Mirror Symmetry

Lectures: Professor Bernd Siebert

Fall 2019; Notes by: Jackson Van Dyke; All errors introduced are my own.

Contents

Overview of mirror symmetry

The course is topics in algebraic geometry. We will be doing some sort of mirror symmetry. We will start with some historical overview. Lecture 1, August 29, 2019

1. Enumerative mirror symmetry

Let X be a CY manifold. In particular we will focus on CY 3-folds. This means $K_X = \det T_X^* \simeq \mathcal{O}_X$ is trivial as a holomorphic line bundle. Typically this means we want $b_1 = 0$ and irreducible.

Example 0.1 (Quartic in $\mathbb{P}^4$). Take $f \in \mathbb{C}[x_0, \ldots, x_4]$ homogeneous of degree 5. If it is sufficiently general, the zero locus is smooth inside $\mathbb{P}^4$ and is an example of a CY three-fold.

We have

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{\mathbb{P}^4}|_X \to \mathcal{O}_X \to 0 \tag{0.1}$$

where $\mathcal{I} = (f) \subset \mathcal{O}_{\mathbb{P}^4}$. We also have that $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{O}_X$ is an invertible sheaf so this first map sends $f \mapsto df$. This implies

$$K_{\mathbb{P}^4}|_X = \det \mathcal{O}_{\mathbb{P}^4}|_X = \mathcal{I}/\mathcal{I}^2 \otimes K_X \ , \tag{0.2}$$

and

$$K_{\mathbb{P}^4} = \mathcal{O}_{\mathbb{P}^4}(-5) \ . \tag{0.3}$$

Then $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^4}$ which has a section with poles of order 5. The point is we can make f into a five by dividing by x_0^5, so we have that, as an abstract sheaf,

$$\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^4}(-5) \ . \tag{0.4}$$

Then we have that

$$\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-5) \tag{0.5}$$

and we can just take the tensor product to get

$$K_{\mathbb{P}^4}|_X \simeq \mathcal{O}_X(-5) \tag{0.6}$$

so $K_X \simeq \mathcal{O}_X$ must be trivial.

Now we want to produce a string theory out of this. This is a very delicate process. There are things called $IIA(X)$ and $IIB(X)$ theories. These are the ones relevant in mirror symmetry. These come from the super-symmetric σ-models with target some 10-dimensional space[0.1] $\mathbb{R}^{1,3} \times X$. These are the so-called super conformal field theories $SCFT_A(X)$ and $SCFT_B(X)$. These are different theories which produce observables, e.g. the Hodge number of X can be computed from these theories. In particular we can compute $h_{1,1}(X)$ and $h_{2,1}(X)$ which correspond to some physical variables. On the B-side we make the same computation but get $h_{2,1}(X)$ and $h_{1,1}(X)$. Then we postulate that there is some other X'

[0.1] To have an anomaly free theory.

where these are not flipped. In particular the observation is, for very specific X, we can find a CY Y with

$$SCFT_A(X) = SCFT_B(Y) \qquad SCFT_B(X) = SCFT_A(Y) \ . \tag{0.7}$$

Somehow then the idea is that if this is really a model for string theory, we should really be swapping

$$IIA(X) \text{ ``}=\text{''} \ IIB(X) \qquad IIB(X) \text{ ``}=\text{''} \ IIA(Y) \ . \tag{0.8}$$

But this might be a bit much to ask.

1.1. Topological twists. There is something called a topologically twisted σ-model introduced by Witten in 1988. This produces a completely different theory. We get two theories, one called $A(X)$, and one $B(X)$.

WARNING 0.1. As it turns out, $A(X)$ ends up computing things in certain limits of the $IIB(X)$ theory.

REMARK 0.1. A priori these are unrelated to $SCFT_A(X)$ and $SCFT_B(X)$.

As it turns out, if we have $SCFT_A(X) = SCFT_B(Y)$, then we have

$$A(X) = B(Y) \qquad B(X) = A(Y) \ . \tag{0.9}$$

$A(X)$ and $B(X)$ compute certain limits, called Yukawa-couplings, for $SCFT_B(Y)$ and $SCFT_A(X)$.

Note that by this twisting procedure $A(X)$ sees (X, ω) (where ω is the Kähler form) only as a symplectic manifold, and $B(X)$ depends only on the complex manifold (X, I).

1.2. Useful calculations. The reason people really got excited about mirror symmetry is that it helps us make calculations we couldn't make before.

In [**4**] the Yukawa couplings for the quintic and the mirror quintic were computed. In particular they computed F_B of the mirror quintic Y_t, and claimed this is in fact equal to F_A of the quintic. Geometrically F_A has to do with counts of (genus 0) holomorphic curves. F_B has to do with period integrals

$$\int_\alpha \Omega_{Y_t} = F_B(t) \tag{0.10}$$

for $\alpha \in H_3(Y_t)$. So they predicted some counts, then someone computed it directly and they agreed.

WARNING 0.2. This Ω_{Y_t} is only defined up to scale so really the case is that

$$F_B(t) = \exp\left(\int_{\alpha_1} \Omega_{Y_t} / \int_{\alpha_0} \Omega_{Y_t}\right) \tag{0.11}$$

for $\alpha_i \in H_3(Y_t)$.

Then Morrison/Deligne in 1992 described $F_B(Y)$ in terms of Hodge theory/more parameters for CY moduli. This is when Gromov-Witten theory entered the scene in 1993 to make $F_A(X)$ precise. So at least we had a mathematical statement.

1.3. Homological mirror symmetry. In 1994 Kontsevich gave his legendary ICM talk. This is where homological mirror symmetry took off. He said that as mathematicians we don't really know SCFTs. But what should be true is really:

$$D\,\mathrm{Fuk}\,(X) = D^b\,(\mathcal{O}_Y)\ . \tag{0.12}$$

This is a formulation, not an explanation.

Professor Siebert would like to convince us of a procedure to construct mirror pairs.

1.4. Proving numerical mirror symmetry. In 1996 Givental gave a proof that in the case of hypersurfaces F_A really is F_B of the mirror. This was somehow a computation showing that the sides do in fact agree. This is not very satisfying to Professor Siebert. In 1997 Lian, Liu, and Yan proved it more generally.

1.5. Proving HMS. In 2003 Paul Seidel proved HMS for the quartic in $\mathbb{P}^3$. Essentially he shows that both sides have enough rigidity to do a very minimal computation. This is also not very satisfying to Professor Siebert. It was then proved in 2011 by N. Sheridan for all CY hypersurfaces.

1.6. Modern state. There are many other manifestations of mirror symmetry. As it turns out even geometric Langlands can be viewed as some form of mirror symmetry.

As for HMS, some symplectic people are trying to prove this for so-called SYZ fibered symplectic manifolds with a rigid space as the mirror.

Then there are intrinsic constructions, things which Professor Siebert has worked on (with Mark Gross) with many applications. The idea is to use mirror symmetry as a tool in mathematics rather than just a phenomenon in physics. The point is one has to find a way of producing mirrors.

This entire story is genus 0, what physicists would call tree-level. There is also a higher genus case. From the representation theory side this has something to do with quantum groups. This is called second quantized mirror symmetry.[0.2] There is an entire field called topological recursion related to this.

1.7. Plan for the class.

(1) part of the COGP computation (periods)
(2) Gromov-Witten theory, virtual fundamental class/moduli stacks
(3) toric degenerations and mirror constructions[0.3]
(4) One strategy for proving HMS is to compute homogeneous coordinate rings of both sides. Polishchuk has shown that this ring determines $D^b\,(\mathcal{O}_Y)$. It would be nice to make the analogous symplectic calculation because this would be a very sneaky proof of HMS.
(5) Higher genus: Donaldson-Thomas invariants play some sort of unclear role in MS because they will have something to do with the higher genus story. One can make these computations using "crystal melting". This is some kind of statistical mechanics.

[0.2] This term comes from QFT.

[0.3] This will include some introduction to toric geometry.

CHAPTER 1

Mirror symmetry for the quintic

1. The quintic threefold, its mirror, and COGP

Take some quintic CY in $\mathbb{P}^4$, i.e. $V(f) \subset \mathbb{P}^4$ for some homogeneous degree 5 f. First let's do a dimension count for homogeneous polynomials in $x_0, \ldots, x_4$ of degree 5. This is just drawing with replacement, so we have

$$x_0^2 x_2 \quad \leftrightarrow \quad || \cdot \cdot | \tag{1.1}$$

$$x_0 x_1^2 x_2 \quad \leftrightarrow \quad | \cdot || \cdot | \tag{1.2}$$

and we get

$$\binom{9}{5} = 126 \tag{1.3}$$

which means

$$\dim_{\mathbb{C}} \left\{ Z(f) \subset \mathbb{P}^4 \right\} = 126 - 1 = 125 \ . \tag{1.4}$$

Now we mod out by $\mathrm{PGL}(5)$, which is of dimension $5 \cdot 5 - 1 = 24$. So we get

$$\dim \underbrace{\mathcal{M}_5}_{\text{moduli space of quintics}} = 101 \ . \tag{1.5}$$

Indeed: for a projective CY manifold X, the moduli space of CY manifolds deformation equivalent to X is a smooth orbifold of complex dimension $h^1(\Theta_X)$, where Θ_X is the holomorphic tangent bundle. We will compute this number as an exercise next time. For $V(5) \subset \mathbb{P}^4$ this is 101.

Lecture 2; September 3, 2019

2. Quintic 3-folds

We will be looking at quintic 3-folds $V(5) \subset \mathbb{P}^4$ given by $Z(f)$ for some homogeneous degree 5 polynomial.

Theorem 1.1. *For X a projective CY manifold then the moduli space of CY deformation-equivalent to X is a smooth space of dimension $h^1(\Theta_X)$.*

By an elementary argument we saw that $\dim \mathcal{M}_{V(5)} = 101$.

2.1. Computation of $H^1(\Theta_X)$. We will compute in the case $X = V(5) \subset \mathbb{P}^4$. We start with the Euler sequence. We have $\mathbb{P}^4 = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_4]$ and

$$x_i \partial_{x_i} = x_i \frac{\partial}{\partial x_i} \tag{1.6}$$

are well-define logarithmic vector fields. Then the sequence is

$$\begin{array}{c} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow 0 \\ 1 \longmapsto \sum e_i \\ e_i \longmapsto x_i \partial_{x_i} \end{array} \tag{1.7}$$

Then we have the conormal sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^4}\big|_X \xrightarrow{\mathrm{res}_x} \Omega^1_X \longrightarrow 0 \tag{1.8}$$

where $\mathcal{I}$ is the ideal sheaf of X. This is dual to

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow N_{X/\mathbb{P}^4} \simeq \mathcal{O}_{\mathbb{P}^4}(5) \longrightarrow 0 \tag{1.9}$$

$\mathcal{I}/\mathcal{I}^2$ can be computed because $\mathcal{I} \simeq \mathcal{O}(-5)$. The restriction sequences are:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{1.10}$$

$$0 \longrightarrow \Theta_{\mathbb{P}^4}(-5) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow \Theta_{\mathbb{P}^4}|_X \longrightarrow 0 \tag{1.11}$$

Now (1.9) gives us

$$\begin{tikzcd} H^0(\Theta_X) \arrow[r] \arrow[d, equal] & H^0(\Theta_{\mathbb{P}^4}|_X) \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \arrow[r] \arrow[d, equal] & H^1(\Theta_X) \arrow[r] & H^1(\Theta_{\mathbb{P}^4}|_X) \arrow[d, equal] \\ 0 & \mathbb{C}^{24} \arrow[u, "\simeq"] & \mathbb{C}^{125} & \mathbb{C}^{101} \arrow[u, "\simeq"] & 0 \end{tikzcd} \tag{1.12}$$

For $H^1(\Theta_{\mathbb{P}^4}|_X)$, (1.11) give us:

$$H^1(\Theta_{\mathbb{P}^4}) \longrightarrow H^1(\Theta_{\mathbb{P}^4}|_X) \longrightarrow H^2(\Theta_{\mathbb{P}^4}(-5)) \quad . \tag{1.13}$$

The Euler sequence gives us:

$$\begin{tikzcd} H^1(\mathcal{O}_{\mathbb{P}^4}(1))^{\oplus 5} \arrow[r] \arrow[d, equal] & H^1(\Theta_{\mathbb{P}^4}) \arrow[r] & H^2(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd} , \tag{1.14}$$

then we can tensor the Euler sequence with $\mathcal{O}(-5)$ to get

$$\begin{tikzcd} H^i(\mathcal{O}_{\mathbb{P}^4}(-4)) \arrow[r] \arrow[d, equal] & H^i(\Theta_{\mathbb{P}^4}(-5)) \arrow[r] \arrow[d, equal] & H^i(\mathcal{O}_{\mathbb{P}^4}) \arrow[d, equal] \\ 0 \arrow[r, Rightarrow] & 0 & 0 \arrow[l, Rightarrow] \end{tikzcd} . \tag{1.15}$$

For $H^0\left(\mathcal{O}_{\mathbb{P}^4}(5)|_X\right)$ we can tensor (1.10) with $\mathcal{O}(5)$ to get

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & H^0\left(\mathcal{O}_{\mathbb{P}^4}\right) \arrow[r] & H^0\left(\mathcal{O}_{\mathbb{P}^4}(5)\right) \arrow[r] \arrow[d, equal] & H^0\left(\mathcal{O}_X(5)\right) \arrow[r] \arrow[d, equal] & H^1\left(\mathcal{O}_{\mathbb{P}^4}\right) \\ \mathbb{C} & \mathbb{C}^{126} \arrow[u, "\simeq"] \arrow[r, Rightarrow] & \mathbb{C}^{125} & 0 & \end{tikzcd} \quad . \tag{1.16}$$

Now for $H^0\left(\Theta_{\mathbb{P}^4}|_X\right)$ we have that (1.11) gives us

$$\begin{tikzcd} H^0\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}\right) \arrow[r] \arrow[d, equal] & H^0\left(\Theta_{\mathbb{P}^4}|_X\right) \arrow[r] & H^1\left(\Theta_{\mathbb{P}^4}(-5)\right) \arrow[d, equal] \\ 0 & \mathbb{C}^{24} \arrow[r, Rightarrow] & \mathbb{C}^{24} \arrow[u, "\simeq"] & 0 \end{tikzcd} \quad . \tag{1.17}$$

Then the result is that:

$$h^1\left(\Theta_X\right) = 101 \ . \tag{1.18}$$

3. Hodge diamond

3.1. Dolbeault cohomology. Let X be a compact Kähler manifold. Recall we have the Dolbeault cohomology $H^{i,j}_{\bar{\partial}}$ is the cohomology of the sequence

$$\mathcal{A}^{i,j} \xrightarrow{\bar{\partial}} \mathcal{A}^{i,j+1} \tag{1.19}$$

where this looks like

$$\sum h_{\mu\nu}\, dz_{\mu_1} \wedge \ldots \wedge dz_{\mu_i} \wedge d\bar{z}_{\nu_1} \wedge \ldots \wedge d\bar{z}_{\nu_j} \tag{1.20}$$

where the $h_{\mu\nu} \in \mathcal{C}^\infty$.

Then we have the following facts:

(1) We have a canonical isomorphism from the Dolbeault cohomology

$$H^{i,j}_{\bar{\partial}} = H^j\left(X, \Omega^i_X\right) = \overline{H^{j,i}_{\bar{\partial}}} \tag{1.21}$$

which implies

$$\dim H^{i,j}_{\bar{\partial}} = h_{ij} = h_{ji} \ . \tag{1.22}$$

(2)

$$H^{n-i,n-j}_{\bar{\partial}} = H^{n-j}\left(X, \Omega^{n-i}_X\right) =^{1.1} H^j\left(X, K_X \otimes \left(\mathcal{O}^{n-i}_X\right)^*\right)^* H^{i,j}_{\bar{\partial}} \tag{1.23}$$

which, in particular, means $h_{ij} = h_{n-i,n-j}$.

(3)

$$H^k(X, \mathbb{C}) = H^k_{dR}(X) = \bigoplus_{i+j=k} H^{i,j}_{\bar{\partial}} \ . \tag{1.24}$$

Recall $b_i = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$. For CY n-folds, $b_1 = 0$ by the definition of CY. This implies $h_{10} = h_{01} = 0$. Moreover

$$H^{n,0} = H^0\left(X, \underbrace{K_X}_{\mathcal{O}_X}\right) \simeq \mathbb{C} \ . \tag{1.25}$$

[1.1] By Serre duality

This implies that

$$h_{n,0} = h_{0,n} = 1 \ . \tag{1.26}$$

We say a CY is irreducible if the universal cover $\tilde{X} \to X$ is not a nontrivial product of CY. This is equivalent to $H^{k,0} = 0$ for $k = 1, \ldots, n-1$.

COUNTEREXAMPLE 1. The product $K3 \times K3$ is not irreducible.

3.2. Hodge diamond. The hodge diamond is the following:

$$\begin{array}{ccccccccc}
 & & & h_{33} & & & & & \leftarrow H^6 \\
 & & h_{23} & & h_{32} & & & & \leftarrow H^5 \\
 & h_{13} & & h_{22} & & h_{31} & & & \vdots \\
h_{01} & & h_{12} & & h_{21} & & h_{30} & & \\
 & h_{02} & & h_{11} & & h_{20} & & & \\
 & & h_{01} & & h_{10} & & & & \\
 & & & h_{00} & & & & &
\end{array} \tag{1.27}$$

The only interesting part is the center:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & {\color{blue}h_{22}} & & 0 & \\
 & & & \| & & & \\
1 & & {\color{red}h_{12}} & = & {\color{red}h_{21}} & & 1 \\
 & & & \| & & & \\
 & 0 & & {\color{blue}h_{11}} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{1.28}$$

since

$$H^1\left(\Theta_X\right) = H^1\left(\Theta_X \otimes \underbrace{K_X}_{\mathcal{O}_X}\right) = H^1\left(\mathcal{O}_X^{n-1}\right) = H^{n-1,1} \ . \tag{1.29}$$

So the question is reduced to making these calculations. In the $V(5)$ case $h_{21} = 101$ as we saw.

3.3. Lefschetz theorem on $(1,1)$-classes. Now we work at the generality of compact Kähler manifolds. The Néron-Severi group $NS(X)$, is the preimage of $H^{1,1}$ under $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$. So somehow morally $NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}$. Now we have the exponential sequence of abelian sheaves:

$$1 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1 \tag{1.30}$$

$$\begin{array}{ccccccccc} H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1\left(X,\mathcal{O}_X^\times\right) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}_C) \\ & & \| & & \| & & & & \\ & & H^{0,1} & & \mathrm{Pic}(X) & & & & \end{array} . \tag{1.31}$$

So $c_1(\mathrm{Pic}(X)) = NS(X)$.

Now we know how to compute $\mathrm{Pic}(X)$ for CY n-folds. In particular, as long as $n \geq 3$, $h_{11} = \mathrm{rank}_{\mathbb{Z}} NS(X)$ and $\mathrm{Pic}(X) \simeq NS(X)$ since $\mathrm{Pic}^0(X) = 0$ in the CY case.

3.4. Hard Lefschetz theorem. Let $X \subset \mathbb{P}^n$ be a Kähler manifold. This tells us that

$$H^k(\mathbb{P}^n,\mathbb{Z}) \to H^k(X,\mathbb{Z}) \tag{1.32}$$

is an isomorphism for $k < n-1 = \dim X$ and surjective for $k = n-1$.

Now for a CY 3-fold we have $0 = H^1(\mathbb{P}^n,\mathbb{Z}) = H^1(X,\mathbb{Z})$ and $H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z} \twoheadrightarrow H^2(X,\mathbb{Z})$ which means $NS(X) \simeq \mathbb{Z}$. This is generated by the hyperplane class $c_1(\mathcal{O}(1))$ and then restricting to X gives the ample line bundle on X and this generates the group.

So today we learned that the interesting part of the diamond in the quintic case looks like

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array} \tag{1.33}$$

and then as it turns out, the mirror quintic will look like

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array} \tag{1.34}$$

so we have a huge Picard group.

Lecture 3;
September 5, 2019

4. Lefschetz hyperplane theorem

This is an addendum to last lecture. Let $X \subset \mathbb{P}^n$ be a hypersurface. Then

$$H^k\left(\mathbb{P}^{n+1},\mathbb{Z}\right) \xrightarrow{\mathrm{res}} H^k(X,\mathbb{Z}) \tag{1.35}$$

is an isomorphism for $k < n-1$ and surjective for $k = n-1$.

5. The mirror quintic

We should have mirrored Hodge numbers. Recall the interesting part of the Hodge diamond for the original quintic is

$$\begin{array}{ccc} & 1 & \\ 101 & & 101 \\ & 1 & \end{array} \tag{1.36}$$

and then the mirror diamond should be:

$$\begin{array}{ccc} & 101 & \\ 1 & & 1 \\ & 101 & \end{array} . \tag{1.37}$$

So $h_{21}(Y) = 1$, i.e. it has a one-dimensional moduli space, but $h_{11}(Y) = 101$ so it has a big $\mathrm{Pic}(Y)$.

5.1. Construction. This is physically motivated by the orbifold of the "minimal CFT" related to the Fermat quintic, i.e. the one given by $x_0^5 + \ldots + x_4^5 = 0$. $(\mathbb{Z}/5)^5$ acts diagonally on $\mathbb{P}^4$. This gives an effective action of $(\mathbb{Z}/5)^4 = (\mathbb{Z}/5)^5 / (\mathbb{Z}/5)$ since one copy of $\mathbb{Z}/5$ acts trivially. Now we take the finite quotient

$$\bar{Y} = X/(\mathbb{Z}/5)^4 \ . \tag{1.38}$$

The action is not free, so this is not a manifold, i.e. it has some orbifold singularities. (X is smooth by Jacobi criterion.) The stabilizer $G_Y \subset (\mathbb{Z}/5)^4$ is nontrivial. There are two cases:

- $x_i = x_j = 0$ ($i \neq j$), in which case $G_Y \simeq \mathbb{Z}/5$. This gives quintic curves $\tilde{C}_{ij} = Z(x_i, x_j) \subset X$. The local action is given by $\zeta(z_1, z_2, z_3) = \left(\zeta z_1, \zeta^{-1} z_2, z_3\right)$. This gives rise to the singularity given by $uv = w^5$, the A_4 singularity in $\mathbb{C}^3$ (with coordinates u, v, w).
- $x_i = x_j = x_k = 0$ (i, j, k pairwise disjoint), where we get $G_Y \simeq (\mathbb{Z}/5)^2$. This gives us

$$\tilde{P}_{i,j,k} \to P_{ijk} \in \bar{Y} \ . \tag{1.39}$$

Now the local action looks like

$$(\zeta, \xi) \cdot (z_1, z_2, z_3) = \left(\zeta \xi z_1, \zeta^{-1} z_2, \xi^{-1} z_3\right) \ . \tag{1.40}$$

Remark 1.1. Inside $\bar{Y}$ we have

$$C_{01} = Z\left(x_0, x_1, x_2^5 + x_3^5 + x_4^5\right) / (\mathbb{Z}/5)^3 \simeq Z(u + v + w) \simeq \mathbb{P}^1 \subset \mathbb{P}^2 \ . \tag{1.41}$$

Any C_{ij} looks like a $\mathbb{P}^1$.

We want to blow these singularities up locally, but this is delicate if we want to stay in the world of projective algebraic varieties, i.e. we might just not have an ample line bundle after blowing up. So we have to prove something extra.

Proposition 1.2. *There exists a projective resolution $Y \to \bar{Y}$.*

This is done most efficiently by toric methods, but can be done by hand.

Let's count the independent exceptional divisors in T. We have 4 over each C_{ij} and 6 over each P_{ijk}. So we have 40 from the C_{ij} and 60 from the P_{ijk} and we have 100 in total. Together with the hyperplane class they span the H^2.

Proposition 1.3. $h_{11}(Y) = 101$, $h_{21}(Y) = 1$.

The proof was done directly by S.S. Roan and done by by toric methods by Batyrev.

5.2. Mirror families. This mod G construction generalizes to what is called the "Dwork family". In particular we have $X_\psi = V(f_\psi)$ where

$$f_\psi = x_0^5 + \ldots + x_4^5 - 5\psi x_0 x_1 \ldots x_4 \tag{1.42}$$

and ψ is a complex parameter. So we have

$$\begin{array}{ccc} (\mathbb{Z}/5)^4 = \{(\zeta_0,\ldots,\zeta_4) \mid \prod \zeta_i = 1\} & & \subset (\mathbb{Z}/5)^5 \\ \downarrow & & \downarrow \\ G = (\mathbb{Z}/5)^3 & & (\mathbb{Z}/5)^4 \end{array} \tag{1.43}$$

and

$$\begin{array}{ccc} X_\psi = Z(f_\psi) & & \\ \Big\downarrow{\scriptstyle /(\mathbb{Z}/5)^3} & & \\ \bar{Y}_\psi & \longleftarrow & Y_\psi \end{array} . \tag{1.44}$$

Note that $Y\psi \simeq Y_{\zeta\psi}$ for $\zeta^5 = 1$. Then we have

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{/\mathbb{Z}/5} & \mathcal{Y} \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{P}^1_\psi & \xrightarrow{/\mathbb{Z}/5} & \mathbb{P}^1_z \end{array} \tag{1.45}$$

where $z := (5\psi)^{-5}$. This is a family of CY 3-folds which are smooth for $z \neq 0, \infty$.

The special fibers are as follows.

- $z = 0$: $x_0 \ldots x_4 = 0$ implies
$$Y_{z=0} \simeq \bigcup_5 \mathbb{P}^3 \subset \mathbb{P}^4 \tag{1.46}$$
is a union of coordinate hyperplanes.
- $z = 5^{-5}$, i.e. $\psi = 1$. This corresponds to one 3-fold A_1 singularity. X_1 has 125 three dimensional A_1-singularities. Which locally look like $x^2 + y^2 + z^2 + w^2 = 0$. They all lie in one $(\mathbb{Z}/5)^3$-orbit. $Y_{5^{-1}}$ has one three dimensional A_1 singularity sometimes called the "conifold".
- $z = \infty$, i.e. $\psi = 0$: this is the Fermat quintic. This has an additional $\mathbb{Z}/5$ symmetry because we drop the condition that the product of the x_i has to be 1. So this is really an orbifold point.

Now, at least from a physics point of view we are done.[1.2]

[1.2] Professor Siebert says that maybe mathematicians would be better off like this: not worrying so much and just seeing what life brings.

6. Yukawa couplings

6.1. A-model. The A-model (symplectic) will deal with the quintic. So we have $H^2(X,\mathbb{Z}) = \mathbb{Z}\cdot h$ for $h = PD\,(\text{hyperplane})$. Then the Yukawa coupling is

$$\langle h,h,h\rangle_A = \sum_{d\in\mathbb{N}} N_d d^3 q^d \ . \tag{1.47}$$

At this point (historically) it was not clear what the N_d actually were because Gromov-Witten theory was sort of being developed in parallel. Nowadays we know that the N_d are Gromov-Witten (GW) counts of rational curves ($g=0$) of degree d. If we write n_d for the primitive counts, then

$$5 + \sum_{d>0} d^3 n_d \underbrace{\frac{q^d}{1-q^d}}_{\text{multiple cover of }\deg d\text{ curves}} \in \mathbb{Q}[\![q]\!] \ . \tag{1.48}$$

6.2. B-model. Now we consider the mirror quintic Y_z, $z=(5\psi)^{-5}$. Now the Yukawa coupling is given by:

$$\langle\partial_z,\partial_z,\partial_z\rangle_B \coloneqq \int_{Y_z}\Omega^\nu(z)\wedge\partial_z^3\Omega^\nu(z) \tag{1.49}$$

where Ω^ν is a "normalized" holomorphic volume form:

$$\int_{\beta_0}\Omega^\nu = \text{constant} \tag{1.50}$$

where $\beta_0\in H_2(Y,\mathbb{Z})$.

We need some kind of mirror to the vector field h on the moduli space of symplectic structures. What we really want is actually the exponentiated thing $e^{2\pi i h}$. So as it turns out on this side of mirror symmetry this looks like $\partial/\partial w$ which corresponds to a vector field on the complex moduli space of Y.

It turns out

$$w = \int_{\beta_1}\Omega^\nu(z) \tag{1.51}$$

where $\beta_1\in H_3(Y,\mathbb{Z})$.

6.3. Mirror symmetry. Now the actual statement of mirror symmetry is that

$$\langle h,h,h\rangle_A = \langle\partial_z,\partial_z,\partial_z\rangle_B \tag{1.52}$$

where $q = e^{2\pi i w(z)}$ $\left(w = c\cdot z + \mathcal{O}\left(z^2\right)\right)$.

So now we have to:

(1) write down the holomorphic 3 form (not too bad)
(2) do the normalization period integral (not too bad)
(3) computing this second integral (more bad)
(4) computing the Yukawa coupling.

6.4. Computation of the periods. There is an account of this in lecture notes by Mark Gross (Nordfjordeid). Recall we have:

$$\begin{array}{ccc} & & Y_\psi \\ & & \downarrow \\ X_\psi & \xrightarrow{/G} & \bar{Y}_\psi \end{array} \quad . \tag{1.53}$$

We know $H_3(Y_\psi, \mathbb{Z}) \simeq \mathbb{Z}^4$. Near $\psi = \infty$ (large complex structure limit) we have a vanishing cycle.

This looks like Professor Siebert's favorite picture of a degeneration. Consider $zw = t$. At $t = 0$ this looks like two disks meeting at a point. For $t \neq 0$ this looks like a cylinder. But now if we do a Dehn twist, we see that there is an S^1 which gets collapsed in this degeneration. A similar story holds in higher dimension.

In particular, our vanishing cycle looks like $\beta_0 = T^3$. Locally

$$u_1 \dots u_4 = z, \beta_0 = \left\{ |u_1| = \dots = |u_4| = |z|^{1/4}, \operatorname{Arg} u_1 \dots u_4 = 0 \right\} \tag{1.54}$$

where the u_i are holomorphic coordinates.

If we lift to X_ψ we get an explicit three-torus

$$T = \left\{ |x_0| = |x_1| = |x_2| = \delta \ll 1, x_3 = x_3(x_0, x_1, x_2) \text{ soln of } f_\psi(x_1, x_2, x_3, 1) = 0, \exists! \xrightarrow{z \to 0} 0 \right\} \tag{1.55}$$

Recall we have

Lecture 4; September 10, 2019

$$\begin{array}{ccc} & & Y_\psi \\ & & \big\downarrow \\ X_\psi & \xrightarrow{\quad /G \quad} & \bar{Y}_\psi \end{array} \tag{1.56}$$

where $G = (\mathbb{Z}/5\mathbb{Z})^3$. Recall last time we discussed:

1. Vanishing cycle T^3, $\beta_0 = \in H_3(Y_\psi, \mathbb{Z})$,
2. Holomorphic 3-form,
3. Normalization,
4. Further periods, and
5. Canonical coordinate/mirror map.

7. Holomorphic 3-form

We will construct the holomorphic 3-form as the residue of a meromorphic/rational 4-form on $\mathbb{P}^4$ with zeros along X_ψ:

$$\Omega(\psi) = 5\psi \operatorname{Res}_{X_\psi} \frac{\tilde{\Omega}}{f_\psi} \in \Gamma\left(X_\psi, \Omega^3_{X_\psi}\right) \tag{1.57}$$

where

$$\tilde{\Omega} = \sum x_i \, dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_4 \quad . \tag{1.58}$$

Locally $x_4 = 1$, $\partial_{x_3} f \neq 0$,

$$\Omega(\psi) = 5\psi \left. \frac{dx_0 \wedge dx_1 \wedge dx_2}{\partial_{x_3} f_\psi} \right|_{X_\psi} \quad . \tag{1.59}$$

8. Normalization

Now we deal with normalization. We have $\varphi_0 := \int_{\beta_0} \Omega(\psi)$, then

$$\tilde{\Omega} = \varphi_0^{-1} \Omega(\psi) \tag{1.60}$$

is normalized with resides

$$2\pi i \int_{\beta_0} \Omega(\psi) = \int_{T^4} 5\psi \, dx_0 \ dx_1 \ dx_2 \ dx_3 \tag{1.61}$$

$$= \int_{T^4} \frac{dx_0 \ \ldots \ dx_3}{x_0 \ldots x_3} \frac{1}{\frac{(1+x_0^5+\ldots+x_3^5)}{5\psi x_0 \ldots x_3} - 1} \tag{1.62}$$

$$= -\sum_{n \geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1 + x_0^5 + \ldots + x_3^5\right)^n}{(5\psi)^n \left(x_0 \ldots x_3\right)^n} \tag{1.63}$$

$$= -\sum_{n \geq 0} \int_{T^4} \frac{dx_0 \ldots dx_3}{x_0 \ldots x_3} \frac{\left(1 + x_0^5 + \ldots + x_3^5\right)^{5n}}{(5\psi)^{5n} \left(x_0 \ldots x_3\right)^{5n}} \tag{1.64}$$

(1.65)

where all summands in both the numerator and denominator must be 5th powers to contribute. So from some combinatorics we have

$$2\pi i \int_{\beta_0} \Omega(\psi) = -(2\pi i)^4 \sum \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} =: \varphi_0(z) \tag{1.66}$$

where $z = (1/5\psi)^5$. The number $(5n)!/(n!)^5$ is the number of terms

$$x_0^{5n} \ldots x_3^{5n} \left(1 + x_0^5 + \ldots + x_3^5\right)^{5n} \ . \tag{1.67}$$

9. Further periods

There is a procedure called Griffith's reduction of pole order. This involves the Picard-Fuchs equation.

