms and Eisenstein series.) We can write $y = 1$ as the orbit:

$$\int_{y=1} \varphi \, dt = \int_{N(\mathbb{Z}) \backslash N(\mathbb{R})} \varphi \, dt \tag{17.102}$$

so this is also called the N-period where

$$N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \ . \tag{17.103}$$

The coefficient a_1 is the *Whittaker period*.

The other example, integrating over the blue line, gives rise to the Hecke period. I.e. we integrate it over the imaginary axis:

$$\int_0^\infty \varphi\left(iy\right) \frac{dy}{y} y^s \ . \tag{17.104}$$

We can rewrite this as:

$$\int_0^\infty \varphi\left(iy\right) \frac{dy}{y} y^s = \sum \frac{a_n}{n^s} \ , \tag{17.105}$$

where the a_n are the coefficients of the q-expansion of φ. This is the L-function of φ:

$$\int_0^\infty \varphi\left(iy\right) \frac{dy}{y} y^s = \sum \frac{a_n}{n^s} = \frac{\Gamma\left(s\right)}{\left(2\pi\right)^s} L\left(\varphi, s\right) \ . \tag{17.106}$$

We can write:

$$L\left(\varphi, s\right) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}} \ . \tag{17.107}$$

This is exactly the product of the local L-factors given by Hecke eigenvalues when φ is an eigenform. The factor $\Gamma\left(s\right) / \left(2\pi\right)^s$ is the L-factor at ∞.

Now we might wonder: how does this relate to the Galois side? Why did we know to do this integral? What is the data required to define this integral? And how does it relate to the data we needed on the other side? A more abstract way to define these periods is: given a subgroup $H \subset G$, we can calculate the H-period as follows. Recall we have a locally-symmetric space associated to G, which was a version of Bun_G:

$$\left[G\right]_F = G\left(F\right) \backslash G\left(\mathbb{A}\right) / \text{level} \ . \tag{17.108}$$

Inside of this we have:

$$\left[H\right]_F = H\left(F\right) \backslash H\left(\mathbb{A}\right) / \ . \tag{17.109}$$

Then we can define the H-period as:

$$\mathcal{P}_H\left(\varphi\right) = \int_{H(F) \backslash H(\mathbb{A})} \varphi \ . \tag{17.110}$$

The Eisenstein period comes from $H = N \subset \mathrm{PSL}_2$, and the Hecke period comes from $H = T \subset \mathrm{PSL}_2$.

Remark 123. Really, we only want spherical subgroups H. This is a finiteness condition which yields good analytic properties.

This comes from a blog post of Terry Tao. Consider abelian L-functions, i.e. the L-functions for $G = \mathrm{GL}_1 = G^\vee$. For example, the Riemann and Dedekind ζ-functions are abelian L-functions. There is also the classical Dirichlet L-function, and the generalization due to Hecke. They correspond to characters of $\mathrm{Gal}\left(\overline{F}/F\right)$. From class field theory we know that these correspond to characters χ of the idèle class group:

cite

$$\mathrm{GL}_1(F) \setminus \mathrm{GL}_1(\mathbb{A}) \ . \tag{17.111}$$

These are sometimes called Hecke grössencharacters.

Riemann wrote a formula for the ζ-function as an integral. In particular, the Ξ function

$$\Xi(s) = \frac{\Gamma(s/2)}{\pi^{s/2}} \xi(s) \ , \tag{17.112}$$

where we can think of the front factor as being the local factor at ∞, can be written as:

$$\Xi(s) = \int_0^\infty y^{s/2} \sum_{n=1}^{\infty} e^{-n^2 \pi y} \, dy \ . \tag{17.113}$$

I.e. this is the Mellin transform of the θ-function.

Locally (so fix a finite prime p) at a place $v = p$ we had (in local class field theory) that

$$K_v^\times / \mathcal{O}_v^\times \simeq \mathbb{Z} \ . \tag{17.114}$$

For example $\mathbb{Q}_p^\times / \mathbb{Z}_p^\times \simeq \mathbb{Z}$. Inside of here we have $\varpi \in K_v^\times / \mathcal{O}_v^\times$ which corresponds to $1 \in \mathbb{Z}$, i.e. $p \in \mathbb{Q}_p^\times / \mathbb{Z}_p^\times$.