Locally $H^3(Y_\psi, \mathbb{C})$ is constant with dimension 4. This gives a trivial holomorphic vector bundle

$$\begin{array}{ccc} & & E = \mathcal{U} \times \mathbb{C}^4 \\ & & \downarrow \\ \mathcal{M}_Y & \hookleftarrow & \mathcal{U} \end{array} \ . \tag{1.68}$$

This has a flat connection ∇^{GM} called the Gauß-Manin connection. Pointwise

$$E_\psi = \bigoplus_{p+q=3} H^{p,q}(X_\psi) \tag{1.69}$$

and we have seen $\Omega(\psi) \in H^{3,0}$. Now consider:

$$\Omega(z) \qquad \partial_z \Omega(z) \qquad \partial_z^2 \Omega(z) \qquad \partial_z^3 \Omega(z) \qquad \partial_z^4 \Omega(z) \tag{1.70}$$

which are related by a fourth order ODE with holomorphic coefficients called the Picard-Fuchs equation.

9.1. Derivation of the equation. Take $X = Z(f_\psi) \subset \mathbb{P}^4$. We can produce more 3-forms from forms with higher-order poles. Consider the long-exact sequence:
(1.71)

$$\begin{tikzcd}
H^4\left(\mathbb{P}^4,\mathbb{C}\right) \arrow[r] & H^4\left(\mathbb{P}^4\setminus X,\mathbb{C}\right) \arrow[r] & H^5\left(\mathbb{P}^4,\mathbb{P}^4\setminus X;\mathbb{C},\right) \arrow[r] \arrow[d, equal, "\text{excision}"'] & H^5\left(\mathbb{P}^4,\mathbb{C}\right) \\
& & H^5\left(\mathcal{U},\mathcal{U}\setminus X;\mathbb{C}\right) \arrow[d, equal] & \\
& & H^5\left(\mathcal{U},\partial\mathcal{U};\mathbb{C}\right) \arrow[d, equal, "LD"'] & \\
& & H_3\left(\mathcal{U}\setminus\partial\mathcal{U};\mathbb{C}\right) \arrow[d, equal] & \\
& & H_3\left(X,\mathbb{C}\right) = H^3\left(X,\mathbb{C}\right) &
\end{tikzcd}$$

where $X \subset \mathcal{U}$ is a tubular neighborhood and we are using the form of Lefschetz duality which states that $H^q(M, \partial M) = H_{n-q}(M \setminus \partial M)$ and in the last step we use Poincare duality. So we start with things of high pole order, this gives us some class in H^3, then in our case we take derivatives, and for certain classes we know they should be zero and this gives us some equations.

9.2. Griffiths' reduction of pole order. If we have

$$\frac{g\tilde{\Omega}}{f^l} \in H^0\left(\mathbb{P}^4, \Omega^4_{\mathbb{P}^4\setminus X}\right) \tag{1.72}$$

then we must have $\deg g = 5l - 5$ ($l = 0$ earlier so we had no g). The exact forms look like

$$d\left(\frac{1}{f^l}\left(\sum_{i<j}(-1)^{i+j}(x_i g_j - x_j g_i)\, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots dx_4\right)\right) \tag{1.73}$$

$$= \left(l\sum g_j \partial_{x_j} f - f \sum \partial_{x_j} g_j\right) \frac{\tilde{\Omega}}{f^{l+1}}\,. \tag{1.74}$$

If $l\sum g_j \partial_{x_j} f \in \mathcal{J}(f) = (\partial_{x_i} f)$ then up to an exact form, it is of lower order since one copy of f cancels. I.e. the first term over f^{l+1} is of order $l+1$, and the second term over f^{l+1} is order l. The upshot is that the numerator $g \in \mathcal{J}(f)$ can reduce l.

So the algorithm is as follows: Compute $\Omega(z)$, $\partial_z\Omega(z)$, $\partial_z^2\Omega(z)$, …, $\partial_z^4\Omega(z) = g\tilde{\Omega}/f_\psi^5$ where $g \in \mathcal{J}(f_\psi)$. Then we express g modulo (1.74) as a linear combination of the $\partial_z^i\Omega(z)$.

Proposition 1.4. *Any period*

$$\varphi = \int_\alpha \Omega(\psi) \tag{1.75}$$

fulfills the ODE

$$\left[\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\right]\varphi(z) = 0 \tag{1.76}$$

where $\theta = z\partial_z$.

REMARK 1.2. This is easy to check for

$$\varphi = \varphi_0 = \sum_{n \geq 0} \frac{(5n)!}{(n!)^5} z^n \ . \tag{1.77}$$

(1.76) is an ODE with a regular singular pole:

$$\theta \varphi (z) = A (z) \varphi (z) \tag{1.78}$$

for $\psi (z) \in \mathbb{C}^s$.

THEOREM 1.5. (1.78) *has a fundamental system of equations of the form*

$$\Phi (z) = S (z) z^R \tag{1.79}$$

with $S (z) \in M (s, \mathcal{O}_0)$, $R \in M (S, \mathbb{C})$, *and*

$$z^R = I + (\log z) R + (\log z)^2 R^2 + \dots \ . \tag{1.80}$$

If the eigenvalues do not differ by integers, we may take $R = A (0)$.

For (1.76)

$$A (0) \simeq \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \tag{1.81}$$

and $S = (\psi_0, \dots, \psi_3)$ where $\psi_i \in \mathcal{O}^4_{\mathbb{C},0}$. This gives us a fundamental system of solutions:

$$\varphi_0 (z) = \psi_0 \text{ (single-valued)} \tag{1.82}$$

$$\varphi_1 (z) = \psi_0 (z) \log z + \psi_1 (z) \tag{1.83}$$

$$\varphi_2 (z) = \psi_0 (z) (\log z)^2 + \psi_1 (z) \log z + \psi_2 (z) \tag{1.84}$$

$$\varphi_3 (z) = \psi_0 (z) (\log z)^4 + \dots + \psi_3 (z) \ . \tag{1.85}$$

This has something to do with monodromy. In particular, the monodromy of $z^{A(0)}$ reflects the monodromy T of $H^3 (Y_z, \mathbb{C})$ about $z = 0$ (or $\psi = \infty$). In fact, one can show that there exists a symplectic basis β_0, β_1, α_1, $\alpha_0 \in H_3 (Y_z, \mathbb{Q})$ with $N = T - I$. Then we have

$$\alpha_0 \mapsto \alpha_1 \mapsto \beta_1 \mapsto \beta_0 \mapsto 0 \tag{1.86}$$

which means

$$\varphi_0 = \int_{\beta_0} \Omega (z) , \varphi_1 = \int_{\beta_1} \Omega (z) , \varphi_2 = \int_{\alpha_1} \Omega (z) , \varphi_3 = \int_{\alpha_0} \Omega (z) \ . \tag{1.87}$$

10. Canonical coordinate/mirror map

Looking at the solution set, we don't have much choice. The solution, when exponentiated should behave like z. Indeed, the canonical coordinate is

$$q = e^{2\pi i w} \tag{1.88}$$

where

$$w = \frac{\int_{\beta_1} \Omega (z)}{\int_{\beta_0} \Omega (z)} = \int_{\beta_1} \tilde{\Omega} (z) \ . \tag{1.89}$$

Then $\varphi_1(z) = \varphi_0(z)\log z + \psi_1(z)$ which is easy to obtain as series solution of (1.76).

$$\psi_1(z) = 5\sum_{n\geq 1}\frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right)z^n \tag{1.90}$$

(up to constant c_2).

Last time we learned how to do these period calculations, get canonical equations, and reduce pole order with the Picard Fucks equation. The remaining topic is the Yukawa coupling. Lecture 5; September 12, 2019

11. Yukawa coupling

We want to compute

$$\langle\partial_z,\partial_z,\partial_z\rangle_B = \int_{Y_z}\tilde{\Omega}(z)\wedge\partial_z^3\tilde{\Omega}(z) \tag{1.91}$$

where $\tilde{\Omega}(z) = \frac{1}{\varphi_0(z)}\Omega(z)$.

We introduce the auxiliary terms

$$W_k = \int_{Y_z}\Omega(z)\wedge\partial_2^k\Omega(z) \tag{1.92}$$

for $k = 0,\ldots,4$. So really we just want W_3. Rewrite the PF equation as

$$\left(\frac{d^4}{dz^4}+\sum_{k=0}^{3}c_k\frac{d^k}{dz^k}\right)\Omega(z) = 0\ . \tag{1.93}$$

This gives us

$$W_4 + \sum_{k=0}^{3}c_kW_k = 0\ . \tag{1.94}$$

11.1. Griffiths-transversality. Now we need to put some important information in called Griffiths-transvr. This has to do with how one defines the variation of Hodge structures. Let $\mathcal{U}$ be open inside the moduli space. Now define a decreasing filtration

$$\mathcal{F} = \underbrace{H^3(Y_z,\mathbb{C})}_{\mathbb{C}^4}\otimes_{\mathbb{Z}}\mathcal{O}_{\mathcal{U}} = \mathcal{F}^0\supset\mathcal{F}^1\supset\mathcal{F}^2\supset\mathcal{F}^3 \tag{1.95}$$

where

$$\mathcal{F}^k = \bigoplus_{q\geq k}R^q\pi_*\Omega^{3-q}\ . \tag{1.96}$$

This is the Hodge filtration. Note that $\Omega\in\Gamma\left(\mathcal{U},\underbrace{R^3\pi_*\Omega^0}_{=\mathcal{F}^3\simeq\mathcal{O}}\right)$. So why do we write it this way instead of using the direct sum decomposition we seem to have? Abstractly we have that

$$\nabla^{GM}\mathcal{F}^k\subseteq\mathcal{F}^{k-1}\otimes\Omega^1_{\mathcal{M}}\ . \tag{1.97}$$

This inclusion comes from the definition/construction of ∇^{GM} and Hodge/Dolbeault theory. Moreover $H^{p,q}\perp H^{p',q'}$ unless $p+p' = 3 = q+q'$ (from $\int_Y\alpha\wedge\beta = 0$).

Together this gives us that $W_0 = W_1 = W_2 = 0$. In particular

$$0 = \frac{d^2 W_2}{dz^2} = \ldots = 2W_3' - W_4 \tag{1.98}$$

and (1.94) tells us that

$$W_3' + \frac{1}{2} c_3 W_3 = 0 \ . \tag{1.99}$$

Now we compute

$$c_3\left(z\right) = \frac{6}{2} - \frac{25^5}{1 - 5^5 z} \tag{1.100}$$

which by separation of variables gives us

$$W_3 = \frac{c_1}{\left(2\pi i\right)^3 z^3 \left(5^5 z - 1\right)} \tag{1.101}$$

for c_1 some integration constant. Finally, reexpress in $q = e^{2\pi i w}$ where $w = \varphi_1\left(z\right) / \varphi_0\left(z\right)$. When we expand we get

$$\langle \partial_w, \partial_w, \partial_w \rangle_B = -c_1 - 575 \frac{c_1}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^3 - \frac{10277490000}{6} \frac{c_1}{c_2^3} q^3 + \ldots \tag{1.102}$$

$$\stackrel{!}{=} 5 + \sum_{d \geq 1} d^3 n_d \frac{q^d}{1 - q^d} \tag{1.103}$$

$$= 5 + n_1 q + \left(8n_2 + n_1\right) q^2 + \left(27 n_3 + n_1\right) q^3 + \ldots \ . \tag{1.104}$$

This predicts that $n_1 = 2875$ (which was classically known) and $n_2 = 6092500$ was also correct (as was shown just a few years earlier than this development by Katz in 1986). Then $n_3 = 317206375$ which originally disagreed with the result, but they found out there was an error in the computation so it also agreed. The first proof of this was in 1996 by Givental.

Lessons learned:[1.3]

(1) The prediction depends on the large complex structure limit because this has something to do with the Kähler cone:

$$\mathcal{K}_X = d\left\{ [\omega] \in H^2_{dR}\left(X\right) \,\middle|\, \omega \text{ Kähler} \right\} \ . \tag{1.105}$$

In particular the monodromy in $H^3\left(Y\right) = \bigoplus_{p=0}^{3} H^{p,3-p}$ corresponds to $\smile [\omega_X]$ on $\bigoplus_{p=0}^{3} H^{p,p}\left(X\right)$. Note however that $\langle\rangle_B$ is defined on all of $\mathcal{M}_B$!

(2) Orbifolding construction of the mirror is special to the quintic. Batyrev/Borisov consider a mirror for complete intersections in toric varieties.

[1.3] Besides that computations are hard...

CHAPTER 2

Stacks

We will learn some things about moduli spaces and stacks to motivate a discussion of Gromov-Witten theory. Hopefully also the logarithmic version and Donaldson-Thomas theory.

The general task is to make sense of curve counting, e.g. the number of genus 0 holomorphic curves in a quintic. Then we have the following problems:

- Classical enumerative algebraic geometry: "general position arguments" needed to make counts work. Transversaily can be very difficult.
- Translate into problem of topology, e.g. intersection theory in Grassmannian $\mathrm{Gr}\,(k,n)$ (Schubert calculus).
- Generally, spaces of curves, e.g. on a given quintic don't have the right dimension.

Example 2.1. Consider the Dwork family $f_\psi = 0$. There are 375 isolated lines ($\simeq \mathbb{P}^1$), e.g. $\left(u, v, -\zeta^k u, -\zeta^k v, 0\right)$ for $u, v \in \mathbb{P}^1$, $\zeta^5 = 1$, $0 \leq k$, $l \leq 4$ and then two irreducible families. In degree > 1 we also always have multiple covers that come in families.

So how do we count in the absence of general deformations? The solution is that there is a virtual formalism. This is exactly what Gromov-Witten theory does. Invariants are constant in families of targets.

1. Moduli spaces

So we are interested in a set of closed points. In particular, this consists of isomorphism classes of certain algebraic geometric objects e.g. varieties, subvarieties. Then we want so somehow give it some extra structure. The best scenario would be to view it as a variety.

Let $T \to \mathcal{M}$ be the structure sheaf. Then the point is that holomorphic maps correspond to families of objects over T.

Example 2.2. Fix some N. The Hilbert scheme $\mathbf{Hilb}\left(\mathbb{P}^N\right)$ will somehow classify closed subschemes. In particular it has the universal property that for any

$$\begin{array}{l} Z \;\subset\; T \times \mathbb{P}^N \\ \big\downarrow{\scriptstyle \text{flat, proper}} \\ T \end{array} \tag{2.1}$$

we have a unique map φ such that

$$\begin{array}{ccc} Z = \mathcal{Z}_T & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ T \times \mathbb{P}^N & \longrightarrow & \mathbf{Hilb}\left(\mathbb{P}^N\right) \times \mathbb{P}^N \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & \mathbf{Hilb}\left(\mathbb{P}^N\right) \end{array} \quad . \tag{2.2}$$

In categorical terms this says that we have a functor

$$\begin{aligned} \mathbf{Sch} & \xrightarrow{\quad F \quad} \mathbf{Set} \\ T & \longmapsto \left\{Z \to T, Z \hookrightarrow T \times \mathbb{P}^N \text{ flat, proper}\right\} \end{aligned} \tag{2.3}$$

which is corepresented by $\mathbf{Hilb}\left(\mathbb{P}^N\right)$. This means we have a natural isomorphism $F \to \hom\left(\cdot, \mathbf{Hilb}\left(\mathbb{P}^N\right)\right)$. In particular we get that $\mathrm{id}_{\mathbf{Hilb}(\mathbb{P}^n)} \in \hom\left(\mathbf{Hilb}\left(\mathbb{P}^N\right), \mathbf{Hilb}\left(\mathbb{P}^N\right)\right)$ corresponds to $F\left(\mathbf{Hilb}\left(\mathbb{P}^N\right)\right)$, e.g. the universal family.

Discouraging observation: For families of curves[2.1] we cannot have (co-)representability.

The reason is that there are families of curves where all of the fibers are isomorphic but they are not globally a product. So if we had such a corepresentation then this can't pull back to the identity.

Lecture 6; September 17, 2019

1.1. The problem of moduli for curves. We will kind of follow [**9**]. [**8**] is where stacks were first really worked out.

We want to repeat the story of **Hilb** for complete curves of genus g.

Recall the notion of families. If S is a scheme (think of this as some sort of parameter space), then a *curve* of genus g over S is a morphism $\pi : C \to S$ such that:

(1) π is proper, flat;
(2) recall the geometric fibers

$$C_S = \operatorname{Spec} K \times_S C \tag{2.4}$$

for K algebraically closed, fit into the diagram

$$\begin{array}{ccc} C_S & \longrightarrow & C \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & S \end{array} \quad . \tag{2.5}$$

Then we ask that:
(a) C_S is reduced, connected, $\dim C_S = 1$,
(b) $h^1\left(C_S, \mathcal{O}_C\right) = g$ (arithmetic genus).

One can also restrict the allowed singularities of these fibers, but this is not necessary for us at the moment. E.g. one might ask for C_S to be non-singular.

[2.1] or any other moduli proble with objects with automorphisms

Now we have a fundamental problem. The functor

$$\begin{aligned} \mathbf{Sch} &\xrightarrow{\quad F_g \quad} \mathbf{Set} \\ S &\longmapsto \{X \to S \mid \text{ non-singular curve of genus } g\} \end{aligned} \tag{2.6}$$

cannot be representable since there are nontrivial isotrivial families of curves, i.e. $\pi : X \to S$ which becomes trivial only after (finite) base change. To see that this is the case, proceed by contradiction. Suppose it is representable by some $\mathcal{M}_g$, then

$$\begin{tikzcd} T_X C_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & C_g \arrow[d] \\ T \arrow[r, "\text{finite}"] \arrow[rr, bend right, "\text{constant}"'] & S \arrow[r, "\varphi"] & \mathcal{M}_g \end{tikzcd} \tag{2.7}$$

but this map being constant implies φ is constant which is a contradiction.

EXAMPLE 2.3. Let C_0 be a curve with $\mathrm{Aut}\,(C_0) \neq \{1\}$, e.g. $C_0 \to \mathbb{P}^1$ a two-to-one hyperelliptic curve, e.g. the projective closure of

$$\big(y^2 - (x-g-1)\,(x-g)\ldots(x-1)\,(x+1)\ldots(x+g+1) = 0\big) \ . \tag{2.8}$$

We have one automorphism which swaps the two branches and one which sends $x \to -x$ so we have $(\mathbb{Z}/2)^2$ symmetry. Take φ to be any automorphism such that $\varphi^a = \mathrm{id}$. Recall $\mathbb{G}_M = \mathrm{Spec}\,\mathbb{C}\left[x, x^{-1}\right]$. Take $C_0 \times \mathbb{G}_m / (\mathbb{Z}/a)$. The action is as follows. For $\mathbb{C}^\times \ni \zeta \neq 1$, $\zeta^a = 1$ we have

$$(\varphi, \zeta) : (z, t) \mapsto (\varphi\,(z)\,, \zeta \cdot t) \ . \tag{2.9}$$

So this is a nontrivial bundle over $\mathbb{G}_m$.

REMARK 2.1. If automorphisms are the problem, then why not just stick to ones without them. As it turns out, restricting to C_S with $\mathrm{Aut}\,(C_S) = \{1\}$ would indeed make F_g representable, but it is not very useful.

Instead, we will construct the moduli space as an *algebraic stack*, which is a generalization of the notion of a scheme accomodating automorphisms from the beginning.

2. Stacks

Another good reference (which is unfortunately only in French[2.2]) is [**20**].

The idea here is to formalize the notion of a "family of objects parameterized by a scheme along fibrewise automorphisms".

Fix a base scheme S (think $\mathbb{C}$). Write $\mathcal{S} = \mathbf{Sch}/S$ for the category of schemes over S.

DEFINITION 2.1. (1) A *category over* $\mathcal{S}$ is a category $\mathcal{F}$ together with a functor $p_\mathcal{F} : \mathcal{F} \to \mathcal{S}$. For $B \in \mathrm{Obj}\,(\mathcal{S})$, we have a fibre category $\mathcal{F}\,(B)$ which is a subcategory of $\mathcal{F}$ with objects

$$\{X \in \mathrm{Obj}\,(\mathcal{F}) \,|\, p_\mathcal{F}\,(X) = B\} \tag{2.10}$$

and morphisms

$$\{\varphi \in \mathrm{Hom}\,(\mathcal{F}) \,|\, p_\mathcal{F}\,(\varphi) = \mathrm{id}_B\} \ . \tag{2.11}$$

[2.2] Professor Siebert says it isn't a big deal since French is basically English.

(2) A category over $\mathcal{S}$ is a groupoid over $\mathcal{S}$ (or fibered groupoid) if
  (a) For all $f : B' \to B$ in $\mathcal{S}$ and $X \in \mathrm{Obj}\,(\mathcal{F})$ there exists $\varphi : X' \to X$ in $\mathcal{F}$ with $p_{\mathcal{F}}(\varphi) = f$:

$$\begin{tikzcd} X' \arrow[r, dashed, "\varphi"] \arrow[d, dashed] & X \arrow[d] \arrow[d, bend left, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \quad . \tag{2.12}$$

  (b) For all commutative diagrams

$$\begin{tikzcd} & X'' \arrow[rr, "\varphi''"] \arrow[dd] & & X \arrow[dd] \arrow[dd, bend left, "p_{\mathcal{F}}"] \\ X' \arrow[ur, dashed, "\exists!"] \arrow[urrr, "\varphi'"] \arrow[dd] & & & \\ & B'' \arrow[rr, "f''"] & & B \\ B' \arrow[ur, "h"] \arrow[urrr, "f'"'] & & & \end{tikzcd} \tag{2.13}$$

(i.e. $p_{\mathcal{F}}(\varphi') = p_{\mathcal{F}}(\varphi'') \circ h$) there exists unique $\chi : X' \to X''$ and $\varphi' = \varphi'' \circ \chi$.

Remark 2.2. (i) (ii) implies that $\varphi : X' \to X$ is an isomorphism iff $p_{\mathcal{F}}(\varphi)$ is an isomorphism.
(ii) $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{S}$ groupoid over $\mathcal{S}$ implies $\mathcal{F}(B)$ are groupoids.
(iii) (ii) implies

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "p_{\mathcal{F}}"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \tag{2.14}$$

X' is unique up to unique isomorphism. Write $f^*X := X'$. This is called the *pull-back*. This construction is functorial in the sense that $\psi : X'' \to X'$ in $\mathcal{F}(B)$ yields a canonical morphisms $f^*\psi : f^*X'' \to f^*X'$, i.e. $f : B' \to B$ gives us

$$f^* : \mathcal{F}(B) \to \mathcal{F}(B') \ . \tag{2.15}$$

Example 2.4 (Representable functors). Let $\mathcal{F} : \mathcal{S} \to \mathbf{Set}$ be a contravariant functor. This yields a groupoid. The objects are (B, β) such that $B \in \mathrm{Obj}\,(\mathcal{S})$ and $\beta \in \mathcal{F}(B)$. The idea is that $p_{\mathcal{F}} : (B, \beta) \mapsto B$. Then

$$\hom\left((B', \beta'), (B, \beta)\right) = \{f : B' \to B \,|\, F(f)(\beta) = \beta'\} \ . \tag{2.16}$$

For example if X is an S-scheme then this defines (is equivalent to) the functor $F(B) := \mathrm{Hom}_S(B, X)$ and

$$F(\varphi : B' \to B) : (f : B \to X) \mapsto (f \circ \varphi : B' \to X) \ . \tag{2.17}$$

The associated groupoid $\underline{X} = \mathcal{F}$ has objects $f : B \to X$ in $\mathcal{Y} = \mathbf{Sch}/S$ with morphisms

$$\begin{tikzcd} B' \arrow[d, "\varphi"] \arrow[dr, "f'"] & \\ B \arrow[r, "f"] & X \end{tikzcd} \tag{2.18}$$

and $p_X : (B \to X) \mapsto B$.

Example 2.5 (Quotient stack). Let X/S be a scheme with an action of a (flat) group scheme G/S (e.g. GL_n). Then we can take the quotient $[X/G]$. The objects are diagrams

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & X \\ B & \end{tikzcd} \tag{2.19}$$

where E/B is a G-principal bundle and f is G-equivariant. The morphisms are given by

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[rr, bend left] & E \arrow[r, "f"] \arrow[d] & X \\ B' \arrow[r] & B & \end{tikzcd} \tag{2.20}$$

where the square on the left must be cartesian.

Fact 1. *G acts freely on X and X/G exists as a scheme so* $[X/G] = \underline{X/G}$.

Example 2.6 (Classifying spaces of principal G-bundles). For $X = \mathrm{pt}$, $BG \coloneqq [\mathrm{pt}/G]$.

So we have three main examples. First, if X is scheme we get Lecture 7; September 19, 2019

$$\underline{X}(S) \coloneqq \mathrm{Hom}(S, X) \ . \tag{2.21}$$

(Note that $\underline{S} = \mathcal{S}$).

Then for $G \circlearrowright X$ we get $[X/G]$. For $F : \mathbf{Sch}/S \to \mathbf{Set}$ we get $\mathcal{F}$ an S-groupoid.

Third, we have the moduli groupoid $\mathcal{M}_g$. The objects are curves $X \to B$, for B any scheme, X_s non-singular for all s. The morphisms are given by cartesian diagrams:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \ . \tag{2.22}$$

Note this is *not* the groupoid associated to F_g.

Similarly, one defines the *universal curve* $\mathcal{C}_g$ over $\mathcal{M}_g$. The objects are pairs $(X \to B, \sigma)$ where $\sigma : B \to X$ is a section.

2.1. Morphisms of groupoids.

Definition 2.2. A morphism between groupoids F_1, F_2 over S is a functor

$$\begin{tikzcd} F_1 \arrow[rr, "p"] \arrow[dr, "p_{F_1}"'] & & F_2 \arrow[dl, "p_{F_2}"] \\ & \mathcal{S} & \end{tikzcd} \tag{2.23}$$

where $p_{F_1} = p_{F_2} \circ p$. Note this is *equality* of functors.

EXAMPLE 2.7. We have the forgetful functor $\mathcal{C}_g \to \mathcal{M}_g$ which simply forgets the section.

EXAMPLE 2.8. Let $f : X \to Y$ be a morphism of schemes. This is equivalent to $p : \underline{X} \to \underline{Y}$ being a morphism of the associated groupoids.

PROOF. $\Longrightarrow$: On objects, $p\left(B \xrightarrow{u} X\right) := X \xrightarrow{f\circ u} Y$. On morphisms

$$p\left(\begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "u"] & \\ B' \arrow[r, "u'"] & X \end{tikzcd}\right) := \begin{tikzcd} B \arrow[d, "s"'] \arrow[dr, "f\circ u"] & \\ B' \arrow[r, "f\circ u'"'] & Y \end{tikzcd}. \tag{2.24}$$

($\Longleftarrow$): If we have

$$\begin{tikzcd} \underline{X} \arrow[rr, "\underline{p}"] \arrow[dr] & & \underline{Y} \arrow[dl] \\ & \mathcal{S} & \end{tikzcd} \tag{2.25}$$

then $p\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{f} Y\right) \in \mathrm{Obj}\,\underline{Y}(X)$ for some f. This is exactly the setup of Yoneda's Lemma: p can be viewed as a natural transformation between

$$\mathrm{Hom}_{\mathcal{S}}(\cdot, X) : \mathcal{S} \to \mathbf{Set} \tag{2.26}$$

and another functor $G : \mathcal{S} \to \mathbf{Set}$. Then Yoneda says that

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathcal{S}}(\cdot, X), G\right) \xrightarrow{\equiv} G(X) \tag{2.27}$$

where $\Phi \mapsto \Phi(\mathrm{id}_X)$. This shows us that p is induced by f. $\square$

EXAMPLE 2.9. Similarly, for a scheme B and a groupoid over $\mathcal{S}$ F, we get that

$$\{p : \underline{B} \to F\} = F(B) \tag{2.28}$$

where $p \mapsto p(\mathrm{id}_B)$.

EXAMPLE 2.10. $\mathcal{S} = \underline{S}$ and for any groupoid F over S, we can view $p_F : F \to \mathcal{S}$ as a morphism of groupoids $F \to \underline{S}$.

EXAMPLE 2.11. Let X/S be a scheme with the action of a group scheme G/S. This yields a quotient morphism $q : \underline{X} \to [X/G]$.

On objects:

$$\left(B \xrightarrow{s} X\right) \mapsto \left(\begin{tikzcd} & (g,b) \arrow[r, mapsto] & g\cdot s(b) \\ G \circlearrowright & G\times B \arrow[r] \arrow[d] & X \\ & B & \end{tikzcd}\right). \tag{2.29}$$

On morphisms

$$
\begin{tikzcd}
B' \arrow[rr, "f"] \arrow[dr, "s'"'] & & B \arrow[dl, "s"] \\
& X &
\end{tikzcd}
\mapsto
\left(
\begin{tikzcd}
& & & X \\
G \times B' \arrow[r, "\mathrm{id} \times f"] \arrow[d] \arrow[urrr, bend left] & G \times B \arrow[d] \arrow[urr, bend left] & & \\
B' \arrow[r, "f"] & B & &
\end{tikzcd}
\right)
\tag{2.30}
$$

REMARK 2.3. Isomorphisms of groupoids are given by equivalences of categories over S. In particular, $p_1 : F_1 \xrightarrow{\text{iso}} F_2$ may not have an inverse, just a quasi-inverse $q : F_2 \to F_1$ such that pq is naturally isomorphic to id_{F_2} and qp naturally isomorphic to id_{F_1}.

S-groupoids in fact form a 2-category $\mathbf{Grpd}/S$. The objects are groupoids over S. The 1-morphisms are functors over S between the groupoids. But now we in fact have another kind of morphism, called a 2-morphism which are morphisms between morphisms.

REMARK 2.4. We can't even define what a cartesian diagram is without talking about these 2-morphisms so it really is necessary to understand them.

Proposition 2.1. *Let X and Y be schemes. Then $X \simeq Y$ as schemes iff $\underline{X} \simeq \underline{Y}$ as groupoids over S.*

PROOF. ($\implies$): Let $f : X \to Y$ be an isomorphism. Then the induced map $p : \underline{X} \to \underline{Y}$ is a strong equivalence. Indeed, f^{-1} induces $q : \underline{Y} \to \underline{X}$ with $pq = \mathrm{id}_{\underline{Y}}$ and $qp = \mathrm{id}_{\underline{X}}$.

($\Longleftarrow$): Let $p : \underline{X} \to \underline{Y}$ be an equivalence, $q : \underline{Y} \to \underline{X}$ a quasi-inverse. As we have seen this means p, q are induced by $f : X \to Y$ and $g : Y \to X$ which implies $qp\left(X \xrightarrow{\mathrm{id}_X} X\right) = \left(X \xrightarrow{gf} X\right)$ as objects in $\underline{X}$. Hence q is a quasi-inverse of p, which implies there exists an isomorphism

$$
\begin{tikzcd}
X \arrow[rr, dashed, "\simeq"] \arrow[dr, "gf"'] & & X \arrow[dl, "\mathrm{id}_X"] \\
& X &
\end{tikzcd}
\tag{2.31}
$$

and similarly for fg, so f is an isomorphism. □

REMARK 2.5. So this tells us we have some kind of subcategory of schemes inside of groupoids. In the future we will write X instead of $\underline{X}$. Similarly we will also write $\mathcal{S}$, $\underline{S}$, and S for the same thing.

2.2. Fibre products and cartesian diagrams.

DEFINITION 2.3. Consider three groupoids F, G, and H over S and morphisms $f : F \to G$, $h : H \to G$. The *fiber product* is as follows. The objects (over a base B) are triples (x, y, ψ) where $x \in F(B)$, $Y \in H(B)$, and $\psi : f(x) \to h(y)$ is an isomorphism in $G(B)$.

The morphisms (over B) $(x, y, \psi) \to (x', y', \psi')$ are pairs

$$\left(x' \xrightarrow{\alpha} x, y' \xrightarrow{\beta} Y\right) \tag{2.32}$$

such that

$$\psi \circ f(\alpha) = h(\beta) \circ \psi' \ . \tag{2.33}$$

So now this fits in the diagram

$$\begin{tikzcd} F \times_G H \arrow[r, "q"] \arrow[d, "p"] & H \arrow[d, "h"] \\ F \arrow[r, "f"] & G \end{tikzcd} \tag{2.34}$$

which only commutes up to 2-morphisms.

WARNING 2.1. This diagram does not commute in general. We have that

$$fp\,(x, y, \psi) = f\,(x) \qquad\qquad gq\,(x, y, \psi) = h\,(y)\ . \tag{2.35}$$

But there is is a natural isomorphism of functors $fp \simeq hq$, i.e. the diagram Eq. (2.34) is 2-commutative. So we need this ψ twisting to get them to agree.

Then $F \times_G H$ has the universal property for 2-commutative diagrams:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & F \times_G H \arrow[r] \arrow[d] & H \arrow[d] \\ & F \arrow[r] & G \end{tikzcd}\ . \tag{2.36}$$

EXAMPLE 2.12. For X, Y, and Z schemes we have

$$\underline{X} \times_{\underline{Z}} \underline{Y} = \underline{X \times_Z Y}\ . \tag{2.37}$$

EXAMPLE 2.13 (Base change). If $T \to S$ is a morphism of schemes then $T \times_S F$ is a groupoid over T. Actually, for all $B \to T$, $F\,(B)$ and $(T \times_S F)\,(B)$ are equivalent

$$\begin{tikzcd} X \arrow[d] & \in\ F\,(B) \\ B \arrow[r] \arrow[d] & T \arrow[dl] \\ S & \end{tikzcd}\ . \tag{2.38}$$

2.3. Definition of stacks. We want to get closer to something which allows us to do algebraic geometry.

DEFINITION 2.4 (Iso-functor). Let (F, p_F) be a groupoid over S, B a scheme over S, and $X, Y \in \mathrm{Obj}\,(F\,(B))$. Then

$$\mathrm{Iso}_B\,(X, Y) : \mathbf{Sch}/B \to \mathbf{Set} \tag{2.39}$$

is the following contravariant functor. On objects:

$$\left(B' \xrightarrow{f} B\right) \mapsto \left\{ f^* X \xrightarrow{\varphi} f^* Y \,\middle|\, \varphi \text{ iso} \right\}\ . \tag{2.40}$$

On morphisms we get:

$$\begin{tikzcd} B'' \arrow[rr, "h"] \arrow[dr, "g"'] & & B' \arrow[dl, "f"] \\ & B & \end{tikzcd} \mapsto \left((f^* X \to f^* Y) \mapsto \left(\underbrace{h^* f^* X}_{= g^* X} \to \underbrace{h^* f^* Y}_{= g^* Y} \right) \right)\ . \tag{2.41}$$

THEOREM 2.2 (Deligne-Mumford). *Take two curves X/B, Y/B of genus 2. The iso-functor* $\mathrm{Iso}_B(X,Y)$ *is represented by a scheme.*

PROOF. We know we have the relative holomorphic cotangent bundles $\omega_{X/B}$ and $\omega_{Y/B}$. These are ample, so they give us an embedding into projective space over B. These are canonical bundles, so any isomorphism $f^*X \to g^*Y$ (for any $f : B' \to B$) preserves this polarization. Now we can use the relative Hilbert scheme (for the graph of $f^*X \to f^*Y$). $\square$

REMARK 2.6. $\mathrm{Iso}_B(X,Y)$ is finite and unramified over B, but not in general flat (e.g. fibre cardinalities can jump).

Lecture 8; September 24, 2019

DEFINITION 2.5 (Stack). A groupoid $(\mathcal{F}, p_{\mathcal{F}})$ over S is a *stack* if:

(1) for any B over S and any $X, X' \in \mathrm{Obj}(\mathcal{F}(B))$, $\mathrm{Iso}_B(X,X')$ is a sheaf in the étale topology;
(2) for $\{B_i \to B\}$ an étale covering of B, $X_i \in \mathrm{Obj}(F(B_i))$, isomorphisms

$$\varphi_{ij} : X_j|_{B_i \times_B B_j} \to X_i|_{B_i \times_B B_j} \tag{2.42}$$

satisfying the cocycle condition, they glue, i.e. there exists $X \in \mathrm{Obj}(\mathcal{F}(B))$ and isomorphisms $X|_{B_i} \simeq X_i$ inducing φ_{ij}.

REMARK 2.7. The first condition could be thought of as some sort of descent for morphisms, and the second can be thought of as descent for objects.

Étale topology. The authoritative reference on this subject is [**18**], [**23**] is easier to read. Replace Zariski open subsets by étale morphisms $U \to X$. The intuition is to think of these as being an unbranched morphism which is locally a diffeomorphism in U. The intersection of $U \to X$, $V \to X$ is given b y $U \times_X V \to X$. This is an example of what is called a *Grothendieck topology.*

Formally, smooth/unramified/étale morphisms $f : X \to Y$ are morphisms of schemes (of finite type). Let $T = \mathrm{Spec}\, A \hookrightarrow T' = \mathrm{Spec}\, A'$ be an infinitesimal extension, i.e. $A = A'/I$ where $I^n = 0$ for some n.

EXAMPLE 2.14. $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\epsilon]/(\epsilon^n) \hookrightarrow \mathrm{Spec}\, k[\epsilon]/\left(\epsilon^{n+1}\right)$. For $n = 2$ this is a square zero extension $\mathrm{Spec}\, k \hookrightarrow \mathrm{Spec}\, k[\epsilon]/\left(\epsilon^2\right)$.

In this situation, we look at all diagrams:

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d, "f"] \\ T' \arrow[r, "\psi"] \arrow[ur, dashed, "\exists\tilde{\psi}?"] & Y \end{tikzcd} . \tag{2.43}$$

Then we ask for the properties:

- $\exists\tilde{\psi}$
- uniqueness of $\tilde{\psi}$.

DEFINITION 2.6. f is formally

(1) *smooth* iff this exists,
(2) *unramified* iff this is unique, and
(3) *étale* iff it exists and is unique.

EXAMPLE 2.15. The $2:1$ cover in Fig. 1 is étale but is not smooth.

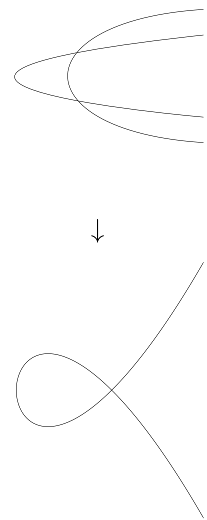

FIGURE 1. The two to one cover of the curve $V\left(y^2 - x^3 - x^2\right)$.

The smooth case should look like a projection. So we can lift a tangent vector but it won't be unique. The étale case looks like mapping two curves onto one, then we choose a point upstairs using φ and the lift is unique. Then a ramified example (which is not smooth) is the cover $\mathbb{A}^1 \to \mathbb{A}^1$ corresponding to the algebraic map $z \mapsto z^2$. The only vector with unique lift is the 0 vector, and the others can somehow lift to anything.

2.4. Examples of stacks.

EXAMPLE 2.16. Let $\mathcal{F}$ be the groupoid associated to a functor

$$F : \mathcal{S} \to \mathbf{Set}\ . \tag{2.44}$$

Then $\mathcal{F}$ is a stack iff F is a sheaf (of sets) in the étale topology. If we take $F = \mathrm{Hom}_{\mathcal{S}}\,(-, X)$ and $X \in \mathcal{S}$ then this is a stack (needs étale descent for morphisms). Note that $F = F_g$ (moduli functor) is not a stack since families might not glue, i.e. (2) from the definition might not be satisfied.

EXAMPLE 2.17. The moduli groupoid $\mathcal{M}_g$ is a stack.

Proposition 2.3. *If G/S is a flat (affine/separated) group scheme $\circlearrowright X$ then $[X/G]$ is a stack.*

PROOF. For (2) we need an étale descent for principal bundles (G-torsors).

For (1) we claim that $\underline{\mathrm{Iso}_B}\,(f, f')$ is represented by a scheme where

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr] & X & E' \arrow[l, "f'"'] \arrow[dl] \\ & B & \end{tikzcd} \ . \tag{2.45}$$

It is enough to check étale locally on B, so we may assume $E = B \times G = E'$. So the situation we have $\sigma : B \to E$ and $\sigma' : B \to E'$. Then we get $f \circ \sigma$ and $f' \circ \sigma'$ are two morphisms

$B \to X$, and then

$$\underline{\mathrm{Iso}_B}(f, f') = B \times_X B \tag{2.46}$$

which fits into the fiber diagram:

$$\begin{array}{ccc} B \times_X B & \longrightarrow & B \\ \big\downarrow & & \big\downarrow{\scriptstyle f'\circ\sigma'} \\ B & \xrightarrow{f\circ\sigma} & X \end{array} . \tag{2.47}$$

□

REMARK 2.8. The étale descent for schemes themselves (not their morphisms) leads to the notion of algebraic spaces (Artin, Knutson).

EXAMPLE 2.18. If $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are stacks with maps $\mathcal{F} \to \mathcal{G}$ and $\mathcal{H} \to \mathcal{G}$ then $\mathcal{F} \times_{\mathcal{G}} \mathcal{H}$ is a stack.

2.5. Representable morphisms. Some morphisms of stacks are "scheme-like":

DEFINITION 2.7. A morphism $f : \mathcal{F} \to \mathcal{G}$ of stacks is *representable* if for any scheme B and morphism $B \to \mathcal{G}$ the fiber product $\mathcal{F} \times_B \mathcal{G}$ is (the groupoid associated to) a scheme (better: an algebraic space).

EXAMPLE 2.19. Consider $X \to [X/G]$. E such that the following diagram is fibered:

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & [X/G] \end{array} \tag{2.48}$$

is a scheme.

EXAMPLE 2.20. Recall we have $\mathcal{M}_g$ the moduli space of genus g curves over B, and then we have $\mathcal{C}_g \to \mathcal{M}_g$ which consists of curves along with sections. Then we have a fibered diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{C}_g \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{M}_g \end{array} . \tag{2.49}$$

The idea is that if we have properties of morphisms of schemes stable under base change (so they are compatible with the philosophy of stacks) then we can move these properties to the world of stacks.

DEFINITION 2.8. Let $\mathcal{F} \to \mathcal{G}$ be representable. Then this has property P (of morphisms of schemes), stable under base change,[2.3] if for any $B \to \mathcal{G}$, for B a scheme, $B \times_{\mathcal{G}} \mathcal{F} \to B$ has this property.

EXAMPLE 2.21. For G smooth over S, $G \circlearrowright X$, $X \to [X/G]$ is smooth.

[2.3]E.g. finite type, separated, flat, affine, proper, ...

2.6. Definition of DM-stack.

DEFINITION 2.9. A stack $\mathcal{F}$ is a *Deligne-Mumford* stack if

(1) $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable, quasi-compact, and separated;[2.4] If $\Delta_{\mathcal{F}}$ is in addition proper, one says $\mathcal{F}$ is *separated.*
(2) There is an étale surjective morphism (an étale atlas) $\varphi : U \to \mathcal{F}$ with U a scheme.

Now we have some comments. Lecture 9; September 26, 2019

(1) The first is the representability of Δ_X.

Proposition 2.4. *Δ_X is representable iff any morphism $B \to C$ where B is a scheme and is representable.*

PROOF. ($\Longrightarrow$): So we for two schemes B and B' we want to show that $B' \times_X B$ is a scheme:

(2.50)
$$\begin{tikzcd} B' \times_X B \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ B' \arrow[r, "g"] & X \end{tikzcd} .$$

But this is implied by Δ_X being representable:

(2.51)
$$\begin{tikzcd} B' \times_X B \arrow[r] \arrow[d] & B \arrow[d, "\Delta_X"] \\ B' \times_S B \arrow[r, "f \times g"] & X \times_S X \end{tikzcd} .$$

□

Another aspect is that $X, Y \in \mathrm{Obj}(\mathcal{F}(B))$ implies $\mathrm{Iso}_{\underline{B}}(X, Y)$ is representable by a scheme.

PROOF. $\mathrm{Iso}_{\underline{B}}(X, Y)$ is represented by the following. X and Y correspond to two maps $f, g : \underline{B} \to X$. Then the fiber product

(2.52)
$$\begin{tikzcd} B \times_{X \times_S X} X \arrow[d] & X \arrow[d] \\ B \arrow[r, "(f,g)"] & X \times_S X \end{tikzcd}$$

represents the iso-functor and hence it is a scheme. □

(2) Δ_X is quasi-compact: we don't want $\mathrm{Iso}_{\underline{B}}(X, Y)$ to be too wild. This can be relaxed (as in the Stacks project).
(3) Δ_X separated: an isomorphism is the identity if it is so generally.
(4) When is Δ_X proper? This should be related to the separatedness of the stack X. For schemes we act for $X \to X \times_S X$ to be a closed embedding.

[2.4] The latter two conditions are sometimes relaxed.

(5) The atlas provides what are called versal deformation spaces for deformations over Artin rings. Write $\bar{A} = A/I$ such that $I^n = 0$ The idea is that we have

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, dashed] \arrow[dr, dashed] & \mathcal{U} \arrow[d, two heads] \\ \operatorname{Spec} \bar{A} \arrow[u, hook] \arrow[r] \arrow[ur] & X \end{tikzcd} \tag{2.53}$$

and we get unique maps.

(6) Atlas is étale: makes Δ_F unramified, i.e. automorphisms are somehow discrete. If we only have a smooth atlas this is what is called an *Artin stack* (or *algebraic stack*).

EXAMPLE 2.22. If we take $BG = [\mathrm{pt}/G]$ for G/S flat (not étale) this is an Artin stack.

Separatedness of stacks.

Lemma 2.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of stacks fulfilling condition 1 Then $\Delta_{F/G} : F \to F \times_G F$ is representable.*

PROOF. See [**20**]. □

A criterion for an Artin stack to be DM.

THEOREM 2.6. *Let $\mathcal{F}$ be an algebraic stack over a Noetherian scheme S with a smooth atlas $U \to \mathcal{F}$ of finite type over S. Then $\mathcal{F}$ is DM iff $\Delta_{\mathcal{F}}$ is unramified.*

PROOF. See [**20**] or [**9**] Theorem 2.1. □

Corollary 2.7. *Let $\mathcal{G}/S$ be a smooth affine group scheme acting on a Noetherian scheme X/S, both of finite type over S, such that the geometric points have finite and reduced stabilizers.*

Then

(i) $[X/G]$ is a DM-stack (for trivial stabilizers, an algebraic space)

(ii) $[X/G]$ is separated iff action is proper.

PROOF. (1) Having finite reduced stabilizers implies that for all

$$B \to [X/G] = \left(\begin{array}{ll} E \longrightarrow X \\ \downarrow \\ B \end{array} \right) \tag{2.54}$$

which means $\mathrm{Iso}_{\underline{B}}(E, E)/B$ is unramified. Then this means $\Delta_{[X/G]}$ is unramified and $X \to [X/G]$ is smooth which implies (by the above theorem) that $[X/G]$ is DM.

□

3. Stable curves

For a complete (read: compact) moduli stack of curves we need to add singular curves. The insight of Deligne-Mumford was that adding node[2.5] suffices.

[2.5] This means that étale locally it looks like $V(zw) \subset \mathbb{A}^2$.

DEFINITION 2.10. A (DM) *stable curve* over a scheme S is a proper flat morphism $\pi : C \to S$ such that

(i) the geometric fibers C_s are reduced, connected, one-dimensional with at most nodes as singularities,
(ii) if $E \subset C_S$ is non-singular rational, $\nu : \tilde{E} \to E$ the normalization, then the number of the preimage of singular points under ν is ≥ 3.

We also want C_s to not be smooth of genus 0 or 1.

REMARK 2.9. Condition (ii) is equivalent to $\mathrm{Aut}\left(C_S\right)$ being finite.

Note that the genus is $g = h^1\left(C_S, \mathcal{O}_{C_S}\right)$.

This gives us immediately that this is a groupoid of stable curves of genus g: $\bar{\mathcal{M}}_g$ which is (at this point) a stack over $\mathbb{Z}$.

3.1. $\overline{\mathcal{M}_g}$ is a DM-stack. Recall that for a stable curve, $\pi : C \to S$ is an étale locally complete intersection morphism and hence has a relative dualizing invertible sheaf $\omega_{C/S}$. Explicitly on geometric fibers we have the normalization:

$$\begin{array}{c} C'_S \xrightarrow{\nu} C_S \\ \\ x_i, y_i \longmapsto i\text{th node} \end{array} \quad . \tag{2.55}$$

Then we can write

$$\nu^* \omega_{C_S} = \omega_{C'_S}\left(x_1 + \ldots + x_r + y_1 + \ldots + y_r\right) \ . \tag{2.56}$$

For $\alpha \in \omega_{C_S}(U)$ (where U is a neighborhood of the ith node) we have that $\nu^*\alpha$ is a rational 1-form with

$$\boxed{\mathrm{Res}_{x_i}\left(\nu^*\alpha\right) + \mathrm{Res}_{y_i}\left(\nu^*\alpha\right) = 0} \ . \tag{2.57}$$

Now use $\omega_{C/S}^{\otimes n}$ to embed C/S into $\mathbb{P}_S^N$ for some N.

THEOREM 2.8. *For $g \geq 2$, $\omega_{C/S}^{\otimes n}$ is relatively very ample for $n \geq 3$. Moreover, $\pi_*\left(\omega_{C/S}^{\otimes n}\right)$ is locally free of rank $(2n-1)(g-1)$.*

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SKETCH PROOF. For C smooth, $\omega_{C/S}^{\otimes n}$ is relatively very amply for all $n \geq 2$. Let's consider the case $S = \mathrm{Spec}\, k$. By Riemann-Roch and Serre duality we can compute:

$$h^0\left(C, \omega_C^{\otimes n}\right) = h^1\left(C, \omega_C^{\otimes n}\right) + \deg\left(\omega_C^{\otimes n}\right) + 1 - g \tag{2.58}$$

$$= h^0\left(C, \left(\omega_C^{\otimes n}\right)^{-1} \otimes \omega_C\right) + n\left(2g-2\right) + 1 - g \ . \tag{2.59}$$

For C smooth we get that $\left(\omega_X^{\otimes n}\right)^{-1} \otimes \omega_C = \omega_C^{1-n}$, and the degree is $(1-n)\left(2g - g_2\right) < 0$, so this gives so $h^0\left(C, \left(\omega_C^{1-n}\right)\right) = 0$ and we get

$$h^0\left(C, \omega_C^{\otimes n}\right) = (2n-1)(g-1) \ . \tag{2.60}$$

We can also twist $h^0\left(C, \omega_C^{\otimes n}\left(-p_1 - p_2\right)\right)$ and this is still zero. So we can separate points and tangents, which is a criterion for being very ample.

For C nodal, we can take the normalization $\nu : C' \to C$ where C' is smooth. Then

$$C_{\mathrm{sing}} = \{z_1, \ldots, z_n\} \qquad \nu^{-1}\left(z_i\right) = \{x_i, y_i\} \ . \tag{2.61}$$

For $\alpha \in H^0(U, \omega_C)$ we have that $\nu^*\alpha$ is a rational section of $\omega_{C'} = \Omega^1_{C'}$ with simple poles at $\{x_i, y_i\}$ (with $\mathrm{Res}_{x'}(\nu^*\alpha) + \mathrm{Res}_{y_i}(\nu^*\alpha) = 0$) so we have

$$\nu^*\omega_C = \Omega_{C'}\left(\sum x_i + \sum y_i\right) . \tag{2.62}$$

This shows, in any case, that $\omega_C^{\otimes n}$ is very ample for $n \geq 3$. Worse case we have a rational $(g = 0)$ irreducible component D of C with 3 special points which gives us

$$\nu^*\omega_C|_D \simeq \mathcal{O}_{\mathbb{P}^1}(1) . \tag{2.63}$$

Finally, use cohomology and base change. □

Corollary 2.9. *Every stable curve C of genus g can be (n-canonically) embedded into $\mathbb{P}^N$ for $N = (2n-2)(g-1) - 1$ $(n \geq 3)$ with Hilbert polynomial*

$$P_{g,n}(t) = (2nt - 1)(g - 1) \tag{2.64}$$

DEFINITION 2.11. $\overline{H}_{g,n} \subset \mathbf{Hilb}^{P_{g,n}}_{\mathbb{P}^N}$ consists of n-canonically embedded nodal curves.

This is an open subscheme (nodes can at most smooth out)

REMARK 2.10. (a) Having a morphism $S \to \overline{H}_{g,n}$ is the same as for a stable curve $\pi : C \to S$ of genus g having an isomorphism $\mathbb{P}\left(\pi_*\left(\omega^{\otimes n}_{C/S}\right)\right) \xrightarrow{\simeq} S \times \mathbb{P}^N$.

(b) $\mathrm{PGL}(N+1)$ acts on $\overline{H}_{g,n}$ by its action on $\mathbb{P}^n$.

THEOREM 2.10. $\overline{\mathcal{M}}_g \simeq \left[\overline{H}_{g,n}/\,\mathrm{PGL}(N+1)\right]$.

PROOF. We will define a functor

$$p : \overline{\mathcal{M}}_g \to \left[\overline{H}_{g,n}/\,\mathrm{PGL}(N+1)\right] . \tag{2.65}$$

On objects:

$$\begin{matrix} C & & E \\ \downarrow\pi & \mapsto & \downarrow \\ B & & B \end{matrix} \tag{2.66}$$

where E is the $\mathrm{PGL}(N+1)$-principal bundle associated to $\mathbb{P}\left(\pi_*\omega^{\otimes n}_{C/S}\right) = \mathbb{P}/B$. We still need a morphism $E \to \overline{H}_{g,n}$ which is $\mathrm{PGL}(N+1)$ equivariant. Consider

$$\begin{tikzcd} C \times_B E \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ E \arrow[r] & B \end{tikzcd} . \tag{2.67}$$

Now

$$E \times_B \mathbb{P} = \mathbb{P}\left(\tilde{\pi}_*\left(\omega^{\otimes n}_{C\times_B E/E}\right)\right) \tag{2.68}$$

is trivial, because it is a principal bundle with a (tautological) section.

On morphisms we have

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ B' \arrow[r, "\varphi"] & B \end{tikzcd} \tag{2.69}$$

which implies

$$\pi'_*\left(\omega_{C'/B'}^{\otimes n}\right)=\varphi^*\pi_*\left(\omega_{C/B}^{\otimes n}\right) \tag{2.70}$$

which leads to a cartesian square:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array} \quad . \tag{2.71}$$

REMARK 2.11. p is fully faithful and essentially surjective. These properties have geometric meaning. The fact that p is faithful tells us that a nontrivial automorphism on C induces a non-trivial automorphism of $\mathbb{P}\left(H^0\left(\omega_C^{\otimes n}\right)\right)$. The fact that p is full tells us that if $\Phi\in\mathrm{PGL}\left(N+1\right)$ with $\Phi\left(C\right)=C$ then Φ is induced by an automorphism of C, i.e. an n-canonical embedding is not contained in a linear subspace.

Essential surjectivity tells us the following. Let

$$\begin{array}{ccc} E & \longrightarrow & \overline{H}_{g,n} \\ \downarrow & & \\ B & & \end{array} \in \mathrm{Obj}\left[\overline{H}_{g,n}/\,\mathrm{PGL}\left(N+1\right)\right] \ . \tag{2.72}$$

Then we have $\pi_E : C_E \to E$ and an isomorphism

$$\mathbb{P}\left(\pi_{E*}\left(\omega_{C_E/E}\right)\right)\simeq E\times\mathbb{P}^n \ . \tag{2.73}$$

Then this descends to

$$B=E/\,\mathrm{PGL}\left(N+1\right),C=C_E/\,\mathrm{PGL}\left(N+1\right) \ . \tag{2.74}$$

□

Corollary 2.11. *$\overline{\mathcal{M}}_g$ is a separated Deligne-Mumford stack of finite type over S (e.g. $S=\mathbb{Z},k$.)*

PROOF. To show this we use a criterion of [**8**], and for separatedness we check finiteness of $\mathrm{Iso}_B\left(C',C\right)\to B$.

See [**8**] for details. □

3.2. Further properties.

Proposition 2.12. *$\overline{\mathcal{M}}_g$ is proper (over $\mathbb{Z}$ or k).*

DIGRESSION 1 (Properties of (morphisms) stacks). For those properties that are local in the smooth (D-M,étale) topology (flat, smooth, unramified, locally Noetherian, normal...) we can just check it on a smooth atlas. Properness is not of this form.

DEFINITION 2.12. A morphism of stacks $f : X \to Y$ is *proper* if it is separated, of *finite type* and *universally closed* (for the Zariski topology[2.6] on the sets $|X|$ and $|Y|$).

We won't use this definition directly. One instead typically uses the valuative criteria for properness.

[2.6] This is generated by representable open embeddings $U\to X$ (resp. $U\to Y$).

First we state the condition for schemes. Let R be a discrete valuation ring. Write K for the quotient field of R. Then we have

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd} . \tag{2.75}$$

Then in this setting uniqueness of such an arrow is separatedness and existence is properness.

Now in the case of stacks, we ask for a finite extension K'/K and then write R' for the normalization of R in K'. Then the question is if this exists/is unique:

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R' \arrow[r] \arrow[urr, dashed, "?"] & \operatorname{Spec} R \arrow[r] & Y \end{tikzcd} \tag{2.76}$$

In moduli theory this goes by the name of stable reduction.

Consider the case $\overline{\mathcal{M}_g}/\operatorname{Spec}\mathbb{Z}$, $g \geq 2$. As it turns out Lecture 11; October 3, 2019

THEOREM 2.13. *$\overline{\mathcal{M}_g}$ is proper.*

By the above valuative criteria it suffices to show the following:

THEOREM 2.14. *Let R be a DVR, $B = \operatorname{Spec} R$, and k be the quotient field of R. Consider some stable curve $f : X_K \to \operatorname{Spec} K$. Then there is a finite extension K'/K and a* unique *stable curve $X' \to B'$, $B' = \operatorname{Spec} R'$ where R' is the normalization of R in K', with*

$$X_{K'} \simeq X_K \times_{\operatorname{Spec} R} \operatorname{Spec} K' \tag{2.77}$$

which fits into

$$\begin{tikzcd} X' \arrow[d] & \supset & X_{K'} \arrow[r] \arrow[d] & X_K \arrow[d] \\ B' & \supset & \operatorname{Spec} K' \arrow[r] & \operatorname{Spec} K \end{tikzcd} . \tag{2.78}$$

PROOF. (In characteristic 0). First extend arbitrary by projectivity as a reduced scheme:

$$\begin{tikzcd} X_K \arrow[r, hook] \arrow[d] & \overline{X} \arrow[d] & \subset & \mathbb{P}^N_R \\ \operatorname{Spec} K \arrow[r, hook] & B \arrow[r, equal] & \operatorname{Spec} R \end{tikzcd} . \tag{2.79}$$

Note that $\overline{X}$ is a surface. Now we desingularize $X \to \overline{X}$. We can do this explicitly by repeatedly blowing up. Taking the fiber at 0 gives us a nodal curve X_0, but this may not be reduced.

The DVR R has a uniformizing parameter $t \in R$, so we have $m = (t)$ maximal. Now complete a base change[2.7] $t \mapsto t^l$ where l is the least common multiple of multiplicities of X_0. This gives us $\tilde{X}' \to B = \operatorname{Spec} R'$ and $\tilde{X}'_0$ is a reduced nodal curve. But this might not be stable. But now we can contract the nonstable cusps. ☐

[2.7] This is where we use characteristic 0. If p divided l, then this would be very nasty.

THEOREM 2.15. *$\overline{\mathcal{M}}_g$ is smooth, connected, and $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (where $\mathcal{M}_g$ parameterizes smooth curves) is a normal crossings divisor with $\left\lfloor \frac{g+1}{2} \right\rfloor$ irreducible components.*

PROOF. Smoothness: We can check this on a smooth atlas $\overline{H}_{g,n} \twoheadrightarrow \overline{\mathcal{M}}_g$. Checking this is really just deformation theory.

Connectedness: We can define the Hurwitz space to be:

$$\mathrm{Hur}_{k,b} = \left\{ C \to \mathbb{P}^1 \;\middle|\; \begin{array}{c} \text{simply branched, } \deg = k, \\ \text{number of branch points } = b, \\ g\left(c\right) = b/2 - k + 1 \end{array} \right\} . \tag{2.80}$$

Hurwitz proved that $\mathrm{Hur}_{k,b}$ is always connected, and for $k > g+1$ we have a surjection $\mathrm{Hur}_{k,b} \twoheadrightarrow \mathcal{M}_g$, so $\mathcal{M}_g$ is connected. □

There are variations of this thing.

(a) $\overline{\mathcal{M}}_{g,k}$ consists of the stable curves of genus g with k marked points. Then the finiteness condition on the automorphisms becomes:

$$\mathrm{Aut}\left(C, \underline{x}\right) < \infty \tag{2.81}$$

where $\underline{x} = (x_1, \ldots, x_k)$. In this case $2g - 2 + k > 0$, so we don't allow:

$$\cancel{\mathcal{M}_{0,0}, \ldots, \mathcal{M}_{0,2}}, \mathcal{M}_{0,3}, \mathcal{M}_{0,5}, \ldots \tag{2.82}$$

$$\cancel{\mathcal{M}_{1,0}}, \mathcal{M}_{1,1}, \ldots \tag{2.83}$$

$$\mathcal{M}_{2,0}, \ldots \ . \tag{2.84}$$

(b) We can also consider pre-stable curves. This just means we drop the stability condition. Fortunately the stacks project comes to our rescue here. Here everything happens in much greater generality. For example, we ask for representability of the Iso functor by algebraic spaces, and this holds very generally. This only forms an Artin stack.

CHAPTER 3

Gromov-Witten theory and virtual techniques

1. Stable maps

Let X be a scheme ($/\mathbb{C}$ or some other field of char $= 0$).

DEFINITION 3.1.

$$\mathcal{M}_{g,k}(X) = \left\{ \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ B \arrow[u, bend left, "\underline{x}"] & \end{tikzcd} \;\middle|\; \begin{array}{c} (C, \underline{x}) \in \tilde{\mathcal{M}}_{g,k}(B) \\ \forall \text{ geom. pts } s \to B : \\ \# \operatorname{Aut}(C_s, \underline{x}_s, f_s) < \infty \end{array} \right\} \tag{3.1}$$

where $\tilde{\mathcal{M}}_{g,k}(B)$ is the stack of nodal curves, and

$$\operatorname{Aut}(C_s, \underline{x}_s, f_s) = \{\varphi \in \operatorname{Aut}(X) \,|\, \varphi(x_i) = x_i \forall i, f \circ \varphi = f\} \ . \tag{3.2}$$

The class of f_s for $s = \operatorname{Spec}\mathbb{C}$ is $f_*[c] \in H_2(X, \mathbb{Z})$ or $H_2(X, \mathbb{Q})$ or $A_1(X)$ where $[c] \in H_2(C, \mathbb{Z})$. For fixed $\beta \in H_2(X, \mathbb{Z})$ (the "degree") we get

$$\mathcal{M}_{g,k}(X, \beta) \subset \mathcal{M}_{g,k}(X) \tag{3.3}$$

is open.

Now we provide an explanation for this definition. Let $f : (C, \underline{x}) \to X/\operatorname{Spec}\mathbb{C}$. Then for any irreducible component $C' \subset C$ we get a corresponding $f : C' \to X$. There are only two possibilities, either this map is either constant or finite. If it is finite, then already $\# \operatorname{Aut}(C', f) < \infty$. So really the contracted components (where f is constant) are the interesting pieces.

For $g = 1$ smooth, everything gets mapped to a point, so we can rule this case out. Besides this, the stability condition is equivalent to saying that any contracted component $C' \subset C$ has at least three special points. Note that a node is two special points on the normalization. For $d = 3$ in $\mathbb{P}^2$ Fig. 1 shows a degeneration as embedded curves to something like $y^2 - x^3 = 0$.

Motivation. For $\mathbb{P}^n$ ($n \geq 3$) the space of embedded curves of fixed arithmetic genus is highly singular, but from the stable maps point of view, at least for the genus 0 case:

THEOREM 3.1. *$\mathcal{M}_{0,k}(\mathbb{P}^n)$ is smooth.*

PROOF. First consider the stack of nodal curves (pre-stable) $\tilde{\mathcal{M}}(0, k)$. This is smooth, so it suffices to prove that the forgetful map $\mathcal{M}_{0,k}(\mathbb{P}^n) \to \tilde{\mathcal{M}}_{0,k}$ (which sends $(C, \underline{x}, f) \mapsto (C, \underline{x})$) is a smooth morphism of stacks.

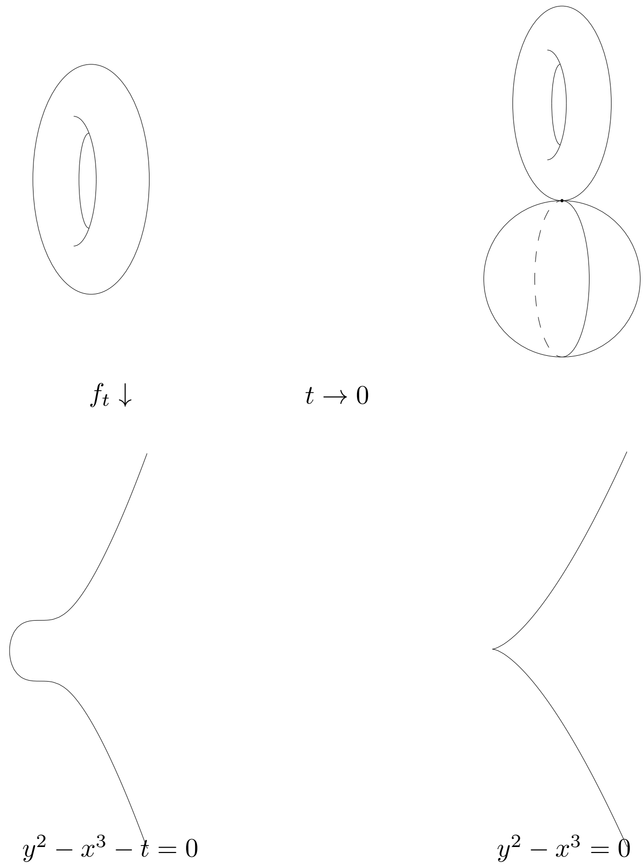

FIGURE 1. Family f_t of stable maps into $\mathbb{P}^1$. When we send $t \to 0$ we are forced to contract the genus one part of the curve which gives us the cusp.