Consider an unramified local character:

$$\chi_v \colon K_v^\times / \mathcal{O}_v^\times \to \mathbb{C}^\times \ . \tag{17.115}$$

This is determined by a single number, i.e. $\chi_v \in \mathbb{C}^\times$. This corresponds to the 1-dimensional Galois representation where F_v acts by χ_v. Crucial observation: the L-value at 0 of this character is:

$$L(\chi_v, 0) = \frac{1}{1 - \chi_v} = \mathrm{Tr}\left(\chi_v, \mathrm{Sym}^\bullet \mathbb{C}\right) = \sum_{n \geq 0} \chi_v^n \ . \tag{17.116}$$

The point is that we can rewrite this as an integral in terms of $K^\times / \mathcal{O}^\times$. In particular we can write:

$$\begin{aligned} L(\chi_v, 0) &= \frac{1}{1 - \chi_v} = \mathrm{Tr}\left(\chi_v, \mathrm{Sym}^\bullet \mathbb{C}\right) && (17.117) \\ &= \sum_{n \geq 0} \chi_v^n && (17.118) \\ &= \sum_{n=0}^{\infty} \chi_v\left(\varpi^n \in K^\times / \mathcal{O}^\times\right) && (17.119) \\ &= \int_{K_v^\times} \chi_v(x) \cdot 1_{\mathcal{O}_v} d^* x \ . && (17.120) \end{aligned}$$

The crucial observation is that this characteristic function is the *local basic function*:

$$\Phi_v = 1_{\mathcal{O}_v} \ . \tag{17.121}$$

The L-factor can be written:

$$L_v\left(\chi_v, s\right) = \int_{K_v^\times} \chi_v \Phi_v\left(x\right) d^* x \cdot |x|_v^s \ . \tag{17.122}$$

This is the Mellin transform of $\Phi_v = 1_{\mathcal{O}}$.

Let's step back. We are doing Fourier theory for $\mathbb{G}_m \circlearrowright \mathbb{A}^1$. Think $\mathbb{A}^1 \simeq K \simeq \mathbb{Q}_p$. Inside of this we have $1_{\mathbb{Z}_p} = 1_{\mathcal{O}}$, and we want to decompose this with respect to the $\mathbb{G}_m$-action. The real version of $\Phi_v = 1_{\mathcal{O}_v} = 1_{\mathbb{Z}_p \subset \mathbb{Q}_p}$ is the Gaussian:

$$\Phi_\infty\left(x\right) = e^{-\pi x^2} \tag{17.123}$$

and the local L-factor at ∞ is:

$$L_\infty = \int_{\mathbb{R}^\times} \Phi_\infty\left(x\right) e^{-\pi x^2} \ . \tag{17.124}$$

17.5.1 Iwasawa Tate

Can rewrite this L-function of a character as:

$$L\left(\chi, s\right) = \int_{F^\times \backslash \mathrm{GL}_1(\mathbb{A})} \chi\left(x\right) \cdot \sum_{\gamma \in F} \Phi\left(\gamma_x\right) d^* x \tag{17.125}$$

where

$$\Phi = \prod_v \Phi_v \ . \tag{17.126}$$

This is a function on:

$$\prod{}' \mathbb{A}^1_{F_v} \ . \tag{17.127}$$

This is a sum over, say $\mathbb{Q}$, so how does this relate to the earliers sum over $\mathbb{Z}$? Well:

$$\sum_{\gamma \in F} \Phi\left(\gamma \cdot x\right) = \sum_{\gamma \in \mathcal{O}_F} \Phi_\infty\left(\gamma_x\right) \ , \tag{17.128}$$

i.e. we get a sum of Gaussians on translates of x. This is the θ-series. There is an operator:

$$\Theta\colon \ \mathrm{Fun}\left(\mathbb{A}^1_{\mathbb{A}}\right) \to \mathrm{Fun}\left(\left[\mathrm{GL}_1\right]\right) \tag{17.129}$$

which is given by summing over all translates:

$$\Theta\left(\Phi\right) = \sum_{\gamma \in F} \Phi\left(\gamma \cdot x\right) \ . \tag{17.130}$$

Note that $\Phi \in \mathrm{Fun}\left(\mathbb{A}^1_{\mathbb{A}}\right)$, and in fact we have that:

$$L\left(\chi\right) = \int \chi \Theta\left(\Phi\right) = \langle \chi, \Theta \rangle \ . \tag{17.131}$$

In the function field setting we have:

$$\mathrm{Pic}\,(C)\,(k) = F^{\times}\backslash\mathbb{A}^{\times}/\mathcal{A}_{\mathbb{A}}^{\times}\ . \tag{17.132}$$

Then:

$$\Theta_{\Phi}\colon\ \mathrm{Pic} \ni L \mapsto \sum_{\gamma\in F^{\times}} \Phi\,(\gamma\cdot L) = \#H^{0}\,(C, L)\ . \tag{17.133}$$

Geometrically,

$$\begin{tikzcd} \mathrm{Pic}^{\mathbb{A}^1} \arrow[r, equal] \arrow[d, "\pi"] & \text{line bundles and section of associated } \mathbb{A}^1\text{-bundle} \\ \mathrm{Pic} \arrow[r, equal] & \text{line bundles} \end{tikzcd} \tag{17.134}$$

and

$$\Theta = \pi_{*}1\ . \tag{17.135}$$

Now we can generalize this very easily to nonabelian groups.