This amounts to checking the following. For $\bar{I} = A/I$ (where $I^2 = 0$) then we have

$$\begin{tikzcd} \operatorname{Spec} \bar{A} \arrow[r] \arrow[d, hook] & \mathcal{M}_{0,k}\left(\mathbb{P}^n\right) \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & \tilde{\mathcal{M}}_{0,k} \end{tikzcd} \tag{3.4}$$

and we want to find the dashed arrow. In other words we want to find:

$$\begin{tikzcd} & & X \\ \bar{C} \arrow[r, hook] \arrow[d] \arrow[urr, bend left, "\bar{f}"] & C \arrow[d] \arrow[ur, dashed, "\exists f?"] & \\ \operatorname{Spec} \bar{A} \arrow[r] & \operatorname{Spec} A & \end{tikzcd} . \tag{3.5}$$

I.e. given $\bar{f}$, we want to find f. The obstruction class is in $H^1\left(\bar{C}, \bar{f}^*\Theta_X\right)$. Stability from before gives us $\bar{f}^*\Theta_X$ is globally generated on each irreducible component $\bar{C}' \subset \bar{C}$. So we have

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\bar{C}'}^{\oplus l} \longrightarrow \bar{f}^*\Theta_{\mathbb{P}^n}\big|_{\bar{C}'} \longrightarrow 0 \quad . \tag{3.6}$$

For $g = 0$ we have

$$0 = H^1\left(\mathcal{O}_{\bar{C}'}\right)^{\oplus l} \longrightarrow H^1\left(\bar{f}^*\Theta_{\mathbb{P}^n}\big|_{\bar{C}'}\right) \longrightarrow H^2 = 0 \quad . \tag{3.7}$$

□

Lecture 12; October 8, 2019

THEOREM 3.2. *$\mathcal{M}_{g,k}(X, \beta)$ is a proper, separated Deligne-Mumford stack.*

PROOF. Write $\mathfrak{m}_{g,k}$ for the Artin stack of nodal curves with k marked points. If we have a diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ T \arrow[u, bend left, "\underline{x}"] & \end{tikzcd} \tag{3.8}$$

this is really the same data as

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"] & X \times T \arrow[r] \arrow[d, "pr_2"] & X \arrow[d] \\ T \arrow[r] \arrow[u, bend left, "\underline{x}"] & T \arrow[r] & \mathrm{pt} \end{tikzcd} . \tag{3.9}$$

This is really a morphism over T, so this tells us the following. We have that

$$\begin{tikzcd} \mathcal{C}_g \simeq \mathfrak{m}_{g,1} \arrow[d] \\ \mathfrak{m}_g \end{tikzcd} \tag{3.10}$$

is a universal curve, so we have

$$\mathcal{M}_g \hookrightarrow \operatorname{Hom}_{\mathfrak{m}_g}\left(\mathcal{C}_g, \mathfrak{m}_G \times X\right) \tag{3.11}$$

is an Artin stack, locally of finite type.

Now since stability is open we want to show:

Lemma 3.3. *This is an open substack.*

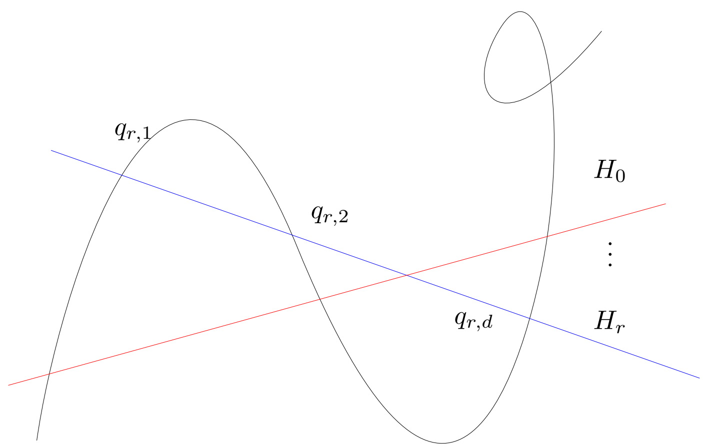

FIGURE 2. $r+1$ hyperplanes in $\mathbb{P}^r$ transversely intersection C_s.

Once we fix $\beta \in H_2(X,\mathbb{Z})$ there are only finitely many "combinatorial types"[3.1] of stable maps. This tells us that $\mathcal{M}_{g,k}(X,\beta)$ is quasi-compact.

The harder part is the stable reduction which tells that this is separated and proper. The idea (due to Fulton-Pandharipande) is that for $X \subset \mathbb{P}^r$ gives us the closed embedding $\mathcal{M}(X,\beta) \hookrightarrow \mathcal{M}(\mathbb{P}^r,d)$, so WLOG $X = \mathbb{P}^r$. The point is that $\mathcal{L} = \omega_c \otimes f^*\mathcal{O}_{\mathbb{P}^r}(1)$ is relatively ample for C/T. Then there exists some number l depending on g, k, and the degree d such that $\mathcal{L}^{\otimes l}$ is relatively very ample. $\square$

Consider the picture in Fig. 2. These hyperplanes $H_i = (t_i = 0)$, for $t_i \in \Gamma(\mathbb{P}^r, \mathcal{O}(1))$ give rise to additional marked points on C. Then locally $\mathcal{M}(\mathbb{P}^r,d)$ is isomorphic to the rigidified space R with objects

$$\begin{tikzcd} C \arrow[d, "\pi"] \\ T \arrow[u, bend left, "q=(q_{ij})"] \end{tikzcd} \tag{3.12}$$

[3.1] By this we mean the intersection pattern of irreducible cusps, generate, and classes β_i of each component with $\beta = \sum_i \beta_i$.

such that we have a bunch of properties. Define

$$\mathcal{H}_i = \mathcal{O}_C\left(q_{i,1} + \ldots + q_{i,d}\right) \ . \tag{3.13}$$

Then for all i we require

$$\pi^*\pi_*\left(\mathcal{H}_i^{-1} \otimes \mathcal{H}_0\right) \to \mathcal{H}_i^{-1} \otimes \mathcal{H}_0 \tag{3.14}$$

is an isomorphism (and for all i, j $\mathcal{H}_i \simeq \mathcal{H}_j$). We also require $\lambda_0, \ldots \lambda_r : T \to \mathbb{G}_m$ to scale the canonical section s_i of $\mathcal{H}_i$.

Now the idea is to do stable reduction on the rigidified level.

2. GW-invariants for hypersurfaces, $g = 0$

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree l, $X = Z(F)$. Then $\mathcal{M}_0(X, d) \subset \mathcal{M}_0\left(\mathbb{P}^{n+1}, d\right)$ is defined by the zero locus of a section of a vector bundle:

$$\begin{tikzcd} \mathcal{C}_0\left(\mathbb{P}^{n+1}\right) \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}^{n+1} \\ \mathcal{M}_0\left(\mathbb{P}^{n+1}\right) & \end{tikzcd} \ . \tag{3.15}$$

On a geometric fiber $f^*\mathcal{O}(l)$ has degree ≥ 0 on each irreducible component. $g = 0$ implies $H^1\left(f_s^*\mathcal{O}(l)\right) = 0$. Then base change gives us

$$R^1\pi_* f^*\mathcal{O}(l) = 0 \ , \tag{3.16}$$

$\mathcal{E} \coloneqq \pi_* f^*\mathcal{O}(l)$ is locally free. Then

$$\operatorname{rank}\mathcal{E} = h^0\left(C_s, f_s^*\mathcal{O}(l)\right) = \deg f_s^*\mathcal{O}(l) + (1 - g) = dl + 1 \tag{3.17}$$

by Riemann-Roch. Then F defines $\sigma \in \Gamma\left(\mathcal{M}_0\left(\mathbb{P}^{n+1}\right), \mathcal{E}\right)$ with $\sigma\left((C_s, f_s)\right) = 0$, i.e. $f_s(C_s) \subset X$.

Definition 3.2 (Kontsevich).

$$\left[\mathcal{M}_0(X, d)\right]_{\text{virt}} \coloneqq c_{dl+1}(\mathcal{E}) \cap \left[\mathcal{M}_0\left(\mathbb{P}^{n+1}, d\right)\right] \ . \tag{3.18}$$

Note that by definition

$$\left[\mathcal{M}_0(X, d)\right]_{\text{virt}} \in A_*\left(\mathcal{M}\left(\mathbb{P}^{N+1}, d\right)\right) \to H_{2*}\left(\mathcal{M}_0\left(\mathbb{P}^{n+1}, d\right)\right) \ . \tag{3.19}$$

First note that

$$\dim \mathcal{M}_0\left(\mathbb{P}^{n+1}, d\right) = \deg f_s^*\Theta_{\mathbb{P}^{n+1}} + (n+1)(1-g) - \underbrace{3}_{\dim \operatorname{Aut}(\mathbb{P}^1)} \tag{3.20}$$

$$= d(n+1) + n - 2 \ . \tag{3.21}$$

For $n = 3$, $l = 5$ we have $\operatorname{rank}\mathcal{E} = 5d + 1$ and by the above computation,

$$\dim \mathcal{M}_0\left(\mathbb{P}^4, d\right) = 5d + 1 \ . \tag{3.22}$$

3. Digression on intersection theory

The book [**11**] is a good reference. Let X be a scheme over a field. We say finite sums

$$\sum n_i [V_i] \tag{3.23}$$

for $V_i \subset C$ irreducible, reduced, closed, subschemes. A rational equivalence is generated by $W \subset X$ an irreducible subvariety, $f \in K(X) \setminus \{0\}$ a rational function. Then we can define the divisor of f to be the sum over $V \subset W$ which is a codimension 1 irreducible subvariety:

$$[\mathrm{div}(f)] := \sum_{V \subset W} \mathrm{ord}_V f \cdot [V] \ . \tag{3.24}$$

We now define the fundamental class of X. Write

$$|X| = \cup_i X_i \tag{3.25}$$

for a decomposition into irreducible components. Define

$$[X] = \sum m_i [X_i] \tag{3.26}$$

with m_i the length of $\mathcal{O}_{X,X_i}$ over $\mathcal{O}_{X,X_i}$.

Example 3.1. Take $X = \operatorname{Spec} A$ to be a thick point where $\dim_k A < \infty$. Then for $M \in A$-**Mod** we want to know the length l. By definition maximal sequence of submodules $M_1 \subset \dots M_l = M$ such that the quotients are just quotients by prime ideals, i.e. $M_{i+1}/M_i \simeq A/\mathfrak{p}_i$ for primes $\mathfrak{p}_i$.

For example we can take $A = k[\epsilon]/\left(\epsilon^{k+1}\right)$. Then the length of A as and A-module is $k+1$ which comes from the length of the sequence:

$$\left(\epsilon^{k+1}\right) = 0 \subset \left(\epsilon^{k}\right) \subset \dots \subset (\epsilon) \subset A \ . \tag{3.27}$$

In general, for an artinian local ring we should obtain the dimension.

Then the Chow groups are

$$A_*(X) = Z_*(X) / \sim_{\text{rat'l}} \tag{3.28}$$

for $* = 0, 1, 2, \dots$.

Example 3.2. The Chow groups vanish for affine space: $A_*(\mathbb{A}^n) = 0$.

For projective space we have

$$A_8(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & i = 0, \dots, n \\ 0 & \text{o/w} \end{cases} \ . \tag{3.29}$$

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Proper push-forward and flat pull-back. We will define a notion of a proper push-forward. Let $f : X \to Y$ be a proper morphism. The definition is very simple. For $V \subset X$ a subvariety we insist that $W = f(Y)_{\text{red}} \subset Y$ is also a subvariety. This gives us a field extension $K(W) \hookrightarrow f^{\#} K(V)$. Then we define:

$$f_*[V] := \begin{cases} 0 & \dim W < \dim V \\ [K(V) : K(W)] \cdot [W] & \dim W = \dim V \end{cases} \ . \tag{3.30}$$

Now we have to prove

Proposition 3.4. *$f_*[V]$ descends to $f_* : A_*(X) \to A_*(Y)$.*

Now we need flat pullback. Let $f : X \to Y$ be a flat morphism of relative dimension n. Then for $V \subset Y$ a subvariety of dimension k, then we can pullback to get $f^{-1}(V)_{\mathrm{red}} \subset X$ which is a union of subvarieties.

DEFINITION 3.3. $f^*[V] \coloneqq \left[f^{-1}(V)\right]$.

Then we show this commutes with rational equivalence.

Excision. We now consider the form of excision we have in this "homology" theory. Consider a closed embedding $i : Y \hookrightarrow X$. Then we get an open embedding $j : X \setminus Y \hookrightarrow X$. Then we get $A_*(X) \xrightarrow{j_*} A_*(X \setminus Y)$, and this actually turns out to be surjective. Then we have kernel $A_*(Y)$, but this might not be injective. So we get an exact sequence:

$$A_*(Y) \longrightarrow A_*(X) \xrightarrow{j_*} A_*(X \setminus Y) \longrightarrow 0 \quad . \tag{3.31}$$

PROOF.

$$Z_k(Y) \to Z_k(X) \to Z_k(X \setminus Y) \to 0 \tag{3.32}$$

□

Weil divisors. Now we have the notion of Weil divisors. Take X be to be n-dimensional. Then the Weil divisors are

$$D = \sum m_i D_i \in Z_{n-1}(X) \quad . \tag{3.33}$$

What we really want to intersect with are not these "homology" classes, but rather some "cohomology" classes called the Cartier divisors. For X any scheme, we first cover X by open subsets U_α. Then in the total ring of quotients we have

$$f_\alpha \in R(U_\alpha) \setminus \{0\} \tag{3.34}$$

for each U_α. These look like g/h for h a non-zero divisor. Then for all α and β we want that $f_\alpha / f_\beta \in \mathcal{O}^\times(U_\alpha \cap U_\beta)$, i.e. it invertible. Then a Cartier divisor is $\{(U_\alpha, f_\alpha)\}$ which gives us an invertible sheaf $\mathcal{O}(D)$.

Let X be pure n-dimensional. We have some kind of map from Cartier divisors to $A_{n-1}(X)$ sending

$$\{(U_\alpha, f_\alpha)\} \mapsto U\left[\operatorname{div}(f_\alpha)\right] \quad . \tag{3.35}$$

We will use the notation that $D \mapsto [D]$.

Cartier divisors. Now we study intersections with Cartier divisors. So we have $D = \{(U_\alpha, f_\alpha)\}$ and some subvariety $j : V \hookrightarrow X$ of pure dimension k. Then we define

$$D \cdot [V] \coloneqq \begin{cases} \left[j^{-1} D\right] & V \not\subset D \\ [C] & C \subset V \text{ Cartier div. s.t. } \mathcal{O}_V(C) \simeq \mathcal{O}_X(D)|_V \end{cases} \tag{3.36}$$

This defines:

$$D \cdot - : A_k(X) \to A_{k-1}(X) \quad . \tag{3.37}$$

First Chern class and Segre classes. Define the first Chern class as follows. Let L be a line bundle. This implies $L = \mathcal{O}_X(D)$ for D some Cartier divisor. Then we define

$$\begin{array}{ccc} A_*(X) & \xrightarrow{c_1(L)} & A_{*-1}(X) \\ z = \sum_i m_i [z_i] & \longmapsto & D \cdot z \end{array} . \tag{3.38}$$

To get higher Chern classes we will define the Segre class of a vector bundle $E \downarrow X$. Write $\mathcal{E}$ for the sheaf of section of E. Write r for the rank. Then

$$E = \underline{\mathrm{Spec}}\,\mathrm{Sym}^\bullet \mathcal{E}^\vee \tag{3.39}$$

where

$$\mathcal{E}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \ . \tag{3.40}$$

Then the lines in E form the projective bundle

$$\mathbb{P}(E) = \underline{\mathrm{Proj}}\,\mathrm{Sym}^\bullet(E^\vee) \ . \tag{3.41}$$

This comes with $\mathcal{O}_{\mathbb{P}(E)}(1)$. The morphism

$$p : \mathbb{P}(E) \to X \tag{3.42}$$

is proper and flat, so we have the above pullback and pushforward for this morphism. Then the Segre classes (for $i \geq 0$) are

$$s_i(E) : A_k(X) \to A_{k-i}(X) \tag{3.43}$$

and we define

$$s_i(E) \frown \alpha \coloneqq p_*\left(c_1\left(\mathcal{O}_{\mathbb{P}(E)}(1)\right)^{r+i-1} \frown p^*\alpha\right) \ . \tag{3.44}$$

Now we want a projection formula for Segre classes, but this just follows from the formula for Cartier divisors. Let $f : X \to Y$ be proper. For $E \downarrow Y$ a vector bundle we have

$$f_*(s_i(f^*E) \frown \alpha) = s_i(E) \frown f_*(\alpha) \ . \tag{3.45}$$

Corollary 3.5. $s_0(E) = \mathrm{id}$.

PROOF. We will use the projection formula for the inclusion of a subvariety $i : V \hookrightarrow X$. It is sufficient to show this for one variety, i.e. WLOG let $V = X$. We can compute

$$f_*(s_i(f^*\alpha) \frown p^*\alpha) = m_i[X] \tag{3.46}$$

since there are no other classes in $A_k(X) = \mathbb{Z} \cdot [X]$. So now we just have to compute the m_i. But this can be done locally, where we have that

$$\mathbb{P}(E) = X \times \mathbb{P}^{r-1} \tag{3.47}$$

and $\mathcal{O}_{\mathbb{P}(E)}(1)$ is just pulled back from $\mathbb{P}^{r-1}$. Then we have

$$c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{k-1} \frown \left[\mathbb{P}^{r-1}\right] = [\mathrm{pt}] \in A_0\left(\mathbb{P}^{r-1}\right) \tag{3.48}$$

which means $m = 1$. □

Chern classes. Now we get Chern classes from Segre classes. We have

$$s_t(E) = \sum s_i(E)\, t^i \in \operatorname{End}(A_*(x)) [\![t]\!] \tag{3.49}$$

$$c_t(E) = s_t(E)^{-1} \ . \tag{3.50}$$

This really needs that $s_i s_j = s_j s_i$ for all i and j.

Then we have the sum formula which tells us that whenever we have

$$0 \to E' \to E \to E'' \to 0 \tag{3.51}$$

we get

$$c_t(E) = c_t(E') - c_t(E'') \ . \tag{3.52}$$

Gysin pullback. Let V be a pure k-dimensional variety, ι a regular embedding of codimension c. Then this is equivalent to being locally given by $I = (x_1, \ldots, a_d) \subset A$. Note each a_i is *not* a zero-divisor in $A/(a_1, \ldots, a_{i-1})$ for $i = 1, \ldots, d$. So we have a cartesian square:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ X \arrow[r, hook, "\iota"] & Y \end{tikzcd} \ . \tag{3.53}$$

For an ideal sheaf $\mathcal{I}$ we have $\mathcal{I}/\mathcal{I}^2$ is locally free of rank d, so a vector bundle of rank d. Define the normal bundle

$$N_{X/Y} = \underline{\operatorname{Spec}} \operatorname{Sym}^\bullet \mathcal{I}/\mathcal{I}^2 \ . \tag{3.54}$$

Similarly, for $\mathcal{I}$ an ideal sheaf of $W \hookrightarrow V$ ($\mathcal{I}/\mathcal{I}^2$ need not be locally free in this case!) we define the normal cone:

$$\begin{tikzcd} C_{W/V} \arrow[d, hook] & \coloneqq & \underline{\operatorname{Spec}}_W \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1} \\ N_{W/V} \arrow[rr, equal] & & \operatorname{Sym}^d \mathcal{I}/\mathcal{I}^2 \arrow[u, two heads] \end{tikzcd} \ . \tag{3.55}$$

Proposition 3.6. *For V pure k dimensional, $C_{W/V}$ is purely k-dimensional.*

Now comes a basic construction. First write

$$s_0 : W \to C_{W/V} \tag{3.56}$$

for the zero section. We know we have

$$C_{W/V} \hookrightarrow N_{W/V} \hookrightarrow g^* N_{X/Y} \tag{3.57}$$

so we can define

DEFINITION 3.4.

$$\iota^! [V] \coloneqq s_0^* \left[C_{W/V} \right] \tag{3.58}$$

where $s_0^* = (\pi^*)^{-1}$ and

$$\pi^* : A_{k-d}(X) \to A_k(E) \ . \tag{3.59}$$

REMARK 3.1. Geometrically we should think of this as intersection with the zero section.

Then this makes sense since we have:

Proposition 3.7. π^* *is an isomorphism.*

PROOF. Surjectivity is easy: it follows from excision. Injectivity is harder. □

Slogan: Take $C_{W/V}$, embed in $g^*\left(N_{X/Y}\right)$, and finally intersect with the zero-section.

This was all for one variety V.

Consider a diagram

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$$Z \overset{\iota}{\hookrightarrow} Y \overset{f}{\leftarrow} X \tag{3.60}$$

where ι is lci. Let $W \subset X$ be a subvariety of dimension d. Then we get

$$\begin{tikzcd} W_Z \arrow[r] \arrow[d] & W \arrow[d, hook, "X"] \\ X_Z \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "\iota"] & Y \end{tikzcd} . \tag{3.61}$$

Then we have

$$C_{W_Z/W} = \operatorname{Spec}_W \bigoplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1} \tag{3.62}$$

pure of dimension d. Then we can pull the normal bundle back and we have:

$$C_{W_Z/W} \hookrightarrow g^* N_{Z/W} \ . \tag{3.63}$$

Then

$$\iota^! [W] \coloneqq s_0^* \left[C_{W_Z/W}\right] \in A_k(W_Z) \to A_*[X] \ . . \tag{3.64}$$

But why the normal cone? We have something called the deformation to the normal cone.[3.2] Consider $M \coloneqq \operatorname{Bl}_{X\times\{0\}} Y \times \mathbb{A}^1$. Over $t \in \mathbb{A}^1 \setminus \{0\}$ we just have $X \hookrightarrow Y$. Over $t = 0$

$$X \hookrightarrow C_{X/Y} \hookrightarrow \mathbb{P}\left(C_{X/Y} \oplus \mathbb{A}^1\right) \cup \operatorname{Bl}_X Y \ . \tag{3.65}$$

The point is that for problems that vary nicely in flat families and are local near X, such as intersection multiplicities, $X \hookrightarrow Y$ is as good as $X \hookrightarrow C_{X/Y}$.

Now we have the following application. We want to calculate the virtual fundamental class (VFC) for $\mathcal{M}_0(X)$ where $X = Z(f) \subset \mathbb{P}^{n+1}$ where $\deg f = l$. Recall we have $\mathcal{M}_0\left(\mathbb{P}^{n+1}\right) \overset{\pi}{\leftarrow} \mathcal{C} \overset{\varphi}{\to} \mathbb{P}^{n+1}$ and then we have a vector bundle of rank r

$$\begin{array}{c} E = \pi_* \varphi^* \mathcal{O}(l) \\ \downarrow \\ \mathcal{M}_0\left(\mathbb{P}^{n+1}\right) = M \end{array} \tag{3.66}$$

called the deformation bundle. Then we have a section $s \in \Gamma(E)$ defined by f with $(s = 0) = \mathcal{M}_0(X) \subset \mathcal{M}_0\left(\mathbb{P}^{n+1}\right)$. Now intersect $\operatorname{im}(s)$ with the zero section:

$$\begin{tikzcd} \mathcal{M}_0(X) \arrow[r, leftrightarrow] \arrow[d] & M \arrow[d, "s"] \\ M \arrow[r, hook, "s_0"] & E \end{tikzcd} . \tag{3.67}$$

[3.2] This should really be called a degeneration, but everyone says deformation.

Then we get:

$$s_0^! [M] \in A_{\dim \mathcal{M}_0(X)-r} \left(\mathcal{M}_0 (X)\right) \ . \tag{3.68}$$

4. Obstruction theories and VFC

4.1. The idea. This idea is from [**21**]. Locally, say on $U \subset \mathcal{M}$, embed U in a smooth space V. This corresponds to an ideal sheaf $\mathcal{I}$ on V. Then we have the diagram:

$$\begin{tikzcd} C_{U/V} = \mathrm{Spec}_U \left(\bigoplus \mathcal{I}^d/\mathcal{I}^{d+1}\right) \arrow[r, hook] & N_{U/V} = \mathrm{Spec}_U \left(\mathrm{Sym}^\bullet \mathcal{I}/\mathcal{I}^2\right) \\ & T_V|_U = \mathrm{Spec}_U \left(\mathrm{Sym}^\bullet \Omega_V|_U\right) \arrow[u, two heads] \end{tikzcd} \ . \tag{3.69}$$

Note that the additive action of $T_V|_U$ leaves $C_{U/V}$ invariant.

Now we want to embed $N_{U/V} \hookrightarrow E_1$ into some vector bundle and globalize. (This is the same as a $T_V|_U$ action on $C_{U/V}$.)

The patching data is as follows. Whenever we have

$$\begin{tikzcd} T_V|_U \arrow[r] \arrow[d] & E_0 \arrow[d] \\ N_{U/V} \arrow[r, hook] & E_1 \end{tikzcd} \tag{3.70}$$

we get a sequence

$$T_V|_U \to E_0 \oplus N_{U/V} \to E_1 \to 0 \tag{3.71}$$

and we ask for it to be exact.

There are two main ways to achieve compatibility. The first is with Artin style obstruction theory as in [**21**]. Alternatively we could use the cotangent complex/derived categories as in [**3**]. We will do this second option.

4.2. Digression on the cotangent complex. For $f : X \to Y$ we have an exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0 \ . \tag{3.72}$$

For f smooth the arrow $f^*\Omega_Y \to \Omega_X$ is injective. In general, we need to replace $\Omega_{X/Y}$, Ω_X, ... by a complex $L^\bullet_{X/Y}$, $L^\bullet_Y$, ... of quasi-coherent sheaves. Then in the derived category we have an exact sequence

$$\ldots \to Lf^*L^\bullet_Y \to L^\bullet_X \to L^\bullet_{X/Y} \to Lf^*L_Y [1] \to L^\bullet_X [1] \to \ldots \tag{3.73}$$

which means this is an exact triangle. This is an extension of what we started in the sense that

$$h^{-1}\left(L^\bullet_{X/Y}\right) \to \underbrace{h^0\left(Lf^*L^\bullet_Y\right)}_{=\Omega_Y} \to \underbrace{h^0\left(L^\bullet_X\right)}_{=\Omega_X} \to \underbrace{\Omega_{X/Y}}_{h^0\left(L^\bullet_{X/Y}\right)} \to 0 = h^1\left(Lf^*L^\bullet_Y\right) \ . \tag{3.74}$$

Now we define the category $D\mathbf{QCoh}\left(\mathcal{O}_X\right)$. The objects are complexes

$$\ldots \to \mathcal{F}^{-2} \xrightarrow{d^{-2}_{\mathcal{F}}} \mathcal{F}^{-1} \xrightarrow{d^{-1}_{\mathcal{F}}} \mathcal{F}^0 \to \ldots \tag{3.75}$$

where

$$d^\bullet = d^\bullet_{\mathcal{F}} : \mathcal{F}^\bullet \to \mathcal{F}^\bullet [1] \ . \tag{3.76}$$

The morphisms $\varphi^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ are Hom spaces as complexes moduli homotopy, i.e. $\varphi^\bullet \simeq \psi^\bullet$ iff there exists $h^\bullet : \mathcal{F}^\bullet : \mathcal{G}^\bullet[-1]$ such that

$$\varphi^\bullet - \psi^\bullet = h^\bullet \circ d^\bullet_{\mathcal{F}} - d_{\mathcal{G}} \circ h^\bullet \ . \tag{3.77}$$

Then we localize (make invertible) quasi-isomorphisms.

Therefore a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ in $D\mathbf{QCoh}(\mathcal{O}_X)$ is

$$\begin{tikzcd} & \tilde{\mathcal{F}}^\bullet \arrow[dl, "\psi^\bullet"'] \arrow[dr, "\tilde{\varphi}^\bullet"] & \\ \mathcal{F}^\bullet \arrow[rr, "\varphi^\bullet"] & & \mathcal{G}^\bullet \end{tikzcd} \ . \tag{3.78}$$

Now consider derived functors. Let $f : X \to Y$. Then we want to define Rf_*. Replace $\mathcal{F}^\bullet$ by quasi-isomorphic complex $\mathcal{I}^\bullet_{\mathcal{F}}$ of injective sheaves [choose one for each $\mathcal{F}^\bullet$]. Then define

$$Rf_*\mathcal{F}^\bullet \coloneqq f_*\mathcal{I}^\bullet_{\mathcal{F}} \ . \tag{3.79}$$

Remark 3.2. The left-derived tensor product $\otimes^L$ and pullback $Lf^* = f^{-1} \otimes^L \mathcal{O}_X$ are more subtle, but they work.

Now the cotangent complex is as follows. First consider the affine case. For a ring map $\varphi : A \to B$, we can resolve B freely as an A algebra. We can even do this canonically. This means we have $p_\bullet \to B \to 0$ with

$$P_\bullet = [\ldots \to A[A[B]] \to A[B]] \ . \tag{3.80}$$

Then in this case:

$$L^\bullet_{B/A} = \Omega^\bullet_{P_\bullet/A} \otimes_{P_\bullet} B \ , \tag{3.81}$$

i.e.

$$L^{-n}_{B/A} = \Omega_{P_n/A} \otimes_{P_n} B \ . \tag{3.82}$$

Let $X \to Y$ be a morphism of algebraic stacks. Then there is a similar "simplicial" resolution for $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. See [**19**] or the stacks project for more information.

The following is important. We can always embed in something smooth:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & Z \arrow[d, "\text{smooth}"] \\ & Y \end{tikzcd} \ . \tag{3.83}$$

Now writing $\mathcal{I} = \mathcal{I}_{X/Z}$ we have

$$\tau_{\geq -1} L^\bullet_{X/Y} = \left[\mathcal{I}/\mathcal{I}^2 \to \underbrace{\mathcal{O}_{Z/Y}\big|_X}_{\iota^*\Omega_{Z/Y}} \right] \ . \tag{3.84}$$

I.e. at the level of linear fibre spaces:

$$\left[T_{Z/Y}\big|_X \to N_{X/Z} \right] \tag{3.85}$$

as needed for the globalization of $C_{X/Z} \hookrightarrow N_{X/Z}$.

4.3. Behrend-Fantechi definition.

Definition 3.5. Let $f : X \to Y$ be a morphism of algebraic stacks. Assume X is Deligne-Mumford. A *perfect obstruction theory* on X/Y is a 2-term complex $\mathcal{F}^\bullet = \left[\mathcal{F}^{-1} \to \mathcal{F}^0\right]$ of locally free coherent sheaves, together with a morphism (in $D\mathbf{QCoh}(\mathcal{O}_X)$)

$$\varphi^\bullet : \mathcal{F}^\bullet \to L^\bullet_{X/Y} \tag{3.86}$$

such that

(1) $h^0(\varphi^\bullet)$ is an isomorphism,
(2) $h^{-1}(\varphi^\bullet)$ is an epimorphism.

The point is the following. Locally embed X in some smooth space Z over Y Lecture 15; October 17, 2019

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & \mathbb{Z} \arrow[dl] \\ Y & \end{tikzcd} . \tag{3.87}$$

Assuming $\varphi^\bullet$ is an honest morphism of complexes, then we already know that $\tau L^\bullet_{X/Y} = \left[\mathcal{I}/\mathcal{I}^2 \to \Omega_{Z/Y}\big|_X\right]$. In this situation we get a commutative diagram

$$\begin{tikzcd} \mathcal{F}^{-1} \arrow[r] \arrow[d] & \mathcal{I}/\mathcal{I}^2 \arrow[d] \\ \mathcal{F}^0 \arrow[r] & \Omega_{Z/Y}\big|_X \end{tikzcd} . \tag{3.88}$$

This gives us a complex

$$0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \oplus \mathcal{I}/\mathcal{I}^2 \to \Omega_{Z/Y}\big|_X \to 0 . \tag{3.89}$$

Then we can ask for exactness at the different terms. At the first term this is equivalent to $h^{-1}(\varphi^\bullet)$ is injective. At the second term this is equivalent to $h^{-1}(\varphi^\bullet)$ surjective and $h^0\left(\varphi^0\right)$ is injective. Then at the third term this is equivalent to $h^0(\varphi^\bullet)$ being surjective.

The conditions in the definition of the definition of a perfect obstruction theory give us that the following sequence is exact:

$$0 \to T_{Z/Y}\big|_X \to \mathcal{F}_0 \oplus N_{Z/X} \to \mathcal{F}_1 \tag{3.90}$$

where $\mathcal{F}_i = L\left(\mathcal{F}^{-i}\right) = \operatorname{Spec}_X \operatorname{Sym}^\bullet \mathcal{F}^{-i}$.

Lemma 3.8. *(a) There exists a unique cone $C_{\varphi^\bullet} \subset \mathcal{F}_1$ with*

$$0 \to T_{Z/Y}\big|_X \to \mathcal{F}_0 \oplus C_{Z/X} \to C_\varphi \to 0 \tag{3.91}$$

exact.

(b) $C_{\varphi^\bullet}$ is independent of the choice of Z and is invariant under homotopy of complexes.

Proof. (a) $\mathcal{F}_1$ is locally free, so we can split the sequence (3.90), then it is trivial.

(b) For two choices Z and Z' we have

$$\begin{tikzcd} X \arrow[r] \arrow[dr] \arrow[dd] & Z' \arrow[d, "\text{smooth}"] \\ & Z \arrow[dl] \\ Y & \end{tikzcd} \tag{3.92}$$

∎

This was the sort of patching picture. There is a more intrinsic story in [**3**]. This is not so bad if we are okay with Artin stacks. Now define

$$\mathcal{M}_{X/Y} = \left[N_{X/Z}/\, T_{Z/Y}\big|_X\right] \tag{3.93}$$

called the intrinsic normal sheaf, and the intrinsic normal cone

$$\mathcal{C}_{X/Y} = \left[\mathcal{C}_{X/Z}/\, T_{Z/Y}\big|_X\right] \tag{3.94}$$

which is pure 0-dimensional.

For $\mathcal{F}^\bullet$ we get also get a stack

$$h^0/h^1\left(\mathcal{F}^\bullet\right) = \left[F_1/F_0\right] \ . \tag{3.95}$$

Then $\varphi^\bullet : \mathcal{F}^\bullet \to L^\bullet_{X/Y}$ is a perfect obstruction theory iff

$$\mathcal{M}_{X/Y} \hookrightarrow h^0/h^1\left(\mathcal{F}^\bullet\right) \tag{3.96}$$

is a monomorphism. Then in this case we construct C_φ as follows:

$$\begin{tikzcd} C_{\varphi^\bullet} \arrow[rr, hook] \arrow[d] & & \mathcal{F}_1 \arrow[d] \\ \mathcal{C}_{X/Y} \arrow[r, hook] & \mathcal{M}_{X/Y} \arrow[r, hook] & h^0/h^1\left(\mathcal{F}^\bullet\right) \end{tikzcd} . \tag{3.97}$$

So why is this called obstruction theory? As usual, consider a lifting problem. Take a square zero extension $T \hookrightarrow \bar{T}$. I.e. an ideal sheaf $\mathcal{I} = \mathcal{I}_{T/\bar{T}}$ such that $\mathcal{I}^2 = 0$. Then consider the diagram

$$\begin{tikzcd} T \arrow[r, "h"] \arrow[d, hook, "\iota"] & X \arrow[d, "f"] \\ \bar{T} \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}", "\exists?"'] & Y \end{tikzcd} . \tag{3.98}$$

Functoriality of $L^\bullet_{X/Y}$ gives

$$\left[Lh^* L^\bullet_{X/Y} \to L^\bullet_{T/\bar{T}} \to \underbrace{\tau_{\geq -1} L^\bullet_{T/\bar{T}}}_{\mathcal{I}[1]} \right] = \operatorname{Obj}(h) \in \operatorname{Ext}^1\left(Lh^\bullet L^\bullet_{X/Y}, \mathcal{I}\right) \tag{3.99}$$

Fact 2. $\operatorname{Obj}(h) = 0$ *iff* $\tilde{g}$ *exists. And then the collection of lifts* $\tilde{g}$ *modulo isomorphism is a torsor under* $\operatorname{Ext}^0\left(Lh^* L^\bullet_{X/Y}\mathcal{I}\right)$.

Proposition 3.9 ([**3**])**.** *$\varphi^\bullet : \mathcal{F}^\bullet \to L^\bullet_{X/Y}$ is a perfect obstruction iff the analogue of the first condition holds for*

$$\varphi^* \operatorname{Obj}(h) \in \operatorname{Ext}^1 \left(\underbrace{h^* \mathcal{F}^\bullet}_{Lh^* \mathcal{F}^\bullet}, \mathcal{I} \right) \tag{3.100}$$

and the analogue of the second condition holds for $\operatorname{Ext}^0(h^* \mathcal{F}, \mathcal{I})$*. This is also equivalent to*

$$\mathcal{M}_{X/Y} \to [\mathcal{F}_n / \mathcal{F}_0] = h^0/h^1(\mathcal{F}^\bullet) \tag{3.101}$$

being an embedding.

THEOREM 3.10. *$\mathcal{M}_{X/Y}$ is a universal (minimal) "obstruction theory", but not a vector bundle stack unless X/Y lci. $\mathcal{F}^\bullet \to L^\bullet_{X/Y}$ provides an embedding into the vector bundle stack $[\mathcal{F}_1/\mathcal{F}_0]$.*

The virtual fundamental class is

$$[X]_{\mathrm{virt},\varphi^\bullet} := s_0^! [C_{\varphi^\bullet}] \in A_*(X) \tag{3.102}$$

for $s_0 : X \to \mathcal{F}_1$ the zero section.

4.4. Obstruction theory for GW-theory.

WARNING 3.1. X is now the target, i.e. not the X from the previous subsection.

Write $\mathcal{M} = \mathcal{M}_{g,k}(X, \beta)$ (and similarly $\mathfrak{m} = \mathfrak{m}_{g,k}$) for the moduli space of stable maps to X of genus g with k marked points, $\beta \in H_2(X, \mathbb{Z})$.

Now we want to construct $\varphi^\bullet : \mathcal{F}^\bullet \to L^\bullet$ for GW theory. We want to work relative to the Artin stack $\mathcal{M}/\mathfrak{m}$.

We have a universal diagram

$$\begin{tikzcd} & X \\ \mathcal{C} \arrow[ur, "f"] \arrow[r] \arrow[d, "\pi"] & \underline{\mathcal{C}} \arrow[d] \\ \mathcal{M} \arrow[u, bend left, "\underline{x}"] \arrow[r] & \mathfrak{m} \end{tikzcd} \tag{3.103}$$

where the square is cartesian. We have a complex in degree -1 and 0:

$$\omega_\pi = \left[\omega_{\mathcal{C}/\mathcal{M}} \to 0 \right] \ . \tag{3.104}$$

We have

$$f^* \Omega_X = Lf^* L^\bullet_X \to L^\bullet_{\mathcal{C}} \to L^\bullet_{\mathcal{C}/\bar{\mathcal{C}}} = \pi^* L^\bullet_{\mathcal{M}/\mathfrak{m}} \tag{3.105}$$

since $\mathcal{C}/\mathfrak{m}$ is flat. Then taking $R\pi_*(\cdot \otimes \omega_\pi)$ we get

$$\varphi^\bullet : R\pi_*(f^*\Omega_X \otimes \omega_\pi) \to R\pi_* \left(\pi^* L^\bullet_{\mathcal{M}/\mathfrak{m}} \otimes \omega_\pi \right) = L_{\mathcal{M}/\mathfrak{m}} \otimes R\pi(\omega_\pi) \xrightarrow[\simeq]{\mathrm{tr}} \mathcal{O}_{\mathcal{M}} \tag{3.106}$$

So we just need to show:

Lemma 3.11. *$R\pi_*(f^*\Omega_X \otimes \omega_\pi)$ is quasi-isomorphic to a two term complex of locally frees $\left[\mathcal{F}^{-1} \to \mathcal{F}^0 \right]$.*

PROOF.

$$\mathcal{L} \coloneqq \omega_{\mathcal{C}/\mathcal{M}}\left(\underline{x}\right) \otimes f^{*}\left(\mathcal{O}_X\left(1\right)\right)^{\otimes 3} \tag{3.107}$$

is relatively ample for $\mathcal{C}/\mathcal{M}$. Write $E \coloneqq f^{*}\Omega_X \otimes \omega_\pi$. For $N \gg 0$ we have

- $\pi^{*}\pi_{*}\left(E \otimes \mathcal{L}^{\otimes N}\right) \to E \otimes \mathcal{L}^{\otimes n}$ is surjective.
- $R^1\pi_{*}\left(E \otimes \mathcal{L}^{\otimes N}\right) = 0$.
- For all geometric points $s \to \mathcal{M}$ $H^0\left(C_s, \mathcal{L}^{\otimes -N}\right) = 0$.

$\square$

Define

$$\mathcal{F} \coloneqq \pi^{*}\pi_{*}\left(E \otimes \mathcal{L}^{\otimes N}\right) \otimes \mathcal{L}^{\otimes -N}\ . \tag{3.108}$$

This is in fact a vector bundle. If we define $H = \ker\left(F \twoheadrightarrow E\right)$, then we have an exact sequence of vector bundles

$$0 \to H \to \mathcal{F} \to E \to 0 \tag{3.109}$$

is exact.

Note that

$$H^0\left(C_s, \mathcal{F}\right) = H^0\left(C_s, \pi_{*}\left(E \otimes \mathcal{L}^N\right)\right) \tag{3.110}$$

$$= \underbrace{H^0\left(C_s, \mathcal{L}_s^{-N}\right)}_{=0} \otimes \pi_{*}\left(E \otimes \mathcal{L}^N\right)_s = 0\ . \tag{3.111}$$

This implies $\pi_{*}\mathcal{F} = 0$, which implies $\pi_{*}H = 0$. Therefore $R^1\pi_{*}\mathcal{F}$ and $R^{-1}\pi_{*}H$ are locally free. By (3.109) we have

$$R\pi_{*}E = \left[R^1\pi_{*}H \to R^1\pi_{*}\mathcal{F}\right]\ . \tag{3.112}$$

4.5. GW-invariants. Let X be smooth, $\dim X = n$. Fix g, k, and β. Then

$$\left[\mathcal{M}_{g,k}\right]_{\mathrm{virt},\varphi^{\bullet}} \in A_d\left(\mathcal{M}_{g,k}\left(X, \beta\right)\right)\ . \tag{3.113}$$

This is invariant udner deformations of X. Explicitly, the dimension is:

$$d = \underbrace{3g - 3 + k}_{\dim \mathfrak{m}_{g,k}} + c_1\left(X\right)\beta + n\left(1 - g\right) \tag{3.114}$$

where the second term comes from Riemann-Roch for vector bundle of rank n, $\deg = c_1\left(X\right)$, $\beta = \deg f_s^{*}\Theta_X$. I.e.

$$d = c_1\left(X\right)\beta + \left(n - 3\right)\left(1 - g\right) + k \tag{3.115}$$

This is called the expected dimension, i.e. it is $\operatorname{rank}\left(\mathcal{F}_n\right) - \operatorname{rank}\left(\mathcal{F}_0\right) + \dim Y$ for $\varphi^{\bullet} : \mathcal{F}^{\bullet} \to L^{\bullet}_{X/Y}$.

For CY 3-folds, $n = 3$ and $k = 0$, so $d = 0$. Lecture 16; October 22, 2019

For other cases, we have two natural maps:

$$
\begin{aligned}
\mathcal{M}_{g,k}(X,\beta) &\xrightarrow{\quad\text{ev}\quad} \underbrace{X\times\ldots\times X}_{k}\\
(C,\underline{x}=(x_1,\ldots,x_k),f) &\xmapsto{\text{ev}} (f(x_1),\ldots,f(x_k))\\
\mathcal{M}_{g,k}(X,\beta) &\xrightarrow{\quad p\quad} \mathcal{M}_{g,k}(C,\underline{x})\\
(C,\underline{x},f) &\longmapsto (X,\underline{x})^{\text{st}}
\end{aligned}
\tag{3.116}
$$

where $\mathcal{M}_{g,k}$ is the DM-stack of stable curves.

Then we get the GW-invariants. For $\alpha_1,\ldots,\alpha_k\in H^*(X)$, $\gamma\in H^*(\mathcal{M}_{g,k})$

$$
\int_{[\mathcal{M}_{g,k}(X,\beta)]_{\text{virt}}} \text{ev}^*(\alpha_1\times\ldots\times\alpha_k)\smile\gamma =: \langle\alpha_1,\ldots,\alpha_k;\gamma\rangle_{\text{GW}}^{g,k,\beta}
\tag{3.117}
$$

4.6. Deformation invariance of GW. Consider a family

$$
\begin{array}{c} X\\ \downarrow^{q}\\ S \end{array}
\tag{3.118}
$$

of proper smooth varieties. We want to compare GW theory of fibers, and morally we want to say they are the same.

$\beta\in H_2(X_s)$ obtain a class in $H_2(X_{s'})$ for all s' (well-defined up to monodromy). Write $\iota_s : X_s \hookrightarrow X$. Take $\mathcal{M}=\mathcal{M}_{g,k}(X,(i_s)_*\beta)$. Then

$$
\begin{tikzcd}
 & \mathcal{M} \arrow[d, "s"] \\
s \arrow[r, hook, "j_s"] & S
\end{tikzcd}
\tag{3.119}
$$

and we can pull back our class

$$
[\mathcal{M}_S]_{\text{virt}} \coloneqq j_s^!\,[\mathcal{M}]_{\text{virt}}\ .
\tag{3.120}
$$

Proposition 3.12. *If S is lci, then for all s, s' we get an algebraic equivalence*

$$
[\mathcal{M}_s]_{virt}\sim[\mathcal{M}_{s'}]_{virt}
\tag{3.121}
$$

in $A_*(\mathcal{M})$. In particular, GW invariants of X_s and $X_{s'}$ agree.

CHAPTER 4

Toric geometry

We will work over a field[4.1] k. The slogan is that toric geometry is

the study of equivariant compactifications of $(\mathbb{G}_m)^n$.

Recall $\mathbb{G}_m = \operatorname{Spec} k\left[z, z^{-1}\right]$ is the algebraic geometer's version of $\mathbb{C}^\times$. The point will be that $(\mathbb{G}_m)^n \subset X$ is captured by combinatorial geometry ($\mathbb{Z}$-cones). References are [**7**, **10**, **24**].

1. Monoids and cones

Let M be a free abelian group of rank d ($\simeq \mathbb{Z}^d$). Write $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ for the dual. Write

$$M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Q}) \qquad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \ . \tag{4.1}$$

Similarly

$$N \subset N_{\mathbb{Q}} \subset N_{\mathbb{R}} \ . \tag{4.2}$$

Definition 4.1. Let V be a vector space over $\mathbb{Q}$ or $\mathbb{R}$. $C \subset V$ is called a *cone* if $0 \in C$, and for all $\lambda > 0$ we have $\lambda C \subset C$. We say C is *convex* if $C + C \subset C$.

Remark 4.1. C is convex iff $\operatorname{Cl}(C) \subset V_{\mathbb{R}}$ is a convex set.

Lemma 4.1. *If $C \subset M_{\mathbb{R}}$ is a convex cone then $(C \cap M, +)$ is a saturated monoid. The additive unit is the origin, and this is commutative.*

Recall saturated tells us that for all $m \in M$ and for all $\lambda \in \mathbb{N} \setminus \{0\}$, $\lambda m \in C$ implies $m \in C$. So the group is trivial.

Example 4.1. $M = \mathbb{Z}$, $P = 2\mathbb{N} + 3\mathbb{N}$. $1 \in M$, $2 \in P$, but $1 \notin P$, so this is not saturated.

Example 4.2. Consider $C = \mathbb{R}_{\geq 0}(0,1) + \mathbb{R}_{\geq 0}(k+1, -1) \subset \mathbb{R}^2$. Then

$$C \cap M = \langle (0,1), (1,0), (k+1, -1) \rangle \ . \cong \mathbb{N}^3 / \left\langle \begin{pmatrix} 0 \\ k+1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \tag{4.3}$$

I.e. the only relation is $u + v = (k+1)w$.

It is a general fact that $M \subset \mathbb{Z}^d$ satisfies $M \simeq \mathbb{N}^d$ iff there exists $m_1, \ldots, m_d \in M$ such that $\det(m_1, \ldots, m_d) = \pm 1$. So the proof follows from

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \det \begin{pmatrix} k+1 & 1 \\ -1 & 0 \end{pmatrix} = 1 \ . \tag{4.4}$$

[4.1] One should think algebraically closed, characteristic 0, but toric geometry will work for many other choices. Even $\mathbb{Z}$.

2. Monoid rings

DEFINITION 4.2. Let S be a monoid, then

$$k\left[S\right] \coloneqq \left(\left\{\sum_{\text{fin.}} a_m \chi^m\right\} \middle| \chi^m \cdot \chi^{m'} = \chi^{m+m'}\right) \tag{4.5}$$

is the *associated monoid ring.*

The first question is when $k\left[S\right]$ is finitely generated. It is sufficient for S to be a finitely generated monoid.

EXAMPLE 4.3. Set $C = 0 \cup (\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0})$. Then $S = C \cap M$ is not finitely generated since $(a, 1)$ are generators for all $a \in \mathbb{N}$.

3. Rational polyhedral cones and the Lemma of Gordan

DEFINITION 4.3. A cone $C \subset M_{\mathbb{R}}$ is a *rational polyhedral* if there are $m_1, \ldots, m_r \in M_{\mathbb{Q}}$ with

$$C = \mathbb{R}_{\geq 0} m_1 + \ldots + \mathbb{R}_{\geq 0} m_r \ . \tag{4.6}$$

Note this implies $C \subset M_{\mathbb{R}}$ is closed and convex.

Proposition 4.2 (Lemma of Gordon)**.** *If $C \subset M_{\mathbb{R}}$ is rational polyhedral, then $C \cap M$ is finitely generated.*

Note this also implies $k\left[C \cap M\right]$ is a finitely generated k-algebra.

PROOF. By the assumption that we have such m_i, we get that

$$K = \left\{\sum_{i=1}^{r} t_i m_i \middle| 0 \leq t_i \leq 1\right\} \subset M_{\mathbb{R}} \tag{4.7}$$

is compact. This means $K \cap M$ is finite.

CLAIM 4.1. $K \cap M$ generates $C \cap M$.

Consider $u = \sum r_i m_i$ for $r_i \geq 0$. Then we can write $r_i = s_i + t_i$ for $s_i \in \mathbb{N}$, $t_i \in [0, 1]$. This implies

$$u = \sum \underbrace{s_i m_i}_{\in K \cap M} + \sum \underbrace{t_i m_i}_{K \cap M} \tag{4.8}$$

so we are finished. □

Lecture 17; October 24, 2019

4. Facts from convex geometry

Now we list some definitions/propositions. Let $\sigma \subset N_{\mathbb{R}}$ be a polyhedral cone.

(a) The *dual cone* is:

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \,|\, \forall v \in \sigma, \langle u, v\rangle \geq 0\} \tag{4.9}$$

FACT 3.
- *σ polyhedral implies $\sigma^{\vee}$*
- $\sigma = \left(\sigma^{\vee}\right)^{\vee}$
- $\left(\sigma_1 \cap \sigma_2\right)^{\vee} = \sigma_1^{\vee} + \sigma_2^{\vee}$

(b) $\sigma + (-\sigma) = \mathbb{R}\sigma \subset N_{\mathbb{R}}$ is the *linear subspace spanned by* σ. We define $\dim_{\mathbb{R}} \sigma := \dim_{\mathbb{R}} \mathbb{R} \cdot \sigma$.

(c) σ is called *sharp* if $\sigma \cap (-\sigma) = \{0\}$.

FACT 4. *TFAE:*
(i) σ *is sharp*
(ii) $\dim \sigma^{\vee} = d$
(iii) $\exists u \in \sigma^{\vee}$, $\sigma \cap u^{\perp} = \{0\}$

In Toric geometry, cones in $N_{\mathbb{R}}$ are always sharp and cones in $M_{\mathbb{R}}$ are always d-dimensional.

(d) Reduction of non-sharp cones: for $C = \sigma^{\vee}$, $L = C \cap (-C) = \sigma^{\perp} = (\mathbb{R}\sigma)^{\perp}$. This is the largest linear subspace contained in C. Moreover, there exists a sharp cone $\bar{C} \subset M_{\mathbb{R}}/L$. Note this is the image $q(C)$ where $q : M_{\mathbb{R}} \to M_{\mathbb{R}}/L$, so $C = q^{-1}(\bar{C})$. Now we split (non-canonically) $M_{\mathbb{R}} \simeq L \oplus M'_{\mathbb{R}}$ and $M_{\mathbb{R}} \xrightarrow{\simeq} M_{\mathbb{R}/L}$. In other words $C \simeq \bar{C} \times L$.

(e) A cone $\sigma \subset N_{\mathbb{R}}$ is polyhedral iff σ is the intersection of finitely many half spaces.

SKETCH PROOF. $\sigma^{\vee} = \sum \mathbb{R}_{\geq 0} u_i$ is equivalent to $\sigma = \bigcap_i H_i$ for $H_i = (\mathbb{R}_{\geq 0} u_i)^{\vee}$. □

(f) $u^{\perp} \subset N_{\mathbb{R}}$, $u \neq 0$, is called a *supporting hyperplane* if $\sigma \subset (\mathbb{R}_{\geq 0} u)^{\vee}$.

(g) A *face* $\tau \subseteq \sigma$ is $u^{\perp} \cap \sigma$ with $u \in \sigma^{\vee}$.

FACT 5.
- *Any face is a rational polyhedral cone.*
- *The topological boundary* $\partial\sigma$ *is the union of faces* $\tau \subsetneq \sigma$ *in* $\mathbb{R} \cdot \sigma \subset N_{\mathbb{R}}$.
- *Let* $\tau_1, \tau_2 \subset \sigma$ *be faces. Then* $\tau_1 \cap \tau_2 \subset \sigma$ *is a face.*
- *For* $\tau_1 \subseteq \tau_2$ *faces and* $\dim \tau_1 = \dim \tau_2$ *we actually have* $\tau_1 = \tau_2$.
- $\min(\dim \tau) = \dim_{\mathbb{R}} \sigma \cap (-\sigma)$ *where* min *ranges over faces* τ.

(h) $\tau \subseteq \sigma$ is a *facet* if $\dim \tau = \dim \sigma - 1$.

FACT 6.
- *Any face* $\tau \subset \sigma$ *is an intersection of facets.*
- *Any face* $\omega \subset \sigma$ *of codimension* 2 *(i.e.* $\dim \omega = \dim \sigma - 2$*) is contained in exactly* 2 *facets.*

(i) A face of dim $= 1$ is called an *extremal ray* If

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_r \tag{4.10}$$

then any extremal ray is of the form $\mathbb{R}_{\geq 0} n_i$ for some i. r is minimal iff each $\mathbb{R}_{\geq 0} n_i$ is an extremal ray.

(j) The map taking faces $\tau \subset \sigma$ to faces $C \subseteq \sigma^{\vee}$ taking $\tau \mapsto \tau^{\perp} \cap \sigma^{\vee}$ is an inclusion reversing and dimension reversing bijection.

5. Affine toric varieties

Fix a field k.

DEFINITION 4.4. A d-dimensional *affine toric variety* is a scheme of the form $\operatorname{Spec} k[\sigma^{\vee} \cap M]$ for $\sigma \in N_{\mathbb{R}}$ a rational polyhedral cone.

Note that since $k[\sigma^{\vee} \cap M] \subset k[M]$ is a localization, we have that $\mathbb{T}_N = \operatorname{Spec} k[M] \subset \operatorname{Spec} k[\sigma^{\vee} \cap M]$ is an open embedding.

REMARK 4.2. The lemma of Gordan implies that these affine toric varieties are of finite type over k.

Proposition 4.3. *If $m_1, \dots, m_s \in M$ generated $\sigma^\vee \cap M$ as a monoid, then*

$$I = \ker\left(q : k\left[x_1, \dots, x_s\right] \twoheadrightarrow k\left[\sigma^\vee \cap M\right]\right) = \ker\left(x_1^{a_1} \dots x_s^{a_s} \mapsto \chi^{a_1 m_1 + \dots + a_s m_s}\right) \tag{4.11}$$

is generated by

$$G = \left\{ x_1 a^1 \dots x_s^{a_s} - x_1^{b_1} \dots x_s^{b_s} \,\middle|\, \sum a_i m_i = \sum b_i m_i ; a_i, b_i \geq 0 \right\} \ . \tag{4.12}$$

Proof. The map q is homogeneous where

$$\deg x_i \coloneqq m_i \qquad\qquad \deg \chi^m = m \ . \tag{4.13}$$

This implies I is an M-homogeneous ideal, which implies I is generated by M-homogeneous expressions.

For $f = \sum \alpha_A x^A \in I$ homogeneous,

$$\sum \alpha_i m_i = m \qquad\qquad q\left(f\right) = \left(\sum \alpha_A\right) \chi^m \tag{4.14}$$

so $f \in I$ iff $\sum \alpha_A = 0$. Descending induction on the number of the nonzero a_is, replace x^A by $x^{A'}$ via $x^A - x^{A'} \in G \subseteq I$. □

Remark 4.3. Ideals generated by elements of the form of the elements of G are called *toric.*

6. k-rational points

Recall for a scheme X, and any field K, we can form the set of K-valued points (the scheme theoretic points)

$$X\left(K\right) \coloneqq \left\{ x \in X \mid \kappa\left(x\right) = \mathcal{O}_{X,x} / \mathfrak{m}_x \simeq K \right\} \ . \tag{4.15}$$

This is sometimes called the maximal spectrum m-Spec.

Remark 4.4. For k algebraically closed, and X/k is of finite type, then $X\left(k\right)$ is just the closed points.

Proposition 4.4. *If $X = \operatorname{Spec} k\left[\sigma^\vee \cap M\right]$ then*

$$X\left(k\right) = \operatorname{Hom}_{\mathbf{Mon}}\left(\sigma^\vee \cap M, \left(k, \cdot\right)\right) \ . \tag{4.16}$$

Note this is the multiplicative monoid $\left(k, \cdot\right)$.

Proof. We know

$$X\left(k\right) = \text{m-Spec}\left(k\left[\sigma^\vee \cap M\right]\right) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}\left(k\left[\sigma^\vee \cap M\right], k\right) \tag{4.17}$$

where we regard k as a ring. Then we map $\Phi \to \varphi$ to get

$$\operatorname{Hom}_{\mathbf{Mon}}\left(\sigma^\vee \cap M, \left(k, \cdot\right)\right) \ . \tag{4.18}$$

□

Example 4.4. Let $C = M_{\mathbb{R}}$. Now we consider $\operatorname{Hom}\left(M, \left(k, \cdot\right)\right)$. Everything in M is invertible, so this is the same as $\operatorname{Hom}_{\mathbf{Mon}}\left(M, k^\times\right) = N \otimes_{\mathbb{Z}} k^\times \simeq \left(k^\times\right)^d$.

Example 4.5. Let $C = \mathbb{R}^2_{\geq 0}$.

$$k\left[\sigma^\vee \cap M\right] = k\left[x, y\right] \tag{4.19}$$

where $x = \chi^{(1,0)}$ and $y = \chi^{(0,1)}$. Each φ takes $x \mapsto \lambda$ and $y \mapsto \mu$. This gives us that

$$\operatorname{Hom}\left(\sigma^\vee \cap M, \left(k, \cdot\right)\right) = k^2 \ . \tag{4.20}$$

Let $X = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. In fact, there exists a map

$$X(\mathbb{C}) \to \sigma^\vee \tag{4.21}$$

which is a kind of "momentum map". The fibers over points $p \in \sigma^\vee$ contained in the interior of a k-dimensional face $\tau \subseteq \sigma$ are isomorphic to $\left(S^1\right)^{\dim \tau}$.

EXAMPLE 4.6. Consider the map $\mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}$. Map $(\lambda, \mu) \mapsto (|\lambda|, |\mu|)$.

7. Toric open subschemes

Recall the following. For $f \in R$, $D_f = \operatorname{Spec} R_f \subset \operatorname{Spec} R$ is a fundamental open subset.

Proposition 4.5. *Let $\sigma \subset N_\mathbb{R}$ be a rational polyhedral cone and $\tau \subseteq \sigma$ a face. Then $\sigma^\vee \hookrightarrow \tau^\vee$ defines a homomorphism*

$$k[\sigma^\vee \cap M] \hookrightarrow k[\tau^\vee \cap M] \ . \tag{4.22}$$

The statement is that this is a localization at one element. In particular

$$U_\tau := \operatorname{Spec} k[\tau^\vee \cap M] \hookrightarrow \operatorname{Spec} k[\sigma^\vee \cap M] =: U_\tau = X \tag{4.23}$$

is a fundamental open subscheme.

PROOF. Write $\tau = \sigma \cap m^\perp$, for $m \in \sigma^\vee \cap M$. This means $m(\sigma \setminus \tau) \subset \mathbb{N} \setminus \{0\}$. Then we claim:

$$\tau^\vee \cap M = \sigma^\vee \cap M + \mathbb{N} \cdot (-m) \tag{4.24}$$

The inclusion $\supseteq$ is clear. So consider $u \in \tau^\vee \cap M$. We know $u + \lambda m \in \sigma^\vee$ for $\lambda \gg 0$. Write $\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s$. So we need to show that for all i $\langle u + \lambda m, n_i \rangle \geq 0$ for $\lambda \gg 0$. Write $\langle u + \lambda m, n_i \rangle = \langle u, n_i \rangle + \lambda \langle m, n_i \rangle$. Then we have two cases. If $n_i \in \tau$ we have that these terms are individually zero, and the whole thing is > 0 for $n_i \in \sigma \setminus \tau$. $\langle u + \lambda m, n_i \rangle \geq 0$. Therefore for all $u \in \tau^\vee \cap M$ we have $u + \lambda m \in \sigma^\vee$ for $\lambda \gg 0$. This implies $k[\tau^\vee \cap M] = k[\sigma^\vee \cap M]_{\chi^m} \subset k[M]$. ☐

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DEFINITION 4.5. $U_\tau := \operatorname{Spec} k[\tau^\vee \cap M] \subset X$ is a fundamental open set.

EXAMPLE 4.7. Let $\sigma = \mathbb{R}^2_{\geq 0} \subseteq N_\mathbb{R} = \mathbb{R}^2$. Write τ_1 for the vertical face, τ_2 for the horizontal face, and 0 for the origin as a face. Then $\tau_1^\vee$ is the upper half plane, and $\tau_2^\vee$ is the right half plane. $0^\vee$ is the whole plane. Then

$$U_{\tau_1} = \operatorname{Spec} k[x, y]_x = \mathbb{G}_m \times \mathbb{A}^1 \tag{4.25}$$

$$U_{\tau_2} = \operatorname{Spec} k[x, y]_y = \mathbb{A}^1 \times \mathbb{G}_m \tag{4.26}$$

$$U_0 = \operatorname{Spec} k[x, y]_{x,y} = \mathbb{G}_m^2 \ . \tag{4.27}$$

8. Rational functions, dimension and normality

Proposition 4.6. *Consider a rational polyhedral cone $C \subset M_\mathbb{R}$. Write*

$$L = \mathbb{R} \cdot C = C + (-C) \ . \tag{4.28}$$

Then

(a)

$$\operatorname{Quot} k[C \cap M] = \operatorname{Quot} k[L \cap M] = k(L \cap M) \tag{4.29}$$

(b) Any affine toric variety is normal.

(c) $\dim(\operatorname{Spec} k[C \cap M]) = \dim_{\mathbb{R}} L$.

PROOF. (a) This is clear.

(b) For $C = v_1^\vee \cap \ldots \cap v_r^\vee$ an intersection of half-spaces we have

$$C \cap M = \bigcap_{i=1}^{r} (M \cap v_i^\vee) \tag{4.30}$$

which implies

$$k[C \cap M] = \bigcap k[M \cap v_i^\vee] \subset k[M] \ . \tag{4.31}$$

The intersection of normal rings is normal, and

$$k[M \cap v_i^\vee] \simeq k[x_1, \ldots, x_{d-1}, x_d]_{x_d} \tag{4.32}$$

is the localization of a polynomial ring and hence normal so we are done.

(c) $\operatorname{Spec} k[C \cap M]$ contains $\operatorname{Spec} k[L \cap M] \simeq \mathbb{G}_m^{\dim_{\mathbb{R}} L}$ as an open dense subset.

□

9. Toric strata

These are the closures of $\mathbb{G}_m^d$ orbits.

Lemma 4.7. *Let $\sigma \subset N_{\mathbb{R}}$ be a rational, sharp polyhedral cone. Write $\dim N_{\mathbb{R}} = d$. Take $\tau \subset \sigma$ to be a face with $L = \mathbb{R}\tau = \tau + (-\tau)$ the linear span of τ. Then*

$$\tau^\perp \cap \sigma^\vee = L^\perp \cap \sigma^\vee \tag{4.33}$$

is the dual cone of σ/L in $N_{\mathbb{R}}/L$. σ has an image σ/L in $N_{\mathbb{R}}/L$ which is again sharp.

So we have $\sigma \twoheadrightarrow \sigma/L$, and $(\sigma/L)^* = \tau^\perp \cap \sigma^\vee$. Therefore, dualizing yields a map

$$j\left[\left(\tau^\perp \cap \sigma^\vee\right) \cap M\right] \hookrightarrow k[\sigma^\vee \cap M] \ . \tag{4.34}$$

This gives rise a dominant[4.2] morphism

$$\underbrace{\operatorname{Spec} k[\sigma^\vee \cap M]}_{=X=U_\sigma} \xrightarrow{\pi} \underbrace{\operatorname{Spec} k\left[\left(\tau^\perp \cap \sigma^\vee\right) \cap M\right]}_{U_{\sigma/L}} \ . \tag{4.35}$$

This map is as follows. For $L \subset N_{\mathbb{R}}$ rational, we get $M_{\mathbb{R}} \twoheadrightarrow M_{\mathbb{R}}/L^\perp$ which gives rise to a torus

$$\mathbb{T}_L = \operatorname{Spec} k\left[M/\left(L^\perp \cap M\right)\right] \subset \mathbb{T} = \mathbb{G}_m^d \ . \tag{4.36}$$

This torus has an action on the fibers of this map. So it turns out, π is the categorical quotient[4.3] of X by $\mathbb{T}_L$.