17.6 General version

The starting data is a group acting on a variety: $G \circlearrowright X$.

Example 119. This reduces to the specific settings we were considering before by taking $G \circlearrowright G/H$, $\mathrm{PSL}_2 \circlearrowright \mathrm{PSL}_2/T$, and $\mathrm{GL}_1 \circlearrowright \mathbb{A}^1$.

Now we have:

$$\prod{}' X\,(K)\,/G\,(\mathcal{O}) = \mathrm{Bun}_G^X = \mathrm{Map}\,(C, X/G)\ , \tag{17.136}$$

which consists of G-bundles with a section of X-bundle. We have a projection map:

$$\begin{tikzcd} \prod{}' X\,(K)\,/G\,(\mathcal{O}) \arrow[r, equal] \arrow[d, "\pi"] & \mathrm{Bun}_G^X \arrow[r, equal] \arrow[d, "\pi"] & \mathrm{Map}\,(C, X/G) \arrow[d, "\pi"] \\ G\,(F)\,\backslash G\,(\mathbb{A})\,/G\,(\mathcal{O}_{\mathbb{A}}) \arrow[r, equal] & \mathrm{Bun}_G \arrow[r, equal] & \mathrm{Map}\,(C, \mathrm{pt}/G) \end{tikzcd}\ . \tag{17.137}$$

Example 120. If $X = G/H$, then $\mathrm{Bun}_G^X = \mathrm{Bun}_H$.

So now we have a natural map. The X-period of an automorphic form is just:

$$\mathcal{P}_X\,(\varphi) = \langle\Theta_X, \varphi\rangle_{\mathrm{Bun}_G} \tag{17.138}$$

where

$$\Theta_X = \pi_{*}1\ . \tag{17.139}$$

Or by adjunction we can write:

$$\mathcal{P}_X\,(\varphi) = \langle\Theta_X, \varphi\rangle_{\mathrm{Bun}_G} = \int_{\mathrm{Bun}_G^{\times}} \pi^{*}\varphi\ . \tag{17.140}$$

We are ignoring the issue of convergence of these integrals and sums. We need to know some analytical properties, which will come from strong finiteness assumptions on X. This is where the theory of spherical varieties comes in.

The punchline of the upcoming work of Ben-Zvi-Sakellaridis-Venkatesh is as follows. Let cite $G \circlearrowright X$ be spherical. This defines a boundary condition for the automorphic theory $\mathcal{A}_G$. The point being that the theory of periods is the three-dimensional part of the theory of boundary data. But, from the physics, we're supposed to use Hamiltonian G-spaces, not just G-spaces. From the data of $G \circlearrowright X$, we get a Hamiltonian G-space $M = T^*X \circlearrowleft G$. The claim is then that the theory of periods is microlocal. I.e. it is really about "automorphic quantization" of M, rather than X. Already in Tate's thesis, the key ingredient was that the functional equation came out of doing the Fourier transform on $\mathbb{A}^1$. But the Fourier transform is not a symmetry of $\mathbb{A}^1$, but rather $T^*\mathbb{A}^1$ given by 90-degree rotation. The fact that the L-function didn't depend on $\mathbb{A}^1$, but rather $T^*\mathbb{A}^1$, implies it is invariant under symmetries of $T^*\mathbb{A}^1$, so in particular the Fourier transform, which yields the functional equation.

There are in fact many examples of periods associated to $G \circlearrowright M$ which are *not* cotangent bundles. For example, the Whittaker period a_1 naturally comes from a twisted cotangent bundle. There is also something called the *theta correspondence* which comes from studying symplectic spaces which are not cotangent bundles. These generally come from general symplectic representations $G \circlearrowright V$, which might not be $W \oplus W^*$. So this is an important source of periods.

Remark 124*.* As mentioned above, the finiteness we need to make all of these integrals converge comes from X being spherical. What is the analogous condition on M? What we insist on is that $\mathcal{O}(M)^G$ is Poisson commutative. These are called multiplicity 1 Hamiltonian spaces. *Hyperspherical* is this plus a few extra conditions.