Now we want to construct a section $\iota : U_{\sigma/L} \to U_\sigma$. We know $\tau^\perp \cap \sigma^\vee \subset \sigma^\vee$ is a face. Then we define the ideal

$$I_\tau = (\chi^m)_{m \in (\sigma^\vee \cap M) \setminus \tau^\perp} \subset k[\sigma^\vee \cap M] \tag{4.37}$$

and the quotient by it defines the map ι^*. We want to see this pre-composes with π to get an isomorphism on $U_{\sigma/L}$. Recall we have

$$k\left[\left(\tau^\perp \cap \sigma^\vee\right) \cap M\right] \xrightarrow{\pi^*} k[\sigma^\vee \cap M] \xrightarrow{\iota^*} k[\sigma^\vee \cap M]/I_\tau \tag{4.38}$$

[4.2] I.e. it has a dense image.

[4.3] The point is we take the closures of the orbits so we at least get a Hausdorff space.

which induces an obvious bijection between the k-vector space basis elements given by monomials $\left(\tau^{\perp}\cap\sigma^{\vee}\right)\cap M$, which implies $\pi\circ\iota=\mathrm{id}$.

Remark 4.5. Note that $k\left[\left(\tau^{\perp}\cap\sigma^{\vee}\right)\cap M\right]=k\left[\sigma^{\vee}\cap M\right]^{\mathbb{T}_L}$ and for $m\in\sigma^{\vee}\cap M$ and $q:M\to M/L$ $\deg_{\mathbb{T}_L}\chi^m=q\left(m\right)$.

Definition 4.6. Closed subschemes of the form $U_{\sigma/L}\subseteq U_\sigma$ are called *toric strata* (in particular, toric varieties themselves for $\mathbb{T}_L=\operatorname{Spec}k\left[\tau^{\perp}\cap M\right]$).

Example 4.8. Take $\tau=\mathbb{R}_{\geq 0}\times\{0\}\subset\mathbb{R}^2_{\geq 0}=\sigma$. $\tau^{\perp}$ is the horizontal half line sitting inside of $\sigma^{\vee}$. Writing the generators of $\sigma^{\vee}$ as x and y we get

$$k\left[\left(\tau^{\perp}\cap\sigma^{\vee}\right)\cap M\right]=k\left[Y\right]\hookrightarrow k\left[x,y\right] \tag{4.39}$$

which corresponds to the map $\mathbb{A}^2\to\mathbb{A}^1$ sending $(x,y)\mapsto y$. In this case $U_{\sigma/L}=Z\left(x\right)$, i.e. the y-axis.

Example 4.9. Take

$$\sigma^{\vee}=\mathbb{R}_{\geq 0}\left(1,0,0\right)+\mathbb{R}_{\geq 0}\left(1,1,0\right)+\mathbb{R}_{\geq 0}\left(1,1,1\right)+\mathbb{R}_{\geq 0}\left(1,0,1\right)\subseteq\mathbb{R}^3\ . \tag{4.40}$$

Write the generators as x, z, y, and w respectively. Then

$$k\left[\sigma^{\vee}\cap M\right]=k\left[x,y,z,w\right]/\left(xy-zw\right) \tag{4.41}$$

since xy and zw are both $\chi^{(2,1,1)}$. We have

$$\sigma^{\vee}\cap\tau^{\perp}=\mathbb{R}_{\geq 0}\left(1,0,0\right)+\mathbb{R}_{\geq 0}\left(1,1,0\right) \tag{4.42}$$

which corresponds to $\tau\subset\sigma$ being an extremal ray. This means

$$k\left[\left(\sigma^{\vee}\cap\tau^{\perp}\right)\cap M\right]\simeq k\left[x,z\right] \tag{4.43}$$

and $I_\tau=(w,y)$. So then explicitly

$$\pi^*:k\left[x,z\right]\to k\left[x,y,z,w\right]/\left(xy-zw\right) \tag{4.44}$$

and

$$\iota^*:k\left[x,y,z,w\right]/\left(xy-zw\right)\xrightarrow{/(w,y)}k\left[x,z\right]\ . \tag{4.45}$$

10. Globalize: fans

Definition 4.7. A fan (in $N_{\mathbb{R}}$) is a non-empty collection Σ of sharp rational polyhedral cones $\sigma\subset N_{\mathbb{R}}$ such that

(i) for all $\sigma\in\Sigma$, and for all $\tau\subseteq\sigma$ a face, $\tau\in\Sigma$
(ii) for all $\sigma,\sigma'\in\Sigma$ implies $\sigma\cap\sigma'\subset\sigma$ (or σ') is a face.

The support of Σ is

$$\left|\Sigma\right|=\bigcup_{\sigma\in\Sigma}\sigma\ . \tag{4.46}$$

We say Σ is complex iff $\left|\Sigma\right|=N_{\mathbb{R}}$.

We will see later that completeness is equivalent to properness of the corresponding variety.

Example 4.10 ($\mathbb{P}^2$). Consider the three cones in Fig. 1. Write σ_0 for the span of $(1,0)$ and $(0,1)$, σ_1 for the span of $(0,1)$ and $(-1,-1)$ and σ_2 for the span of $(1,0)$ and $(-1,-1)$. Write τ_1 for the span of $(0,1)$, τ_2 for the span of $(1,0)$, and τ_3 for the span of $(-1,-1)$. In this case a fan is given by the σ_i, the τ_i, and the origin.

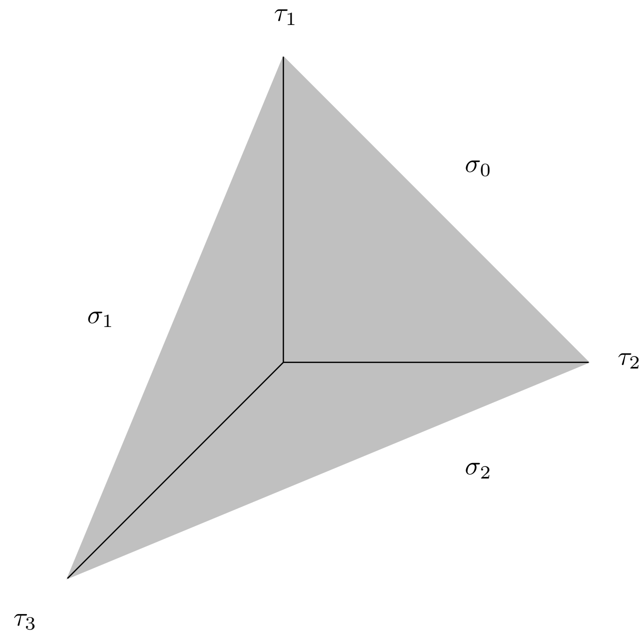

FIGURE 1. The fan in $\mathbb{A}^2$ which will eventually correspond to $\mathbb{P}^2$.

EXAMPLE 4.11. Adopt the same notation as Example 4.10 Now imagine we have the upper right quadrant $\mathbb{R}^2_{\geq 0}$, as well as the span of $(-1,-1)$. Then as a fan this is $\{\sigma_0, \tau_1, \tau_2, \tau_3, 0\}$.

EXAMPLE 4.12. For $d = 1$ Σ is either $\{\bullet\}$, $\left\{ \ \bullet \text{—} \quad \right\}$, or

$$\left\{ \quad \text{—} \bullet \text{—} \quad \right\} . \tag{4.47}$$

We now construct the associated scheme. Let $\Sigma = \{\sigma\}$ be a fan. This will give rise to $\left\{U_\sigma \supset \mathbb{G}_m^d\right\}$ affine toric varieties. Then for all $\sigma, \sigma \in \Sigma$ (WLOG maximal) we get $\sigma \cap \sigma' \subset \sigma$ and $\sigma \cap \sigma' \subset \sigma'$. This gives rise to open embeddings

$$\begin{tikzcd} & U_\sigma \\ U_{\sigma\cap\sigma'} \arrow[ur, hook] \arrow[dr, hook] & \\ & U_{\sigma'} \end{tikzcd} . \tag{4.48}$$

Then

$$X(\Sigma) \coloneqq \varinjlim_{\sigma\in\Sigma} U_\sigma \ . \tag{4.49}$$

is a scheme. We will show this compatibility explicitly for $\mathbb{P}^2$.

11. Example: $\mathbb{P}^2$

Let Σ be as in Example 4.10. Dualizing, we get the cones in Fig. 2.

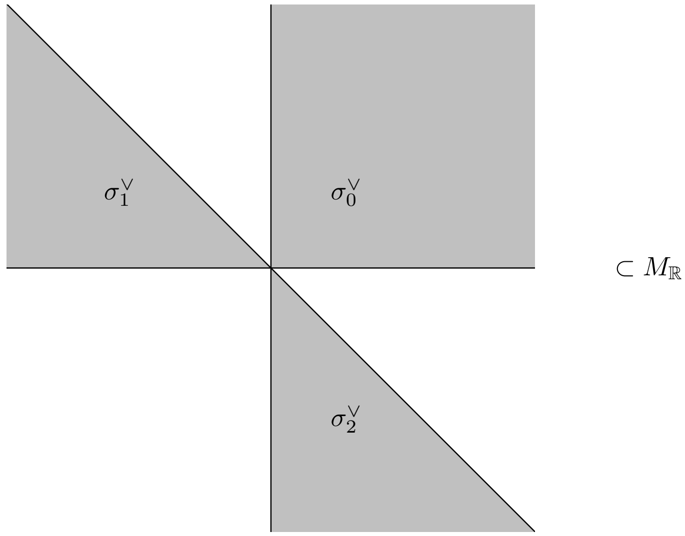

FIGURE 2. The dual cones to the fan in Example 4.10.
.

These cones give us the opens

$$U_{\sigma_0} = \operatorname{Spec} k \left[\underbrace{\chi^{(1,0)}}_{u_0}, \underbrace{\chi^{(0,1)}}_{u_1} \right] \tag{4.50}$$

$$U_{\sigma_1} = \operatorname{Spec} k \left[\underbrace{\chi^{(-1,1)}}_{v_0}, \underbrace{\chi^{(-1,0)}}_{v_1} \right] \tag{4.51}$$

$$U_{\sigma_3} = \operatorname{Spec} k \left[\underbrace{\chi^{(0,-1)}}_{w_0}, \underbrace{\chi^{(1,-1)}}_{w_1} \right] \quad . \tag{4.52}$$

Now we glue these charts together. $\sigma_0 \cap \sigma_1$ is just the positive y axis, $\sigma_0^\vee \cap \sigma_1^\vee$ is the upper half plane, which corresponds to

$$k\left[u_0, u_1\right]_{u_0} = k\left[v_0, v_1\right]_{v_1} \quad . \tag{4.53}$$

But eh n$U_{\sigma_0,\sigma_1} \hookrightarrow U_{\sigma_0}$ corresponds to

$$k\left[u_0, u_1\right] \hookrightarrow k\left[u_0, u_1\right]_{u_0} \tag{4.54}$$

and similarly $U_{\sigma_0,\sigma_1} \hookrightarrow U_{\sigma_1}$ corresponds to

$$k\left[v_0, v_1\right] \hookrightarrow k\left[v_0, v_1\right]_{V_1} \tag{4.55}$$

so these agree, and explicitly

$$v_0 = u_0^{-1} u_1 \qquad\qquad v_1 = u_0^{-1} \quad . \tag{4.56}$$

So we find that

$$X\left(\Sigma\right) \simeq \mathbb{P}^2 = \operatorname{Proj} k\left[x_0, x_1, x_2\right] \tag{4.57}$$

where explicitly:

$$u_0 = x_1.x_0 \qquad u_1 = x_2/x_0 \tag{4.58}$$

$$v_0 = u_0^{-1}u_1 = (x_0/x_1)(x_2/x_0) = x_2/x_1 \qquad v_1 = u_0^{-1} = x_0/x_1 \tag{4.59}$$

$$w_0 = u_1^{-1} = x_0/x_2 \qquad w_1 = x_1/x_2 \ . \tag{4.60}$$

12. $\mathbb{P}^d_k$ as a toric variety

Write

$$v_i = e_i = \begin{pmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{pmatrix} \in \mathbb{R}^d \tag{4.61}$$

for $i \in \{1, \dots, d\}$ and

$$v_0 = -\sum_{i=1}^{d} e_i = \begin{pmatrix} -1 \\ \dots \\ -1 \end{pmatrix} \ . \tag{4.62}$$

Then Σ is the complete fan with rats (one-dimensional cones) $\mathbb{R}_{\geq 0} v_i$. Lecture 19; October 31, 2019

13. Toric strata

Let Σ be a fan, $\tau \in \Sigma$. This gives a closed subset which is the fixed locus of the torus. Explicitly we have $\mathbb{T}_\tau = \operatorname{Spec} k[M_\tau]$, $N_\tau = \mathbb{R} \cdot \tau \subset N$. So we take

$$\operatorname{Spec}\left(k[\sigma^\vee \cap M]^{\mathbb{T}_\tau}\right) = \operatorname{Spec}\left(k\left[\left(\tau^\perp \cap \sigma^\vee\right) \cap M\right]\right) \subset \operatorname{Spec}\left(k[\sigma^\vee \cap M]\right) \ . \tag{4.63}$$

Now we globalize. Let Σ be a fan in N, $\tau \in \Sigma$. Then we get the quotiwt fan in $N/\mathbb{R}\tau$ given by

$$\Sigma_\tau = \{\sigma/\mathbb{R}\tau \subset N/\mathbb{R}\tau \mid \sigma \in \Sigma, \sigma \supseteq \tau\} \ . \tag{4.64}$$

This gives rise to a closed embedding $X(\Sigma_\tau) \subseteq X(\Sigma)$. Note that if $X(\Sigma)$ is dimension d, then $X(\Sigma_\tau)$ is of dimension $d - \dim \tau$.

Example 4.13. If we take the fan of $\mathbb{P}^3$, and let τ be the ray given by the positive z-axis. Then the quotient fan gives us $\mathbb{P}^2$.

There is a fundamental correspondence between rays and toric divisors, i.e. divisors invariant under the torus action. The affine charts are given by $\sigma \supseteq \tau$ since this gives us

$$k[\sigma^\vee] \twoheadrightarrow k\left[\underbrace{\sigma^\vee \cap \tau^\perp}_{(\sigma/\mathbb{R}\tau)^\vee}\right] \ . \tag{4.65}$$

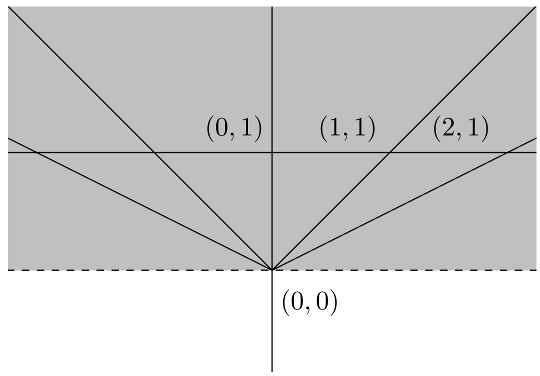

FIGURE 3. An infinite fan in $N_{\mathbb{R}} = \mathbb{R}^2$.

14. Quasi-compactness

EXAMPLE 4.14. Consider the fan in Fig. 3. Note that

$$|\Sigma| = \mathbb{R} \times \mathbb{R}_{>0} \cup \{(0,0)\} \subset \mathbb{R}^2 \tag{4.66}$$

is not closed. This gives rise to

$$X(\Sigma) = \bigcup_{\infty} \mathbb{A}^2 \tag{4.67}$$

which is *not* quasi-compact.

As it turns out, we can blow up infinitely many times to get:

$$\begin{array}{c} X(\Sigma) \\ \downarrow \mathrm{Bl}_{\infty} \\ \mathbb{A}^1 \times \mathbb{G}_m \end{array} . \tag{4.68}$$

Write π for the projection to $\mathbb{A}^1$. Then we have

$$\pi^{-1}(0) = \cup_{\infty} \mathbb{P}^1 \tag{4.69}$$

and for a generic point $\eta \in \mathbb{A}^1$ we have

$$\pi^{-1}(\eta) = \mathbb{G}_m / k(t) \ . \tag{4.70}$$

Then we have the following:

Proposition 4.8. *$X(\Sigma)$ is quasi-compact iff Σ is finite.*

PROOF. ($\Longleftarrow$): This is clear.
($\Longrightarrow$): We know

$$\bigcup_{\tau \in \Sigma \text{ a ray}} X(\Sigma_\tau) \tag{4.71}$$

can only have finitely many irreducible components. Therefore the number of $\tau \in \Sigma$ such that $\dim \tau = 1$ is finite, so Σ is finite. $\square$

15. Morphisms of fans

DEFINITION 4.8. Let Σ and Σ' be fans in N and N' respectively. A morphism of fans $\varphi : \Sigma \to \Sigma'$ is a linear map $\varphi : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ with $\varphi(N) \subseteq N'$ and such that for all $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ such that $\varphi(\sigma) \subseteq \sigma'$.

REMARK 4.6. σ' is not necessarily unique, but there exists a unique maximal one.

DEFINITION 4.9 (Construction of toris morphism). A morphism of fans $\varphi : \Sigma \to \Sigma'$ induces $\Phi : X(\Sigma) \to X(\Sigma')$ which is equivariant wrt the action of $\mathbb{G}_m(N) \to \mathbb{G}_m(N')$ as follows. Locally, for $\sigma \in \Sigma$, $\sigma' \subseteq \varphi(\sigma)$, define $\Phi : U_\sigma \to U_{\sigma'}$ by

$$\begin{array}{ccc} k\left[\sigma^\vee \cap M\right] & \xleftarrow[\varphi^*]{} & k\left[(\sigma')^\vee \cap M'\right] \\ \\ \chi^{\varphi^*(m)} & \xleftarrow[\varphi^*]{} & \chi^m \end{array} , \tag{4.72}$$

and we get

$$\begin{tikzcd} N \arrow[r] & N' & \\ \sigma \arrow[u, hook] \arrow[r, "\varphi"] & \sigma' \arrow[u, hook] \arrow[r, "m"] & \mathbb{R}_{\geq 0} \end{tikzcd} . \tag{4.73}$$

REMARK 4.7. This Φ is really simple at the level of affine opens. We just write it down in a basis.

REMARK 4.8. The map $\mathbb{G}_m(N) \to \mathbb{G}_m(N')$ maps $e = Z(\chi^m - 1) \mapsto e' = Z\left(\chi^{m'} - 1\right)_{m \in M'}$.

16. Examples

EXAMPLE 4.15. Let $N_{\mathbb{R}} = \mathbb{R}$, $N'_{\mathbb{R}} = \mathbb{R}^2$. Define the map by $\lambda \mapsto (p\lambda, q\lambda)$. Then $\varphi^* M' = \mathbb{N}^2 \to M = \mathbb{N}$ by sending $(m_1, m_2) \mapsto pm_1 + qm_2 \in \mathbb{N}$. On coordinate rings this is

$$\begin{array}{ccc} k[u] & \xleftarrow[\varphi]{} & k[x,y] \\ \\ u^p & \longleftarrow\!\shortmid & x \\ \\ u^q & \longleftarrow\!\shortmid & y \end{array} . \tag{4.74}$$

Now if $\gcd(p,q) = 1$, then $\ker(\varphi^*) = (x^q - y^p)$ Then the closure of the image of $\mathbb{T}_N$ in $\mathbb{T}_{N'} = \mathbb{G}_m^2$ is $Z(x^p - y^q)$ in $\mathbb{A}^2$.

EXAMPLE 4.16. Take the cone generated by $(1,0)$ and $(1,1)$ in $\mathbb{R}^2$.

$$N_{\mathbb{R}} = \mathbb{R}^2 \xrightarrow{\varphi = \mathrm{id}} N_{\mathbb{R}} = N'_{\mathbb{R}} = \mathbb{R}^2 \tag{4.75}$$

except the cone $\sigma' = \mathbb{R}^2_{\geq 0}$. The dual cone of σ is now the span of $(1,-1)$ and $(0,1)$ and of course $\sigma'^\vee$ is just $\mathbb{R}^2_{\geq 0}$. The map from $\sigma'^\vee \to \sigma^\vee$ is given by

$$\begin{aligned} uv &\longleftarrow x = \chi^{(1,0)} \\ v &\longleftarrow y = \chi^{(0,1)} \end{aligned} \tag{4.76}$$

This is one of the affine charts of the blowup $\mathrm{Bl}_0 \mathbb{A}^2 \to \mathbb{A}^2$:

$$\mathrm{Bl}_{(x,y)} \mathbb{A}^2 = \operatorname{Proj} k \underbrace{[x,y]}_{\deg=0} \underbrace{[U,V]}_{\deg=1} / (xV - yU) \ . \tag{4.77}$$

Now we de-homogenize, i.e. we consider the charts $U \neq 0$ and $V \neq 0$. To eliminate ym we set $U = 1$, $V/U = v$, and we map

$$\begin{aligned} k[x,y] &\longrightarrow k[x,v] \\ x &\longmapsto x \\ y &\longmapsto xv \ . \end{aligned} \tag{4.78}$$

Similarly, for the $V \neq 0$ chart we take $V = 1$, $u = U/V$. Then to eliminate x we map

$$\begin{aligned} k[x,y] &\longrightarrow k[y,u] \\ x &\longmapsto yu \\ y &\longmapsto y \end{aligned} \quad . \tag{4.79}$$

Definition 4.10. A subdivision of fans is a morphism $\tilde{\Sigma} \to \Sigma$ where $N \xrightarrow{\mathrm{id}} N = N'$, i.e. for all $\sigma' \in \Sigma'$

$$\sigma' = \bigcup_{\sigma \subseteq \varphi^{-1}(\sigma')} \varphi(\sigma) \ . \tag{4.80}$$

The upshot is that a subdivision of fans corresponds to a *modification*, i.e. a birational, proper morphism $X(\Sigma) \to X(\Sigma')$.

Example 4.17. Consider the cone σ spanned by $(0,1)$ and $(k,1)$ in $N_{\mathbb{R}} = \mathbb{R}^2$. Then the projection to the y-axis maps to $N'_{\mathbb{R}} = \mathbb{R}$. This maps to σ' given by the span of 1.

The dual cone $\sigma^\vee$ is given by the span of $(-1,k)$ and $(1,0)$. Then we get a map from $\sigma'^\vee$ (which is still the span of 1) by sending $1 \mapsto (0,1)$. This corresponds to the map

$$\begin{aligned} k[t] &\longrightarrow k[x,y,t]/\left(xy - t^k\right) \\ t = \chi^1 &\longmapsto xy \end{aligned} \tag{4.81}$$

where $x = \chi^{(1,0)}$, $y = \chi^{(-1,k)}$, and $t = \chi^{(0,1)}$. This gives rise to what is called the A_{k-1} surface singularity. This is a fibration of the A_{k-1} surface singularity by a family $\{C_t\}$ of curves with $C_t \simeq \mathbb{G}_m$ smooth for $t \neq 0$, $X_0 = \mathbb{A}^1 \cup \mathbb{A}^1$ a nodal curve .

EXAMPLE 4.18 (Change of lattice). Consider a map $N_{\mathbb{R}} = \mathbb{R}^2 \to N'_{\mathbb{R}} = \mathbb{R}^2$ defined by a matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} . \tag{4.82}$$

This corresponds to a map of rings:

$$\begin{array}{ccc} k[x,y] & \longleftarrow & k[x,y] \\ \\ x^a & \longleftarrow\!\shortmid & x \\ \\ y^a & \longleftarrow\!\shortmid & y \end{array} . \tag{4.83}$$

This corresponds to a covering branched of order b on the x-axis and of order a on the y-axis.

The upshot is that a change of lattice gives us a branched cover of $X(\Sigma')$ branched along toric strata. We can read the branch index off from the cokernel of the map $N \to N'$

EXAMPLE 4.19. Let $\tau \in \Sigma$. Take

$$\Sigma(\tau) = \{(\tau, N_\tau = \mathbb{R}\tau)\} . \tag{4.84}$$

Then we get a map

$$\begin{array}{ccc} \Sigma(\tau) & \longrightarrow & \Sigma \\ \\ N_\tau & \hookrightarrow & N \end{array} . \tag{4.85}$$

This will give an affine toric subvariety. Explicitly,

$$X(\Sigma(\tau)) = \operatorname{Spec} k\left[(\tau \subset N_\tau)^\vee \cap M_\tau\right] \to X(\Sigma) . \tag{4.86}$$

EXAMPLE 4.20. As a subexample, take $\mathbb{P}^2$ with τ a ray. Then we have

$$\mathbb{T}_{N_\tau} = \{g \in TT_N \mid \forall p \in X(\Sigma_\tau) \mid g \cdot p = p\} = \mathrm{Fix}_{X(\Sigma_\tau)} \subseteq T_N \tag{4.87}$$

and we ask for the closure of the distinguished point. This is affine, contains $[1,0,1]$, $[1,1,1]$, but not $[0,1,0]$.

In fact $\mathbb{T}_{N_\tau} \subset X(\Sigma_\tau)$ and

$$X(\Sigma(\tau)) \times \mathbb{T}_{N_\tau} \subset X(\Sigma_\tau) . \tag{4.88}$$

So it is locally a product of a torus and this affine variety.

Lecture 20;
November 5, 2019

17. Properties of toric morphisms

Adopt the notation $X(\Sigma) = TV(\Sigma, N)$. For $\varphi : \Sigma \to \Sigma'$ write $\Phi : X(\Sigma) \otimes X(\Sigma')$.

Proposition 4.9. *(a) Φ is a closed embedding iff*

$$\Sigma = \left\{ \varphi^{-1}(\sigma') \,\middle|\, \sigma' \in \Sigma' \right\} \tag{4.89}$$

(which in particular implies $N_{\mathbb{R}} \otimes N'_{\mathbb{R}}$ is injective) and for all $\sigma_1, \sigma_2 \in \varphi^{-1}(\sigma')$ the maps

$$\sigma_2^{\vee} \cap M' \twoheadrightarrow \sigma^{\vee} \cap M \tag{4.90}$$

are surjective.

(b) Φ is dominant[4.4] *iff $N'/\varphi(N)$ is a torsion group (equivalently $[N', \varphi(N)] < \infty$).*
(c) Φ is always separated.
(d) Φ is proper iff $\varphi^{-1}(|\Sigma'|) = |\Sigma|$.

Note that removing maximal cones certainly will not yield a proper morphism. The first two are quite easy, the last two follow from valuative criteria.

18. The category of toric varieties

At the moment our definition of a toric variety is a variety which comes from this fan construction. Similarly, our definition for morphisms of the is a morphism induced by a morphism of fans. This is a lazy man's definition. We now offer a more intrinsic definition of these.

Definition 4.11. A *toric variety* (over a ground field k) is a separated scheme of finite type over k which is connected and normal (in particular reduced and irreducible), together with the action of an algebraic torus $\mathbb{T}_N = \operatorname{Spec} k[N^*]$ and a $\mathbb{T}_N$ equivariant open embedding $\mathbb{T}_N \hookrightarrow X$.

Let $\mathbb{T}_N \circlearrowright X$ and $T_{N'} \circlearrowright Y$ be toric varieties. Morphisms of toric varieties will be given by:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \circlearrowright & & \circlearrowright \\ \mathbb{T}_N & \longrightarrow & \mathbb{T}_{N'} \end{array} . \tag{4.91}$$

Note the bottom morphism is equivalent to a morphism $N \to N'$.

Theorem 4.10. *The category of toric varieties is equivalent to the category of fans.*

19. Polytopes and polyhedra

Definition 4.12. A (convex) *polyhedron* $\Xi \subseteq V$ is a finite intersection of half-spaces.

Definition 4.13. A *polytope* is the convex hull of finitely many points.

Note this implies that a polytope is bounded.

Let $V = M_{\mathbb{R}}$, $M = \mathbb{Z}^n$. A rational polytope (resp. integral) if it is the convex hull of $v_i \in M_{\mathbb{Q}}$ (resp. $v_i \in M$) A polyhedron is rational if all half-spaces are $u \geq 0$ for $u \in M^* \otimes_{\mathbb{Z}} \mathbb{Q} = N_{\mathbb{Q}}$.

Theorem 4.11. *A polytope is the same as a bounded polyhedron.*

[4.4] Recall this means it has a dense image.

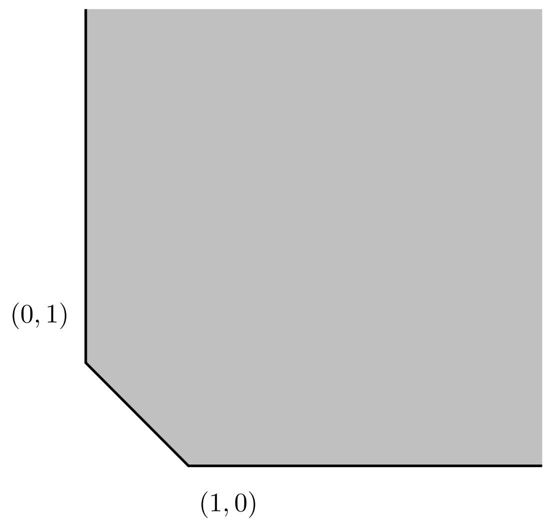

FIGURE 4. Ξ from Example 4.21.

Polyhedron and polytopes have faces as before. Codimension 1 faces are facets. They also have vertices as before. Note that they may not exist for unbounded polyhedra.

Let $\Xi \subset M_{\mathbb{R}}$ be a polyhedron. Then we can form the cone:

$$C(\Xi) = \mathrm{Cl}\left(\{(m,h) \in M_{\mathbb{R}} \times \mathbb{R} \,|\, m \in h \cdot \Xi\}\right) \ . \tag{4.92}$$

Note that if we don't take the closure, then the cone of an unbounded polyhedron would not be closed.

The *recession cone* (or asymptotic cone) of Ξ is the intersection $C(\Xi) \cap M_{\mathbb{R}} \times \{0\} \subseteq M_{\mathbb{R}}$.

EXAMPLE 4.21. Take $M_{\mathbb{R}} = \mathbb{R}^2$, and consider Ξ as in Fig. 4. Now we can form the one over this. The cone at height 0 is just a quadrant.

DEFINITION 4.14 (Dual polytopes/polyhedra). Let $\Xi \subseteq M_{\mathbb{R}}$, $0 \in \mathrm{int}(\Xi) \setminus \partial\Xi$. Define the dual (polar) polyhedron

$$\Xi^{\circ} := \{v \in N_{\mathbb{R}} \,|\, \forall u \in \Xi, \langle u, v\rangle \leq 1\} \ . \tag{4.93}$$

REMARK 4.9. We will eventually construct Calabi-Yau varieties from these, and this dualizing operation will be mirror symmetry.

REMARK 4.10. The idea is that we want $C(\Xi)^{\vee} = C(\Xi^{\circ})$.

REMARK 4.11. There is a choice of convention here where we might instead ask for $\langle u, v\rangle \geq -1$. Then this just gives negative of our dual polyhedron.

Note that proper faces $\tau \subseteq C(\Xi)$ ($\tau \not\subseteq M_{\mathbb{R}} \times \{0\}$) are in one-to-one correspondence with (proper) faces of Ξ.

EXAMPLE 4.22. Consider the polytope in Fig. 5. This has dual polytope in Fig. 5.

EXAMPLE 4.23. Consider the cube

$$\Xi = \mathrm{conv}\{(\pm 1, \pm 1, \pm 1)\} \ . \tag{4.94}$$

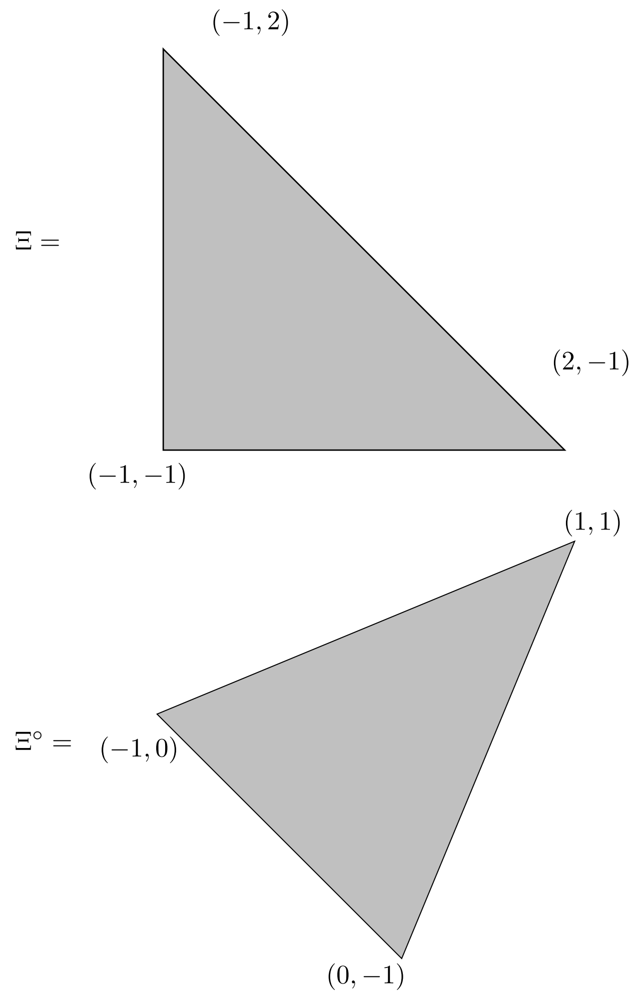

FIGURE 5. Polyhedron in $\mathbb{R}^2$ and its dual polyhedron.

The dual is

$$\Xi^\circ = \operatorname{conv}\{\pm e_1, \pm e_2, \pm e_3\} \tag{4.95}$$

and octahedron.

20. Toric varieties from polyhedra

Consider a convex, rational polyhedron $\Xi \subset M_\mathbb{R}$. For $\Xi' \subset \Xi$ a face, we get

$$\sigma_{\Xi'} = \{v \in N_\mathbb{R} \mid \langle u - u', v\rangle \geq 0, \forall u \in \Xi, u' in \Xi'\} \tag{4.96}$$

$$= \left(\operatorname{Cl}\left(\mathbb{R}_{\geq 0}\{u - u' \in M_\mathbb{R} \mid u \in \Xi, u' \in \Xi'\}\right)\right)^\vee . \tag{4.97}$$

Proposition 4.12. *(a)*

$$\Sigma_\Xi := \{-\sigma_{\Xi'} \mid \Xi' \subset \Xi \text{ a face}\} \tag{4.98}$$

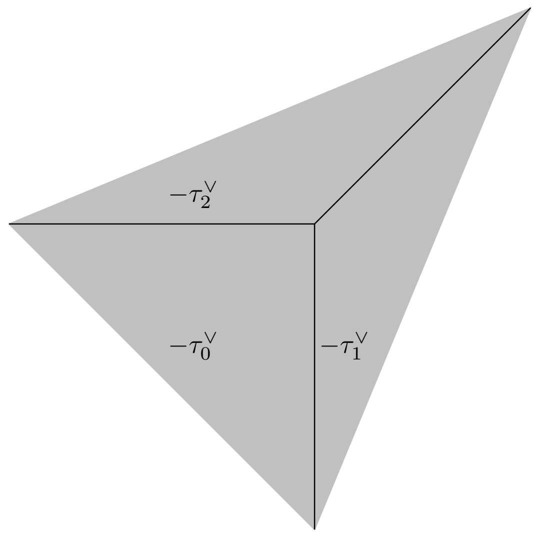

FIGURE 6. Σ_Ξ from Example 4.24.

is a fan in $N_\mathbb{R} = M_\mathbb{R}^*$. *We call this the* normal fan of Ξ. *Note* Σ_Ξ *is complete iff* Ξ *is bounded.*

(b) $0 \in \Xi$ *implies*

$$\Sigma_\Xi = \{0\} \cup \{\mathrm{Cl}\left(\{(\mathbb{R}_{\geq 0} w) \,|\, w \subseteq \Xi^\circ \text{ a face}\}\right)\} \ . \tag{4.99}$$

EXAMPLE 4.24. Consider the polytope

$$\begin{tikzcd} v_2 \arrow[d, dash] \arrow[dr, dash] & \\ v_0 \arrow[r, dash] & v_1 \end{tikzcd} \ . \tag{4.100}$$

We have three corresponding cones τ_i for each v_i. These give us exactly the three cones in Fig. 1. Then Σ_Ξ is as in Fig. 6.

REMARK 4.12. The normal fan is insensitive to rescaling of Ξ and to (small) parallel transport of faces.

DEFINITION 4.15. The *toric variety associated to* Ξ is $X(\Sigma_\Xi)$.

EXAMPLE 4.25. $\mathbb{P}^d = X(\Sigma_\Xi)$ for

$$\Xi = \mathrm{conv}\{0, e_1, \ldots, e_d\} \ . \tag{4.101}$$

For $d = 2$ this was the example Example 4.24 up to scaling. I.e. we take the convex hull of $(0,1)$ and $(1,0)$ and we get $\mathbb{P}^2$ as in Example 4.10. For $d = 3$ this corresponds to the polyhedron which is a tetrahedron.

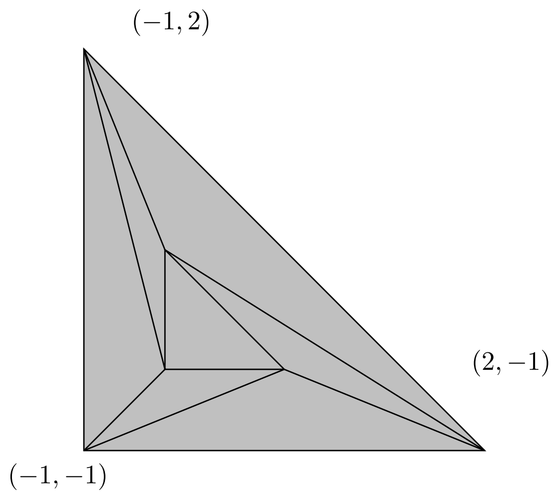

FIGURE 7. A fan is formed by taking the cone over this picture. Note this has 7 maximal cones.

EXAMPLE 4.26. Recall the Hirzebruch surfaces $X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)\right)$ for $a \geq 0$. This is given by $X(\Sigma_\Xi)$ where Ξ is

$$\begin{array}{ccccc} (0,1) & \text{———} & (1,1) & & \\ | & & & \diagdown & \\ (0,0) & \text{—————} & & & (a,1,0) \end{array} \quad . \tag{4.102}$$

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Proposition 4.13. $\operatorname{Proj}(C(\Xi) \cap (M \oplus \mathbb{Z})) = X(\Sigma_\Xi)$.

The proof of this is essentially that if you understand the construction of this Proj, this is a tautology. I.e. we dehomogenize at $\chi^{(v,1)}$ to get an open set $\operatorname{Spec} k[\sigma^\vee \cap M]$. Then the result follows.

REMARK 4.13. We have a canonical bijection

$$\{\Xi \subset M \mid \Xi\mathbb{Q}- \text{ polyhedron}\} \xrightarrow{\simeq} \left\{ (\Sigma, \varphi : |\Sigma| \to \mathbb{R}) \,\middle|\, \begin{array}{c} \Sigma \text{ fan in } N_{\mathbb{R}} \\ \varphi \text{ strictly convex, piecewise linear} \end{array} \right\} . \tag{4.103}$$

In particular the vertices v_i on the LHS correspond to maximal $\sigma \in \Sigma$, and $\varphi|_\sigma = v_i : N_{\mathbb{R}} \to \mathbb{R}$. The other direction sends functions φ to the Newton polyhedron of (Σ, φ), written $\Xi(\Sigma, \varphi)$. Note that $\Xi \mapsto \Xi + m$ corresponds to $(\Sigma, \varphi) \mapsto (\Sigma, \varphi + m)$.

REMARK 4.14. Note every fan arises in this way. In dimension 2 every fan does arise in this way, but not in dimension 3 and higher.

COUNTEREXAMPLE 2. Take the cone over Fig. 7. This does not support a strongly convex piecewise linear function. This is equivalent to the triangulation not being "regular".

EXAMPLE 4.27. Recall Example 4.21. The cone is given by:

$$C(\Xi) = \langle (1,0,0), (0,1,0), (1,0,1), (0,1,1) \rangle \ . \tag{4.104}$$

Then we have

$$k\left[C\left(\Xi\right)\cap\left(M\oplus\mathbb{Z}\right)\right]\simeq k\left[x,y\right]\left[U,V\right]/\left(xV-yU\right) \tag{4.105}$$

and taking Proj we get the blowup

$$\operatorname{Proj}\left(k\left[C\left(\Xi\right)\cap M\oplus\mathbb{Z}\right]\right)=\operatorname{Bl}_0\mathbb{A}^2\ . \tag{4.106}$$

From the point of view of fans, we take the upper right quadrant (which gives us $\mathbb{A}^2$) then we subdivide this fan to get a cone spanned by $(1,0)$ and $(1,1)$ and a cone spanned by $(1,1)$ and $(0,1)$. The point is that chopping off edges of a polyhedron corresponds to (projective[4.5]) subdivision of the corresponding fan Σ_Ξ.

21. Weil divisors on toric varieties

Let X be a d-dimensional toric variety with fan Σ and rays $\mathbb{R}_{\geq 0}v_1,\dots,\mathbb{R}_{\geq}v_r$, where the v_i are primitive vectors. Each v_i gives a $(d-1)$-dimensional toric stratum, which is the Weil divisor D_i. These are called toric prime divisors. Write

$$\operatorname{Div}^T X\subset\operatorname{Div} X \tag{4.107}$$

for the subgroup of $\mathbb{T}$-invariant Weil divisors. Note that

$$\operatorname{Div}^T X=\sum\mathbb{Z}\cdot\left[D_i\right]\ . \tag{4.108}$$

The divisor class group is:

$$\operatorname{Cl} X=A_{d-1}\left(X\right)=|X/\sim_{\text{rat'l}}\ . \tag{4.109}$$

Proposition 4.14. *The sequence*

$$\begin{array}{c} 0\longrightarrow M\longrightarrow\operatorname{Div}^T X\longrightarrow\operatorname{Cl} X\longrightarrow 0\\[1em] m\longmapsto\left(\chi^m\right)\end{array} \tag{4.110}$$

is exact, where

$$\chi^m=\sum_{i=1}^{r}\operatorname{ord}_{D_i}\chi^m D_i\ . \tag{4.111}$$

Proof. $k\left[M\right]$ is factorial (UFD) which means

$$0=\operatorname{Cl}\operatorname{Spec}k\left[M\right]=\operatorname{Cl}\mathbb{T}=\operatorname{Cl}X\setminus D \tag{4.112}$$

which implies $A_{d-1}\left(D\right)=\operatorname{Div}^T\left(X\right)\twoheadrightarrow\operatorname{Cl}X$.

Let $\sum n_iD_i\in\ker\varphi$. Then there exists some $f\in k\left(X\right)^\times$ such that $(f)=\sum n_iD_i$. This implies

$$f|_{X\setminus D}\in k\left[M\right]^\times=k^\times\times M \tag{4.113}$$

i.e. $f=\lambda\chi^m$ for $\lambda\in k^\times$ and $m\in M$. WLOG $\lambda=1$. □

[4.5] I.e. supporting a strictly convex PL function.

Note that $\operatorname{ord}_{D^i}\chi^m = \langle v_i, m\rangle$. To see this, first adapt to a basis for N given by $e_1 = v_1$, and the other e_i arbitrary. Write e_i^* for the dual basis. Then

$$\mathbb{T}_{D_i} = \operatorname{Spec} k\left[\chi^{\pm e_2^*}, \dots, \chi^{\pm e_d^*}\right] \ . \tag{4.114}$$

This sits inside the open set

$$X \supseteq U_i = \operatorname{Spec} k\left[\chi^{e_1^*}, \chi^{\pm e_2^*}, \dots, \chi^{\pm e_d^*}\right] \ . \tag{4.115}$$

Then $\chi^m = \chi^{m_1}\dots\chi^{m_d}$ where $m = \sum m_i e_i$. Then

$$m_1 = \sum \operatorname{ord}_{D_i}\chi^m = \langle m, v_i\rangle \ . \tag{4.116}$$

22. Cl $(\mathbb{P}^n)$

Corollary 4.15. $\operatorname{Cl}(\mathbb{P}^n) = A_{n-1}(\mathbb{P}^n) = \mathbb{Z}\cdot H$ *for H a hyperplane. The degree map* $\operatorname{Cl}(\mathbb{P}^n) \to \mathbb{Z}$ *is given by intersection with a line.*

Proof. We have a SES

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{m\mapsto(v_i,m)} \mathbb{Z}^{n+1} \xrightarrow{(a_i)\mapsto\sum a_i} \mathbb{Z} \longrightarrow 0 \tag{4.117}$$

and then the rays are

$$\Sigma_{\mathbb{P}^n}^{[1]} = \{\mathbb{R}_{\geq 0}e_1, \dots, \mathbb{R}_{\geq 0}e_n, \mathbb{R}_{\geq 0}(-e_1, \dots, -e_n)\} \ . \tag{4.118}$$

□

23. Cartier divisors on toric varieties

Recall the Cartier divisors are

$$\operatorname{CaDiv} X = \Gamma\left(X, \mathcal{K}^\times/\mathcal{O}^\times\right) \tag{4.119}$$

for $\mathcal{K}$ the sheaf of total quotient rings. For X integral, $\mathcal{K}$ is is a locally constant sheaf with stalks $\mathcal{K}(X)$. Then we have

$$\left\{(U_i, f_i) \,\middle|\, f_i \in \mathcal{K}(X)^\times, \forall i, j f_i/f_j \in \mathcal{O}^\times(U_i \cap U_j)\right\} \tag{4.120}$$

and $\operatorname{CaDiv} X \hookrightarrow \operatorname{Div} X$.

Lemma 4.16. *Let X be an affine toric variety given by $X = \operatorname{Spec} k[\sigma^\vee \cap M]$. Then $D \in D^T(X)$ is Cartier iff there exists $m \in M$ such that $D = (\chi^m)$.*

Proposition 4.17. *Let Σ be a fan in $N_\mathbb{R}$ with toric variety X_Σ. Then*

$$\operatorname{CaDiv}(X_\Sigma) = \operatorname{PL}(\Sigma) \coloneqq \{\varphi : |\Sigma| \to \mathbb{R} \,|\, \varphi \textit{ piecewise linear wrt } \Sigma\} \ . \tag{4.121}$$

The map sends $\sum \varphi(v_i \cdot D_i \mapsfrom \varphi)$ where the v_i are the primitive vectors of σ, in particular the rays of Σ are given by $\mathbb{R}_{\geq 0}v_i$.

24. Pic (X_Σ)

Proposition 4.18. *The Picard group* $\operatorname{Pic}(X_\Sigma) = \operatorname{PL}(\Sigma)/M = \operatorname{CaCl} X = \operatorname{CaDiv}(X)/\sim_{\mathit{ret'l}}$ *where* $\operatorname{CaCl} X$ *is the Cartier divisor class group.*

Corollary 4.19. $\operatorname{Pic}(X_\Sigma)$ *is discrete and always torsion free, i.e. for $\varphi \in \operatorname{PL}(\Sigma)$, $\varphi \in M_\mathbb{R}$ iff $\varphi \in M$.*

Lecture 22;
November 12, 2019

25. Ample toric divisors

Proposition 4.20. *Let $\varphi \in \mathrm{PL}(\Sigma)$ with $|\Sigma|$ convex and $D = \sum \varphi(v_i) D_i$ the corresponding Cartier divisor. Then $\mathcal{O}_X(D)$ is ample iff φ is strictly convex.*

PROOF. ($\Longleftarrow$): Let Ξ denote the Newton polytope. Then

$$X = X(\Sigma_\Xi) = \mathrm{Proj}\,(k[C(\Xi) \cap (M \oplus \mathbb{Z})]) \tag{4.122}$$

and $\mathcal{O}_X(D) = \mathcal{O}_X(1)$, which implies very ample.

($\Longrightarrow$): Let $\rho \in \Sigma^{[d-1]}$ be a codimension 1 cone such that $\rho \not\subseteq \partial|\Sigma|$. The "kink of φ along ρ" is $\mathcal{K}_\rho(\varphi) \in \mathbb{Z}$. Write $\rho = \sigma_1 \cap \sigma_2$ for σ_i maximal. Then

$$\varphi|_{\sigma_1} = m_1 \in M \qquad\qquad \varphi|_{\sigma_2} = m_2 \in M\ . \tag{4.123}$$

Write $\check{d}_S \in M$ for the unique primitive generator of $\rho^\perp \simeq \mathbb{Z}$ with $\check{d}_\rho\big|_{\sigma_2} \geq 0$. Then $m_2 - m_1 = \mathcal{K}_\rho(\varphi) \cdot \check{d}_\rho \in \rho^\perp$.

FACT 7. *φ is strictly convex iff for all ρ, $\mathcal{K}_\rho(\varphi) > 0$.*

FACT 8. $\mathcal{K}_\rho(\varphi) = \deg \mathcal{O}_X(D)|_{X(\Sigma_\rho) \subset X}$.

For D ample, this degree is positive. ∎

CHAPTER 5

Toric degenerations and mirror symmetry

1. The Batyrev mirror construction

1.1. Reflexive polytopes.

Definition 5.1. Let $0 \in \Xi \subset M_{\mathbb{R}}$ be a lattice polytope. This is called *reflexive* iff

(i) $\operatorname{int}(\Xi) \cap M = \{0\}$
(ii) each facet of Ξ has $\mathbb{Z}$-affine distance 1 from the origin

or equivalently

(i) $\operatorname{int}(\Xi) \cap M = \{0\}$
(ii) $\operatorname{int}(\Xi^\circ) \cap M = \{0\}$

or equivalently Ξ and Ξ° are lattice polytopes.

Note that Ξ is reflexive iff Ξ° is reflexive.

Example 5.1. Let

$$\Xi = \begin{tikzcd} (-1,2) \arrow[d, dash] \arrow[dr, dash] & \\ (-1,-1) \arrow[r, dash] & (2,-1) \end{tikzcd} . \tag{5.1}$$

Then the dual is

$$\Xi^\circ = \begin{tikzcd} & & (1,1) \arrow[dl, dash] \\ (-1,0) \arrow[urr, dash] \arrow[dr, dash] & & \\ & (0,-1) & \end{tikzcd} . \tag{5.2}$$

Note these both have a unique interior integral point.

Example 5.2. A square of width 2 centered around the origin has dual with given by rotating 45°.

There are only 16 reflexive polytopes in 2-dimensions as in Fig. 1. There are $4,319$ in 3-dimensions, and in dimension 4 we have $473,800,776$. This is where the statement that there are many different string theories comes from.

Counterexample 3. If we take the convex hull of $(-1,-1), (-1,0), (1,0), (0,2)$ this is not reflexive.

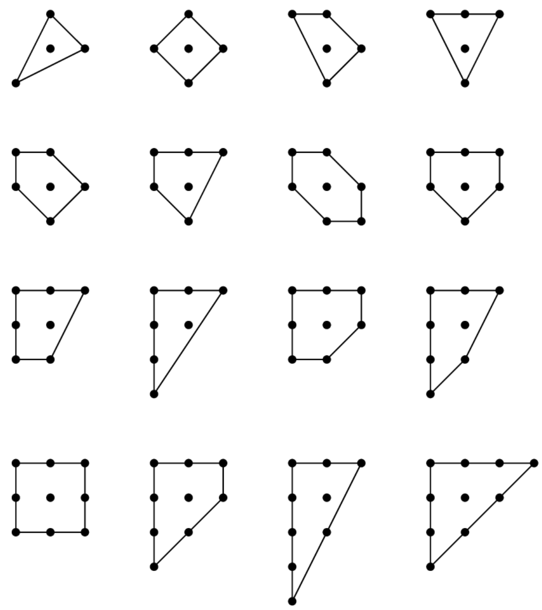

FIGURE 1. The 16 reflexive polytopes in 2-dimensions. Figure from [**1**].

1.2. Fano toric varieties.

DEFINITION 5.2. A variety X is called *Fano* iff X has Gorenstein singularities (this is equivalent to K_X being Cartier, or equivalently that X has a dualizing line bundle (note this means X is projective)) and K_X is ample.

Proposition 5.1. *A complete toric variety X is Fano iff $X \simeq X(\Sigma_\Xi)$ with Ξ reflexive.*

PROOF. Let $\Sigma \subset N_{\mathbb{R}}$. Then $U = \operatorname{Spec} k[\sigma^\vee \cap M]$ has canonical divisor $K_U = -\sum D_i$ for $S_i \subset U$ toric prime divisors. Note the holomorphic volume form $z_1^{-1}\, dz_1 \wedge \ldots \wedge z_d^{-1}\, dz_d$ has simple poles along any toric divisor.

Now we see that K_U is Cartier iff there exists there exists $m_\sigma \in M$ such that for all i we have $\langle m_\sigma, V_i\rangle = 1$. Now we translate this condition to the world of polytopes. In particular, Ξ is the convex hull of the m_σs, and then this condition above is exactly saying that the integral distance is 1. □

1.3. CY hypersurfaces in the smooth case. Let X be a smooth Fano variety woth $\Gamma(X, -K_x) \neq 0$. Then by Bertini, we have that $Z(s) \subset X$ is smooth. The adjunction formula tells us that

$$K_{Z(s)} = (-K_X + K_X)|_{Z(s)} = 0\ . \tag{5.3}$$

The problem is that only very few are smooth toric Fano varieties.

1.4. (Partial) Desingularization. Write Σ_Ξ for the fan over the faces of Ξ°.

DEFINITION 5.3. A *maximal, projective, crepant, partial (MPCP)* subdivision Σ of Σ_Ξ is

- Σ is a subdivision of Σ
- Σ is simplicial and projective.

- The set $\Sigma(1)$ of primitive generators of rays of Σ

$$\Sigma(1) = \Xi^\circ \cap (N \setminus \{0\}) = \partial\Xi \cap N \ . \tag{5.4}$$

I.e. we need to triangulate each face of Ξ° regularly.

REMARK 5.1. For $d \geq 3$ Σ may not be unique since there are all sorts of subdivisions.

Proposition 5.2. *For Σ MPCP then*

(i) X_Σ is a Gorenstein orbifold[5.1]
(ii) Ξ is the polytope associated to $-K_X$
(iii) $f : K_{X_\Sigma} \to K_{X(\Xi)}$ is crepant. *This means that the pullback of the canonical bundle is a canonical bundle, i.e. $K_{X_\Sigma} = f^* K_{X(\Xi)}$.*
(iv) $-K_{X_\Sigma}$ is semi-ample, i.e. it is generated by global sections a big[5.2]

1.5. Batyrev hypersurface mirrors. Let Ξ, Ξ° be a reflexive pair. Let Σ and $\check{\Sigma}$ be MPCP subdivisions of Σ_Ξ and Σ_{Ξ°. Then we get mirror dual families. For a general section $s \in \Gamma(X_\Sigma, -K_{X_\Sigma})$ we can form

$$Z(s) \subset X_\Sigma \tag{5.5}$$

and similarly $\check{s} \in \Gamma\left(X_{\check{\Sigma}}, -K_{X_{\check{\Sigma}}}\right)$ we get

$$Z(\check{s}) \subset X_{\check{\Sigma}} \ . \tag{5.6}$$

These are the mirror pairs. Note that $\Gamma(X, -K_X) = \mathrm{Map}(\Xi^\circ \cap M, k)$. Lecture 23; November 19, 2019

1.6. Mirror theorems.

THEOREM 5.3. $h^{1,1}(X) = h^{n-1,1}\left(\check{X}\right)$

PROOF. See [**6**] Theorem 4.15, or [**2**]. □

THEOREM 5.4 (Batyrev). *There exist "stringy" refinements of $h^{p,q}$ for orbifold singularities, and*

$$h^{p,q}_{st}(X) = h^{n-p,q}_{st}\left(\check{X}\right) \ . \tag{5.7}$$

This has something to do with motivic integration. Then we have that Gromov-Witten theory is the same as variations of Hodge structures.

THEOREM 5.5. $GW_{g=0}(X) = \mathrm{VHS}\left(\check{X}\right)$.

PROOF. See [**5**, **12**, **22**]. □

Then there are some generalizations to toric complete intersections by Batyrev-Borisov. The goal is then to make a more intrinsic construction which applies to a more general situation as in the ongoing project [**15**, **17**].

EXAMPLE 5.3. Consider the polytope in Fig. 2. Now label the vertices as on the right of Fig. 2. This gives us a degeneration in $\mathbb{P}^2 \times \mathbb{A}^1_t$ given by

$$t^0 xyz + t^2\left(x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2\right) + t^3\left(x^3 + y^3 + z^3\right) = 0 \ . \tag{5.8}$$

For $t \neq 0$ this is a CY hypersurface (i.e. an elliptic curve) and for $t = 0$ $xyz = 0$ is the union of three hyperplanes $\mathbb{P}^1$.

[5.1] Smooth in codimension 3, but not in general.
[5.2] This means $-K^d_{X_\Sigma} > 0$.

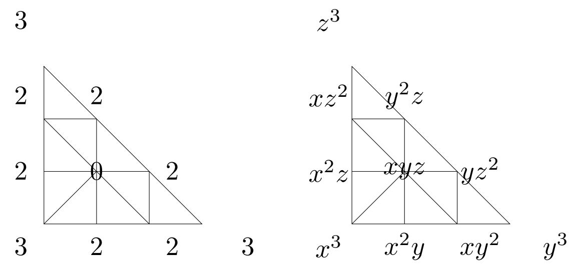

FIGURE 2. A polytope with its unique MPCP subdivision. A PL function is given by the labeled values.

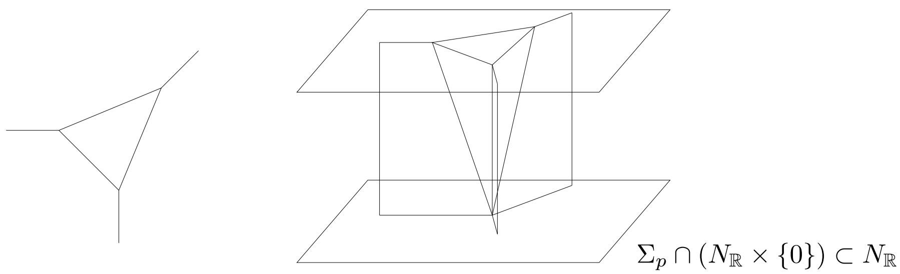

FIGURE 3. Construction of the family $\mathcal{X}$ starting from Ω.

Consider $\mathcal{X} \to S$ where (in deformation theory) S is Spec of some DVR. But really we could take it to be a local ring, or a complete local ring. Then we want the fiber over 0, X_0, to be a union of toric varieties, i.e. a coproduct of toric varieties over a codimension 1 strata.

2. Mumford type toric degenerations

2.1. Construction. Let $\Omega \subseteq N_{\mathbb{R}}$ be convex and closed. Let $\mathcal{P} = \{\sigma\}$ be rational such that

$$\Omega = \bigcup_{\sigma \in \mathcal{P}} \sigma \tag{5.9}$$

locally finite. This gives us a fan

$$\Sigma_{\mathcal{P}} = \{C(\sigma) \subset N_{\mathbb{R}} \oplus \mathbb{R}\} \ . \tag{5.10}$$

The idea is to put this at height one, and form a cone as in Fig. 3.

Proposition 5.6. *(a)* $\pi^{-1}\left(\mathbb{A}^2 \setminus \{0\}\right) = \left(\mathbb{A}^2 \setminus \{0\}\right) \times X\left(\Sigma_{\mathcal{P}} \cap (N_{\mathbb{R}} \times \{0\})\right)$ *which is of dimension* $d+1$.

(b)

$$\pi^{-1}(0) = \bigcup_{v \in \mathcal{P} \ vertex} \underbrace{X(\Sigma_v)}_{D_v} \ . \tag{5.11}$$

(c)

$$\mathrm{mult}_{D_v}\left(\pi^{-1}(0)\right) = \min\{d \in \mathbb{N} \setminus \{0\} \mid d \cdot v \in N\} \ . \tag{5.12}$$

PROOF. (a) $\pi^{-1}\left(\mathbb{A}^1 \setminus 0\right)$ corresponds to the fan $\{C(\sigma) \cap (N_{\mathbb{R}} \times \{0\})\}$ in $N \oplus \mathbb{Z}$.

(b) $\pi^{-1}(0)$ is the union of toric prime divisors D_w with $w \in \Sigma$, $\dim w = 1$, and $w \twoheadrightarrow \mathbb{A}^1$.

(c) We know v is of the form $1/d\,(n,d)$ for $n \in N$ and d minimal. Locally complete implies that

$$\operatorname{ord}_{D_w} \pi^*(t) = d\ . \tag{5.13}$$

□

In the example in Fig. 3, $t \neq 0$ gives us $\pi^{-1}(t) = \mathbb{P}^2$, and $t = 0$ gives us

$$\pi^{-1}(0) = \bigcup_3 X \left(\begin{array}{ccc} & & (2,1) \\ & & \diagup \\ (-1,0) & \text{—} \cdot & \\ & & \diagdown \\ & & (1,-1) \end{array} \right) . \tag{5.14}$$

Note that

$$\begin{array}{c} \mathbb{P}^2 \\ \big\downarrow/(\mathbb{Z}/3) \\ \pi^{-1}(0) \simeq \mathbb{P}(1,1,3) \end{array} . \tag{5.15}$$

2.2. Polytope picture. For mirror symmetry we need a polarized version. For $\Xi \subset M_{\mathbb{R}}$ a $\mathbb{Q}$-polyhedron write $\check{\mathcal{P}}$ for a $\mathbb{Q}$-polyhedron decomposition of Ξ. For a regular subdivision there exists an integral piecewise affine function $\check{\varphi} : \Xi \to \mathbb{R}$ which is strictly convex. This gives a graph $\Gamma_{\check{\varphi}} \subset M_{\mathbb{R}} \times \mathbb{R}$. The upper convex hull is

$$\mathbb{B}_{\check{\varphi}} := \Gamma_{\check{\varphi}} + (0 \oplus \mathbb{R}_{\geq 0}) = \{(m,h) \in \Xi \times \mathbb{R} \mid h \geq \check{\varphi}(m)\}\ . \tag{5.16}$$

This gives us the polarized Mumford degeneration

$$\left(X\left(\Sigma_{B_{\check{\varphi}}}\right)\right) \tag{5.17}$$

where

$$X\left(\Sigma_{B_{\check{\varphi}}}\right) = \operatorname{Proj} k\left[C\left(\mathbb{B}_{\varphi}\right) \cap (M \oplus \mathbb{Z} \oplus \mathbb{Z})\right]\ . \tag{5.18}$$

2.3. Mirror symmetry for Mumford degenerations. The idea is that we have a 1-to-1 correspondence:

$$\begin{gathered} (\mathcal{B} = |\mathcal{P}|, \mathcal{P}, \varphi) \longleftrightarrow \left(\check{\mathcal{B}} = \left|\check{\mathcal{P}}\right|, \check{\mathcal{P}}, \check{\varphi}\right) \\ \text{PA-function} \longmapsto \check{\sigma} \text{ Newton polyhedron of } \varphi|_{\Sigma_\sigma}, \sigma \in \mathcal{P} \\ (\Sigma_{\mathcal{P}}, \psi) \longleftrightarrow (\Xi, \check{\psi}) \end{gathered} \tag{5.19}$$

where ψ is a PL function.

EXAMPLE 5.4. A toric degeneration of $\mathbb{P}^2$ given by $\mathcal{P}$ from Fig. 4 $\check{\mathcal{P}}$ is also shown in Fig. 4.

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November 21, 2019

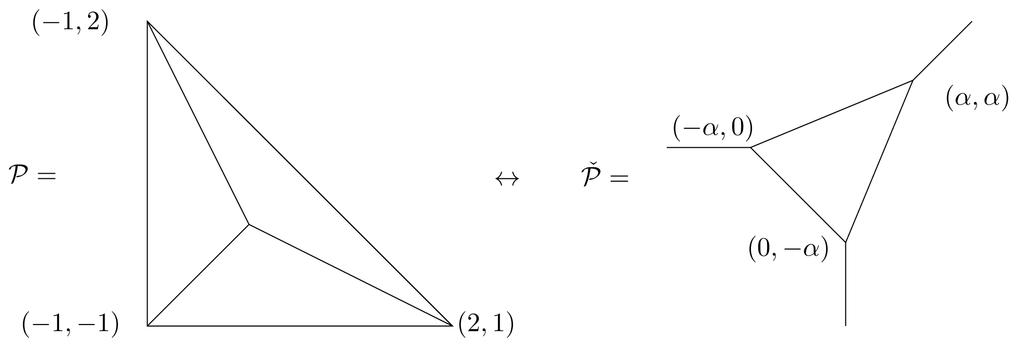

FIGURE 4. An example of $\mathcal{P}$ and $\check{\mathcal{P}}$.

2.4. Toric degenerations. So we have

$$
\begin{tikzcd}
\mathcal{X}_0 \arrow[r, phantom, "\subseteq"] \arrow[d] & (\mathcal{X}, \mathcal{L}) \arrow[d, "\pi"] \\
0 \arrow[r, phantom, "\in"] & S_t = \operatorname{Spec}(\text{a DVR})
\end{tikzcd}
\tag{5.20}
$$

where we want $\mathcal{X}_0$ to be a union of projective toric varieties and $\mathcal{L}|_{\mathbb{P}(\sigma)} = \mathcal{O}_{\mathbb{P}(\sigma)}(1)$. So the data is $(\mathcal{B} = \cup\sigma, \mathcal{P} = \{\sigma\}, \varphi)$ where $\mathcal{B}$ is a manifold with boundary, $\dim_{\mathbb{R}} \mathcal{B} = \dim X = d$, and φ is a polarizing function. So this is the fan picture. φ corresponds to toric models of π in codimension 1.

Then there is an equivalent polyhedron construction. The discrete Legendre transform is mirror duality for $(\mathcal{B}, \mathcal{P}, \varphi)$ which yields $\left(\check{\mathcal{B}}, \check{\mathcal{P}}, \check{\varphi}\right)$.