The name is motivated by hypertoric varieties. This name comes from them being hyperkähler.

The idea of BZ-Sakellaridis-Venkatesh is to organize the theory of periods using the theory of boundary conditions. Then the periods on the automorphic side will have a dual on the $\mathcal{B}$-side. This is a bit nebulous, but from this we can extract a Hamiltonian variety for the dual group using reconstruction:

$$G^\vee \circlearrowright M^\vee \ . \tag{17.141}$$

We said that $G^\times \circlearrowright M^\vee$ is related to L-functions, and $G \circlearrowright M$ is related to periods. So this establishes a bridge between periods and L-functions.

Conjecture 3 (Global numerical/arithmetic)**.** *The norm of the X-period of an automorphic form is given by the corresponding L-function*

$$\left|\mathcal{P}_X(\varphi)\right|^2 = L\left(\rho, M^\vee\right) \ , \tag{17.142}$$

where φ corresponds to $\rho\colon \mathrm{Gal} \to G^\vee$.

Remark 125*.* The $|\cdot|^2$ comes from the fact that this is secretly about $M = T^*X$ rather than X.

Can now run around and make a lot of conjectures.

Conjecture 4 (Global geometric)**.** *Recall we have*

$$\mathcal{L}_{M^\vee} \in \mathbf{QC}^! \left(\mathrm{Loc}_{G^\vee}\right) = \mathcal{B}_{G^\vee}\left(C\right) \tag{17.143}$$

given by the pushforward along:

$$\mathrm{Loc}_{G^\vee}^{X^\vee} \to \mathrm{Loc}_{G^\vee} \ . \tag{17.144}$$

We can do the same thing on the $\mathcal{A}$-side. We have

$$\mathcal{A}_G\left(C\right) = \mathbf{Shv}\left(\mathrm{Bun}_G\left(C\right)\right) \ . \tag{17.145}$$

Inside of here we get the period sheaf*, which is:*

$$\mathcal{P}_X = \pi_* \underline{k} \tag{17.146}$$

where

$$\pi\colon \ \mathrm{Bun}_G^X \to \mathrm{Bun}_G \ . \tag{17.147}$$

What does $M^\vee$ mean from the point of view of automorphic forms? On the automorphic side, Sakellaridis-Venkatesh wrote [SV17], where they study the relationship between the global theory of periods and local questions in representation theory (harmonic analysis). What does one do with a G-variety over a local field? First consider the local arithmetic setting. So we can consider the space of functions on X over K, $X\left(K\right)$. The group $G\left(K\right)$ acts on this, so we can decompose, say, $L^2\left(X\left(K\right)\right)$ as a representation of $G\left(K\right)$. This is the kind of object that appears in the local Langlands correspondence.

A less ambitious thing to do is to look at the notion of *spherical functions*. This means we consider functions on $X\left(K\right)$ invariant under the positive half of $G\left(\mathcal{O}\right)$. This is now a representation of the Hecke algebra:

$$X\left(K\right)/X\left(\mathcal{O}\right) \circlearrowleft k\left[G\left(\mathcal{O}\right)\backslash G\left(K\right)/G\left(\mathcal{O}\right)\right] \simeq \left|\mathbf{Rep}\left(G^\vee\right)\right| \tag{17.148}$$

X being a spherical variety is equivalent to $G\left(\mathcal{O}\right) \circlearrowright X\left(K\right)$ having discrete orbits. E.g. the affine Grassmannian.

So now we're considering the Hecke algebra acting on spherical functions:

$$\mathcal{H} \simeq \left|\mathbf{Rep}\left(G^\vee\right)\right| \circlearrowright L^2\left(X\left(K\right)/G\left(\mathcal{O}\right)\right) \ . \tag{17.149}$$

$\Phi = 1_{X(\mathcal{O})}$ is a spherical function. The Θ operator is:

$$\Theta\colon \ \bigotimes{}' L^2\left(X\left(K\right)/G\left(\mathcal{O}\right)\right) \to L^2\left(\mathrm{Bun}_G\right) \ , \tag{17.150}$$

and

$$\Theta\left(\Phi\right) = \mathcal{P}_X \ . \tag{17.151}$$

So we want to understand the Plancherel measure for $L^2\left(X\left(K\right)\right)$. This is a measure on the set of unitary representations of $G\left(K\right)$. By the local Langlands correspondence, this is matched with local Langlands parameters. Instead we can consider spherical functions and try to decompose under the $\mathcal{H}$-action. Recall that:

$$\mathcal{H} \simeq \left|\mathbf{Rep}\left(G^\vee\right)\right| \simeq \mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) \ . \tag{17.152}$$

We won't even try to decompose the entire space. Just this basic function Φ.

So we're asking a harmonic analysis question, and we will answer it geometrically. Given a state Φ in a Hilbert space, what we get out of it is a spectral measure on the spectrum of the algebra of operators

In our case, we encode all of the different products of the orbit of the $\mathcal{H}$-action containing Φ, i.e.

$$\langle H_v \cdot \Phi, H_w \cdot \Phi \rangle \tag{17.153}$$

for representations $H_v, H_w \in \mathcal{H} \simeq |\mathbf{Rep}\left(G^\vee\right)|$. This determines a measure on the spectrum of the Hecke algebra, which is the Plancherel measure. This is telling us how Φ decomposes, since $\mathcal{H} \simeq |\mathbf{Rep}\left(G^\vee\right)|$.

In fact, the same representation controlled the global L-function appearing in the theory of the X-period. "The period (for X) squared" looks like the Euler product of the Plancherel measure for X.

How can we think about this geometrically? This will explain where the dual boundary condition comes from. The punchline is that $M^\vee \circlearrowleft G^\vee$ is a categorified form of the Plancherel measure for $X(K)/G(\mathcal{O})$.

The geometric version of this same question is as follows. We replace functions by sheaves:

$$\mathbf{Shv}\left(X(K)/G(\mathcal{O})\right) \ . \tag{17.154}$$

Example 121. If $X = G$ and $G \times G \circlearrowright G$, then this recovers the affine Grassmannian $G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$.

The spherical condition mimics the discrete aspect of this example. Inside of this we have a basic sheaf:

$$\Phi = \underline{k}_{X(\mathcal{O})} \in \mathbf{Shv}\left(X(K)/G(\mathcal{O})\right) \ . \tag{17.155}$$

I.e. we replace the characteristic function with the constant sheaf.

What about the Plancherel measure? We can do the same thing geometrically. Replace the pairing with Hom, so consider:

$$\mathrm{Hom}_{\mathbf{Shv}}\left(H_V * \Phi, H_W * \Phi\right) \ . \tag{17.156}$$

The trace of Frobenius on this space will give us the classical harmonic analysis. This is all captured by a single object, which is the analogue of the Plancherel measure: it is captured by

$$\mathcal{A}_X = \underline{\mathrm{End}}_{\mathcal{H}}(\Phi) \ , \tag{17.157}$$

where this is an internal Hom, i.e. there is a functor:

$$\mathcal{H} \to \mathbf{Shv}\left(X(K)/G(\mathcal{O})\right) \tag{17.158}$$

given by acting on Φ. There is an adjoint given by $\mathrm{Hom}(\Phi, -)$. I.e. $\mathcal{A}$ is an algebra object in

$$\mathcal{H} = \mathbf{Shv}\left(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})\right) = \mathcal{A}_G\left(S^2\right) \ . \tag{17.159}$$

It isn't just any old algebra object, but in fact it is an associative factorization algebra. I.e. it is some version of an $\mathbb{E}_3$-algebra object in $\mathcal{A}_G\left(S^2\right)$. But this is exactly the data we started from: starting with a boundary condition, among other things, we found an algebra in line operators. When we think carefully about what we get in the local situation, $\mathcal{A}_X$ is exactly the algebra of local operators on the boundary conditions. The dual Hamiltonian space is then:

$$M^\vee = \mathrm{Spec}_{/(\mathfrak{g}^{\vee *}/G^\vee)}(\mathcal{A}_X) \in \mathcal{A}_G\left(S^2\right) \ , \tag{17.160}$$

and recall that:

$$\mathcal{A}_G\left(S^2\right) = \mathcal{B}_{G^\vee}\left(S^2\right) = \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \ , \tag{17.161}$$

i.e. $M^\vee$ is a geometric version of the support of the Plancherel measure. The support of the moment map is the support of the Plancherel measure.

Sakellaridis, Sakellaridis-Venkatesh, and Sakellaridis-Wang calculate a lot about the Plancherel measure. So we can make a precise description of $M^\vee$. Then one can state some conjectures. The most fundamental piece of this is the following. cite[3]

Conjecture 5 (Local geometric)**.** *The line operators on the boundary match:*

$$\mathbf{Shv}\left(X\left(K\right)/G\left(\mathcal{O}\right)\right) \simeq \mathbf{QC}\left(M^\vee/G^\vee\right) \tag{17.162}$$

compatibly with the actions of:

$$\mathcal{H} \simeq \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \ . \tag{17.163}$$

Note: the latter isomorphism is Satake.

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