3. Towards GS-type toric degenerations

3.1. The Dwork pencil and A_1-singularities. Consider

$$\mathcal{X} = \left(t \cdot \sum_{i=0}^{3} z_i^4 + \prod_{i=0}^{3} z_i\right) \subset \mathbb{P}^3 \times \mathbb{A}^1 \tag{5.21}$$

which projects $\pi : \mathcal{S} \to \mathbb{A}^1$. This is a 3-dimensional analogue of the quintic we saw at the beginning of the course. This is a degeneration of $K3$. Then we get $\mathcal{X}_0 = \pi^{-1}(0)$ which is a union of 4 $\mathbb{P}^2$s given by $\mathbb{P}(\sigma)$ for σ maximal. The total space of $\mathcal{X}$ has $6 \cdot 4 = 24$ three-dimensional A_1 singularities on the singular locus of X_0 which is 6 copies of $\mathbb{P}^1$.

Let's consider the local equation by dehomogenizing at z_0, $z_1 \neq 0$. This gives us

$$z_1 z_2 = \frac{1 + z_1^4 + z_2^4 + z_3^4}{z_1}\ . \tag{5.22}$$

So up to an analytic change of coordinates, this is isomorphic to $uv = wt$, which is an A_1 singularity.

Modulo t^{k+1} we get:

$$\chi \bmod t^{k+1} = \cup_{i=0}^{3} X_{\sigma_i}^k \tag{5.23}$$

where $X_{\sigma_i}^0 = \mathbb{P}(\sigma_0)$. This is essentially the kth order neighborhood of $X_{\sigma_i} \subset \mathcal{X}$. The observation is that these individual terms do not depend on $X_{\sigma_i}^k$ are completely determined by $(\mathcal{B}, \mathcal{P}.\varphi)$. Only the identification in higher codimension is nontoric (the gluing is not toric).

3.2. Key example: one A_1 singularity. Consider two cells σ_1, σ_2 which are both half planes intersecting in a real line ρ. Write w for the coordinate on the intersection, u for σ_1 and v for σ_2. Then we want an A_1 singularity at a point $w = -1$ in the intersection. Then

$$uv = \underbrace{(1+w)}_{=f_\rho} \cdot t \bmod t^{k+1} \quad . \tag{5.24}$$

Now the rings to glue are

$$R_{\sigma_1} = [u, v, w, t] \,/\, \left(uv - t, t^{k+1}, v^{k+1}\right) \tag{5.25}$$

$$R_{\sigma_1} = [u, v, w, t] \,/\, \left(uv - t, t^{k+1}, u^{k+1}\right) \tag{5.26}$$

so $X^k_\rho = \operatorname{Spec} R_{\rho,\sigma_i}$. Now we get a diagram:

$$\begin{array}{lcccc} R_{\sigma_1} \longrightarrow R_{\rho,\sigma_1} = \left(R_{\sigma_1} / \left(u^{k+1}\right)\right) & u & v & f_\rho^{-1} \cdot u & f_\rho v \\ & \downarrow & \downarrow & \uparrow & \uparrow \\ R_{\sigma_2} \longrightarrow R_{\rho,\sigma_2} = \left(R_{\sigma_2} / \left(v^{k+1}\right)\right) & f_\rho \cdot u & f_\rho^{-1} v & u & v \end{array} \tag{5.27}$$

so we can form the fiber product to get

$$R_{\sigma_1} \times_{R_{rho,\sigma_i}} R_{\sigma_2} \;\cong\; \mathbb{C}\,[u, v, w, t] \,/\, \left(uv - (1+w)\, t, t^{k+1}\right)$$

$$\begin{array}{ccc} (u, f_\rho u) & \longleftarrow & u \\ (f_\rho, v) & \longleftarrow & v \\ (w, w) & \longleftarrow & w \end{array} \tag{5.28}$$

So we learned, that outside codimension 2, we have two toric models, given by $(\mathcal{B}, \mathcal{P}, \varphi)$. Namely, if σ is maximal we get

$$R^k_\sigma = A_k \left[\Lambda_\sigma\right] \tag{5.29}$$

where $A_k = \mathbb{C}\,[t] \,/\, \left(t^{k+1}\right)$ and $\Lambda_\sigma = \mathbb{Z}^d$, i.e. M_σ where σ lives. We should think of this has an integral tangent vector on σ. Then for a codimension 1 thing $\rho \in \mathcal{P}^{[d-1]}$ we get

$$R^k_\rho = A_k \left[\Lambda_\rho\right] \left[Z_+, Z_-\right] // \left(Z_+ Z_- - f_\rho t^{\kappa_\rho(\varphi)}\right) \tag{5.30}$$

where $\kappa_\rho(\varphi) \in \mathbb{N} \setminus \{0\}$ is the kink of φ along ρ, and $f_\rho \in A_k[\Lambda_\rho]$. So this is the sense in which it is living on a wall.

4. Reconstruction

This is the question of how to go from $(\mathcal{B}, \mathcal{P}, \varphi)$ to $(\mathcal{X} \to s)$. The point is that we will build $\mathcal{X}$ by gluing (fibered coproduct) of $\operatorname{Spec} R^k_\rho$ and $\operatorname{Spec} R^k_\rho$. Explicitly our starting data is

$$\left(\mathcal{B} = \bigcup_{\sigma \in \mathcal{P}} \sigma, \varphi, f_\rho \in A_k \left[\Lambda_\rho\right]\right) \quad . \tag{5.31}$$

4.1. Explanation without scattering. A fan structure at v_1 is given by

$$\begin{tikzcd} & v_2=(0,1) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ (-1,0) \arrow[r, dash] & v_1 \arrow[r, dash] & (1,0) \end{tikzcd} \quad . \tag{5.32}$$

A fan structure at v_2 is given by

$$\begin{tikzcd} & v_2, W \arrow[r, dash] \arrow[dl, dash] \arrow[d, dash] & Y \arrow[dl, dash] \\ X \arrow[r, dash] & Z & \end{tikzcd} \quad . \tag{5.33}$$

These will have different affine structures. We can fix this by removing one point on the internal edge. Here we have

$$f_{\rho,v_1} = \alpha + w \tag{5.34}$$

for $\alpha \in \mathbb{C}^\times$ and $w = W/Z$.

For R^k_{ρ,v_1} we get

$$Z_+ = y = Y/Z \tag{5.35}$$

$$Z_- = x = X/Z \tag{5.36}$$

with the equation

$$Z_+ Z_- = (\alpha + w) \cdot t \ . \tag{5.37}$$

Then homogenizing we get

$$\frac{Y}{Z}\frac{X}{Z} = (\alpha + W/z)\, t \ . \tag{5.38}$$

This is (non-toric) a generation family of conics. If we did a Mumford type degeneration this $\alpha z + w$ would be a w.

Similarly, for R^k_{ρ,v_2} we get

$$Z_+ = y = \frac{Y}{w} \qquad\qquad Z_- = xz^{-1} = \frac{X}{W}\frac{W}{Z} = \frac{X}{Z} \tag{5.39}$$

so

$$Z_+ Z_- = (z\alpha + 1)\, t \tag{5.40}$$

and after homogenizing we get

$$\frac{Y}{W}\frac{X}{Z} = \left(\frac{Z}{W}\alpha + 1\right) t \tag{5.41}$$

so

$$XY = (Z\alpha + W)\, Z \cdot t \ . \tag{5.42}$$

The upshot is that we obtain consistent models

$$f_{\rho,v_1} = z^{m^\rho_{v_1,v_2}} f_{\rho,v_2} \tag{5.43}$$

for $m^\rho_{v_1,v_2} \in \Lambda_\rho$. In the example $m^\rho_{v_1,v_2} = (0,1)$. So add this to our data:

$$(\mathcal{B}, \mathcal{P}, \varphi, \{f_{\rho,v}\}) \ . \tag{5.44}$$

4.1.1. *Propagation.* Consider four copies of the quadrant glued together:

$$\begin{tikzcd} & w & \\ y & v \arrow[l, "\rho_1"'] \arrow[u] \arrow[r, "\rho_2"] \arrow[d] & x \\ & z & \end{tikzcd} \tag{5.45}$$

The toric model at v is

$$A_k\left[x,y,z,w\right]/\left(xy-t,zw-t\right) \ . \tag{5.46}$$

Now take $f_{\rho_1}=1+y$. The local model for $R^k_{\rho_1}$ is $zw=(1+y)\,t$. But then for $x\neq 0$ we have $y=x^{-1}t$, which implies we need a local model

$$zw=\left(1+x^{-1}t\right)t \ . \tag{5.47}$$

This forces us to set

$$f_{\rho_2}=\left(1+x^{-1}t\right) \ . \tag{5.48}$$

We can visualize this by saying that each wall carries the data of a monomial.

4.1.2. *Walls leaving the codimension* 1 *locus.*

$$\begin{tikzcd} & & z \\ x & \cdot \arrow[l, "\rho_1"'] \arrow[ur] \arrow[d] & \\ & y & \end{tikzcd} \tag{5.49}$$

The toric model is $xyz=t$ and

$$f_{\rho_1}=1+x=1+(yz)^{-1}\,t \ . \tag{5.50}$$

To make this consist, take two copies of R^k_σ and glue by a (non-toric) "wall-crossing" automorphism Φ. In particular it maps

$$\begin{aligned} y &\overset{\Phi}{\longmapsto} \left(1+y^{-1}z^{-1}t\right)t \\ z &\longmapsto \underbrace{\left(1+y^{-1}z^{-1}t\right)^{-1}}_{\equiv 1 \bmod t} t \end{aligned} \ . \tag{5.51}$$

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5. Wall structures

5.1. Walls. Fix $A=\mathbb{C}\left[t\right]/\left(t^{k+1}\right)$ for fixed k. Eventually $k\to\infty$. As usual we have the starting data

$$\left(\mathcal{B}=\bigcup_{\sigma\in\mathcal{P}}\sigma,\mathcal{P}=\left\{\sigma\right\},\varphi\right) \tag{5.52}$$

with an $\mathbb{Z}$-affine structure. The singular locus $\Delta\subset\mathcal{B}$ has $\mathrm{Codim}_{\mathbb{R}}\,\Delta=2$.

Definition 5.4. A wall $\mathfrak{p}$ on $\mathcal{B}$ is

- a rational polyhedral subset $\mathfrak{p} \subseteq \sigma$ (for $\dim \sigma = d$) such that $\dim \mathfrak{p} = d - 1$ and $\operatorname{int}(\mathfrak{p} \cap \Delta) = \emptyset$,
- a function $f_{\mathfrak{p}} = \sum c_m z^m$ where $m \in \Lambda_{\mathfrak{p}} \subset \Lambda_\sigma \simeq \mathbb{Z}^d$, $c_m \in A$.

In codimension 0, $\mathfrak{p} \cap \operatorname{int}(\sigma) \neq \emptyset$ implies $f_{\mathfrak{p}} = 1 + c_m z^m$ for $c_m = 0 \bmod t$. The m is telling us the propagation direction. We call these proper walls. In codimension 1, i.e. $\mathfrak{p} \subseteq \rho$ (with $\dim \rho = d-1$) we have $f_{\mathfrak{p}} = f_{\underline{\rho}}$ for some $\underline{\rho} \subset \rho$. We call these slabs. Then a wall structure $\mathscr{S}$ is a finite collection of walls (for *finite* k).

5.2. Wall crossing. Let $(\mathfrak{p}, f_{\mathfrak{p}})$ denote a codimension 0 wall where $\mathfrak{p} \subset \sigma$. Write

$$R_\sigma = A\left[\Lambda_\sigma\right] \simeq A\left[z_1^{\pm 1}, \dots z_d^{\pm 1}\right] \quad . \tag{5.53}$$

Then the wall gives an automorphism

$$\Theta_{\mathfrak{p}} : R_\sigma \to R_\sigma \tag{5.54}$$

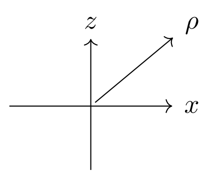
which sends

$$z^m \mapsto f_{\mathfrak{p}}^{\langle n_{\mathfrak{p}}, m\rangle} \cdot z^m \tag{5.55}$$

where $n_{\mathfrak{p}} \in \Lambda_\sigma^*$ and $n_{\mathfrak{p}}^\perp = \Lambda_{\mathfrak{p}}$. This is an automorphism since $\theta_{\mathfrak{p}} \equiv \operatorname{id} \bmod t$.

Example 5.5. Consider the wall:

(5.56)



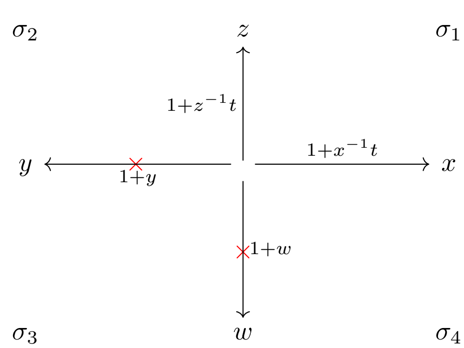
.

Now we will cross from x towards z. First we have $f_{\mathfrak{p}} = 1 + t^2 x^{-1} z^{-1}$. Then the map $\theta_{\mathfrak{p}}$ maps

$$x \mapsto \left(1 + t^2 x^{-1} z^{-1}\right) x \qquad z \mapsto \left(1 + t^2 x^{-1} z^{-1}\right)^{-1} z \ . \tag{5.57}$$

Example 5.6. Propagating walls intersect along codimension 2 polyhedral subsets called joins. Take the following notation:

(5.58)



.

Then by crossing the walls we get:

$$
\begin{aligned}
x &\longmapsto (1+w)\,x \\
&\longmapsto \Big(1+(1+y)^{-1}w\Big)\,x \\
&\longmapsto \Big(1+(1+(1+w)\,y)^{-1}w\Big)(1+w)^{-1}\cdot x \\
&\longmapsto \underbrace{\Big(1+(1+(1+(1+y)\,w)\,y)^{-1}(1+y)\,w\Big)(1+(1+y)\,w)^{-1}x}_{g}
\end{aligned} \tag{5.59}
$$

Now this look terrible but really

$$g = \Big(1+((1+y)+(1+y)\,wy)^{-1}(1+y)\,w\Big)(1+(1+y)\,w)^{-1}x \tag{5.60}$$

$$= \Big(1+(1+wy)^{-1}w\Big)(1+w+wy)^{-1}x \tag{5.61}$$

$$= (1+wy)^{-1}(1+wy+w)(1+w+wy)^{-1}x \tag{5.62}$$

$$= (1+wy)^{-1}x\ . \tag{5.63}$$

This tells us that to make this consistent, we need to insert a wall in the upper right diagonal direction with the function

$$(1+wy)\ . \tag{5.64}$$

Then the theory is so beautiful, that this works no matter which monomial we start with.

The observation (due to Kontsevich-Soibelman) is as follows. Assuming all walls carry function congruent to 1 mod t and

$$\theta = \underbrace{\theta_{\mathfrak{p}_1}\ldots\theta_{\mathfrak{p}_r}}_{\text{counterclockwise}} \equiv \mathrm{id} \bmod t^k \tag{5.65}$$

then there is a unique way to insert walls with functions $1+ct^{k+1}z^m$ to achieve

$$\prod \theta_{\mathfrak{p}} = \mathrm{id} \bmod t^{k+1}\ . \tag{5.66}$$

5.3. Construction of $\mathcal{X}^\circ = \mathcal{X}\setminus \mathrm{Codim}\,2$. Recall the goal is to go from $(\mathcal{B},\mathcal{P},\varphi)$ and $f_{\underline{\varrho}}$ to

$$
\begin{matrix}
\mathcal{X}_k = \mathcal{X}_0 = \bigcup \text{ toric varieties} \\
\downarrow \\
S_k = \operatorname{Spec}\mathbb{C}[t]\,/\,\big(t^{k+1}\big)
\end{matrix}\ . \tag{5.67}
$$

Assume we have consistency of $\mathscr{S}$ in codimension 0 and codimension 1. Then define

$$\mathcal{X}^\circ := \varinjlim \{\operatorname{Spec} R_{\mathfrak{e}}, \operatorname{Spec} R_{\mathfrak{u}}\} \tag{5.68}$$

via wall crossings and canonical localizations $R_{\mathfrak{e}} \to R_{\mathfrak{u}}$ for $\mathfrak{e}\subset\mathfrak{u}$. Then $\mathcal{B}\setminus\bigcup_{\mathfrak{p}\in\mathcal{S}}\mathfrak{p}$ gives rise to closures of connected components (WLOG we can assume convex) $\mathfrak{u}\subseteq\sigma$, and $R_{\mathfrak{u}} := R_\sigma$. Then we have

$$R_{\mathfrak{e}} = A\,[\Lambda_{\mathfrak{e}}]\,[z_+,z_-]\,/\,(z_+z_- - f_{\mathfrak{e}}\cdot t^{\kappa_{\underline{\varrho}}})\ . \tag{5.69}$$

5.4. Consistency in all codimensions/broken lines. Now we have broken lines β. Inductively we define

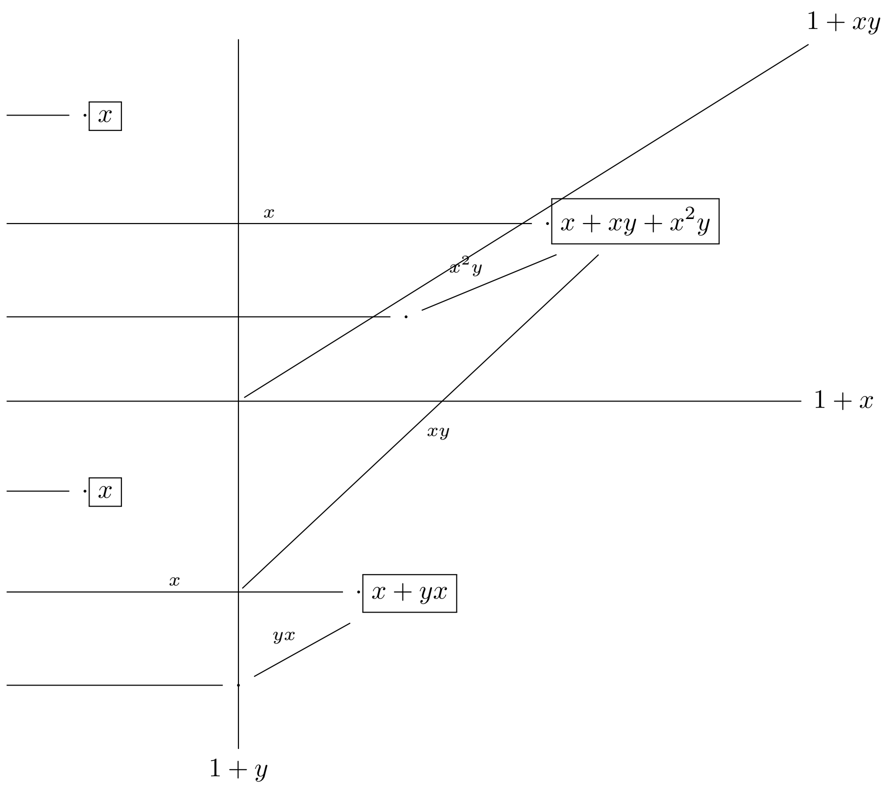
$$\theta_{\mathfrak{p}_j}\left(b_{j-1}z^{m_{j-1}}\right) = \sum a_i z^{m_i} \tag{5.70}$$

where $b_j z^{m_j} \in \{a_i z^{m_i}\}$.

Definition 5.5. $\mathscr{S} = \{\mathfrak{p}\}$ is consistent if locally sums over broken lines give well-defined functions.

Example 5.7. Let $m_0 = (-1, 0)$ and $z^{m_0} = x$. Then the broken walls look like:

(5.71)



5.5. Canonical functions - generalized theta functions. Assume $\mathscr{S}$ is consistent (in codimension ≤ 2), m asymptotic integral vector field. By the same construction, we get

$$\vartheta \in \mathcal{O}\left(\mathcal{X}^{\circ}\right) \tag{5.72}$$

defined by sums over broken lines. On $R_{\mathfrak{u}} = A\left[\Lambda_\sigma\right]$, for $\mathfrak{u} \subset \sigma$, define

$$\vartheta_m = \sum_{\substack{\beta \text{ with asymptotic } z^m \\ \text{endpt } p}} b_\beta z^{m_\beta} \ . \tag{5.73}$$

This is independent of p.

5.6. Construction of $\mathcal{X} \to \operatorname{Spec} A$.

THEOREM 5.7. *For sS a consistent wall structure on* $(\mathcal{B}, \mathcal{P}, \varphi)$ *there exists a partial completion* $\mathcal{X} \supset \mathcal{X}^{\circ}$ *with*

$$\mathcal{X}_0 = \bigcup_{\sigma \in \mathcal{P}} \mathbb{P}(\sigma) \quad . \tag{5.74}$$

PROOF. In the affine case (i.e. $\mathcal{P}$ without bounded cells, i.e. each cell is a cone) then

$$\mathcal{X} = \operatorname{Spec} \oplus_{m \in \mathcal{B}(\mathbb{Z})} A \cdot \vartheta_m \ . \tag{5.75}$$

In the general case, apply the affine case to $\operatorname{Tot}(\mathcal{O}_{\mathcal{X}^{\circ}}(-1))$. So we use the fact that for a projective variety

$$\bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}_X(d)) \Gamma(L, \mathcal{O}_X) \tag{5.76}$$

where $L \downarrow X$ is the total space of $\mathcal{O}_X(-1)$.

On the level of $\mathcal{B}$, we take the truncated cone over $\mathcal{B}$. □

5.7. Construction of consistent wall structures. A first approach is done in [**16**]. $\mathcal{B}$ has affine charts near vertices $\left(\Delta \cap \left|\mathcal{P}^{[0]}\right| = \emptyset\right)$. $\mathcal{X}$ is mirror to $\mathcal{Y}/\check{S}$ with $\mathcal{Y}_0$ the union of toric varieties. This is an inductive construction via scattering. This needs "local rigidity" in codimension 2 of initial data $f_{\rho,v} = f_{\underline{\rho}}$.

The second approach in [**14**] uses the approach that this scattering corresponds to relative Gromov-Witten theory on toric surfaces. Then $\mathcal{X}$ is mirror to (Y, D) a log CY surface on $\mathcal{Y}/\check{S}$ where $\dim \mathcal{Y}_0 = 2$. This let to the results of [**13**]. So this is some kind of hybrid approach.

Then in an upcoming work, by log GW-theory on (Y, D) (or $(\mathcal{Y}, \mathcal{Y}_0)$) where $\mathcal{Y}$ is the union of tropicalizations of holomorphic curves.

6. Intrinsic mirror map

Insert here Professor Siebert's talk from the geometry seminar a few weeks ago. The point here is to construct (homogeneous) coordinate ring (mirror to (Y, D) (or $\mathcal{Y}$)) via punctured Gromov-Witten theory. So this is log GW invariants with contact orders which can be negative. This comes without an explicit use of wall structures. The point is that we get complete control of the mirror construction: $(X, D) \rightsquigarrow \mathcal{Y} \to S'$ (or $\mathcal{X} \to S \rightsquigarrow \mathcal{Y} \to S'$) via log GW theory on X or $\mathcal{X}$ (really this is clsoed string theory $g = 0$).

In the affine case we start with a pair (Y, D) where $D = \cup D_i \subseteq Y$ is a normal crossings divisor. Then we require that we have a numerical equivalence

$$K_Y + D \sim \sum a_i D_i \tag{5.77}$$

for $a_i \geq 0$. This is all we will require of a log CY.[5.3]

EXAMPLE 5.8. Consider $\mathbb{P}^2$. Recall $\mathcal{B}(\mathbb{Z})$ is the union of three copies of $\mathbb{N}^2$. Then for some (a, b) for $a, b > 0$ this gives us $u \mapsto \left(u^a, u^b\right)$. The generators of the ring are in one-to-one correspondence with contact orders. The multiplication rule is given by:

$$\vartheta_{m_1} \cdot \vartheta_{m_2} \sum_p N_{m_1, m_2, p} \vartheta_p \tag{5.78}$$

where $N_{m_1, m_2, p}$ is the number of genus 0 curves with contacts m_1, m_2, $-p$ with D. The hard part is showing associativity.

[5.3] This is a rather liberal notion of a log CY.

For proper CY we have

$$\begin{array}{ccc} \mathcal{Y} & \supset & \mathcal{Y}_0 \\ \downarrow & & \downarrow \\ S & \supset & \{0\} \end{array} \tag{5.79}$$

and then

$$\mathcal{X} = \operatorname{Proj}\left(\bigoplus A \cdot \vartheta_m\right) \ . \tag{5.80}$$

The challenges are as follows:

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- extract variations of Hodge structures and closed Gromov-Witten invariants (genus 0)
- Homological Mirror Symmetry (HMS)
- other mirror phenomenon (wide open)
  - $g > 0$ (non-commutative, phantom integrability)
  - hyper-Kähler (AAB branes, geometric Langlands)
  - Donaldson-Thomas theory

CHAPTER 6

A peek into Homological Mirror Symmetry

This is a bit more subtle than just saying the derived category is the Fukaya category. Throughout, (X, ω) will be a compact CY.

1. Overview

So far we have looked at closed strings in these pictures as in Fig. 1.

Then we have an A-model (GW) and a B-model (VHS), given by topological twists. HMS is the open string version of this pictorially shown in Fig. 2.

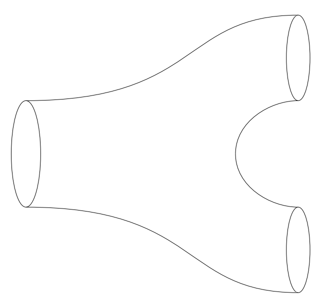

FIGURE 1. A closed string propagating through time and splitting into two strings.

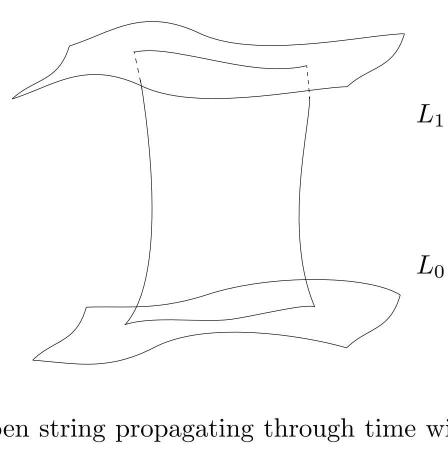

FIGURE 2. An open string propagating through time with boundary conditions L_0 and L_1.

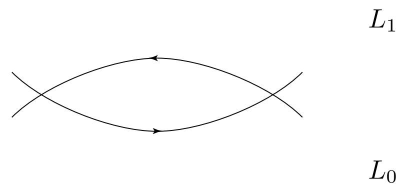

FIGURE 3. A map $u : \Delta \to X$ from the open disk to our manifold which sends the boundary to Lagrangians L_i. This should be thought of as an open string being born and dying.

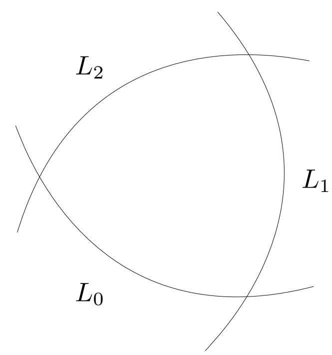

FIGURE 4. A map $u : \Delta \to X$ where we have three marked points and send the portions of the boundary to three different Lagrangians. We will eventually count things like this to put an algebraic structure on the sets of intersection points between Lagrangians.

So we have some open strings propagating through space with some boundary conditions L_1 and L_2. Then we twist this to get the A-model, given by the Fukaya category $\mathcal{F}(X)$ in genus 0.[6.1] The B-model is given by the derived category $D^b(\mathcal{O}_X)$.

1.1. ***A* and *B*-models.** Pictorially, we have strings being born and dying as in Fig. 3. So this is some kind of quantum correction between these intersection points. Then in general we have polygons as in Fig. 4.

The boundary conditions are given by the set of "objects" $\mathrm{Obj}(\mathcal{A})$ and the intersections of boundary conditions given by the set of "morphisms" $\mathrm{Hom}(\mathcal{A})$. Propagation of strings is then given by composition of morphisms m_k. Pictures as in Figs. 3 and 4 give algebraic structure for the m_k.

Let (X, ω) be a symplectic manifold with $\dim_{\mathbb{C}} X = d$. When $\mathcal{A} = \mathcal{F}(X)$, the objects are Lagrangian submanifolds, i.e. submanifolds $L \subset X$ such that $\dim L = d$ and $\omega|_{L=0}$. Then propagating strings correspond to maps $u : \Delta \to X$ where $\Delta = \{|z| \leq 1\} \subset \mathbb{C}$ is the closed unit disk. For $L \pitchfork L'$ Lagrangians intersecting transversely, we have

$$\hom(L, L') = \mathbb{C}^{L \cap L'} \,. \tag{6.1}$$

[6.1] No one has dared define this in the higher genus case.

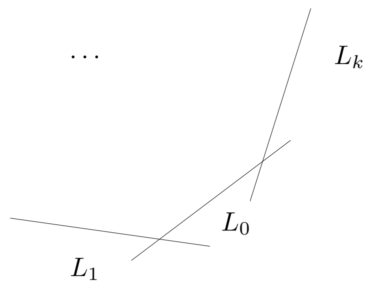

FIGURE 5. The type of disk we count when defining m_k.

Really we want to equip these Lagrangians with $\mathrm{U}(1)$-bundles and then instead of $\mathbb{C}$ we want sections. Then the

$$m_k : \mathrm{Hom}\,(L_k, L_{k-1}) \otimes \ldots \mathrm{Hom}\,(L_1, L_0) \to \mathrm{Hom}\,(L_0, L_k) \tag{6.2}$$

have to do with counting maps $u : \Delta \to X$ (such that $u\,(\partial D) \subset L_0 \cup \ldots \cup L_k$) as in Fig. 5.

In the B-model we have the category $D^b\,(\mathcal{O}_X)$. The objects are complexes

$$\mathcal{F}^\bullet = \left(\cdots \to \mathcal{F}^{-r} \to \mathcal{F}^{-r-1} \to \cdots\right) \tag{6.3}$$

with bounded[6.2] cohomology

$$h^i\,(\mathcal{F}^\bullet) = 0 \tag{6.4}$$

if $|i| \gg 0$.

Why does this matter? The closed story is really enumerative, which not so many people care about. But this open story brings in new objects that many people care about. Lagrangians are the holy grail of symplectic geometry; derived categories matter for geometric representation theory, algebraic geometry, etc.

1.2. Algebraic structure. The moduli space of domains (stable disks) has a recursive structure.

EXAMPLE 6.1. Consider a disk with 4 marked points on the boundary. Maps defined on this are what we count for m_3. This can degenerate in many ways, two of which are shown in Fig. 6. We want to view 0 as the outgoing point, and the others as incoming. We can encode these degenerations as rooted (metric) ribbon[6.3] trees. The associated trees are shown in Fig. 6.

The result is that for $L_0, \ldots, L_k$ mutually transverse, we can define

$$m_k : \mathrm{Hom}\,(L_k, L_{k-1}) \otimes \ldots \otimes \mathrm{Hom}\,(L_1, L_0) \to \mathrm{Hom}\,(L_0, L_k)\,[2-k] \tag{6.5}$$

by (virtually) counting J-holomorphic maps from disks to X. The virtual dimension of this moduli space is

$$(k-2) + d_0 - \sum_{i=1}^{k} d_i\ . \tag{6.6}$$

[6.2]This is what the b is good for.

[6.3]A ribbon graph is a graph where at each vertex the edges are cyclically ordered. The idea is that ribbon graphs can always be embedded in the plane.

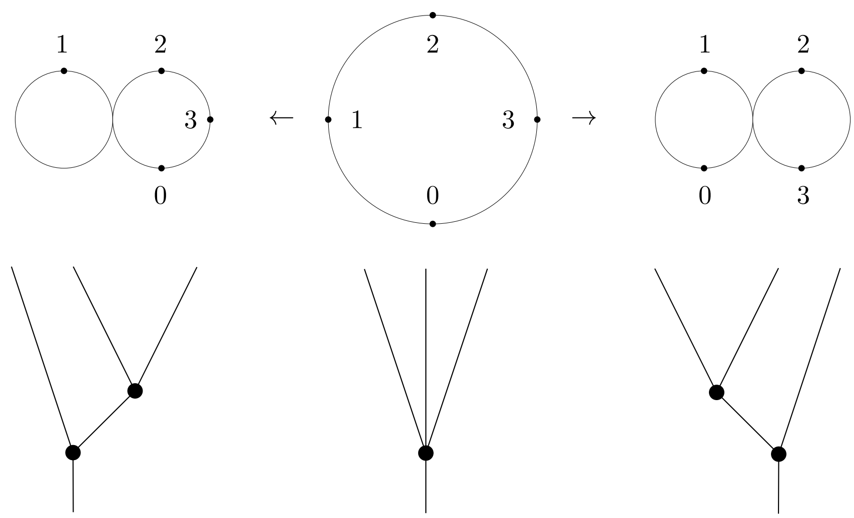

FIGURE 6. Two of the ways this disk can degenerate and the associated rooted ribbon trees.

The intersection points have (in good situations) a $\mathbb{Z}$-grading, so this is what this $[2-k]$ means. Then the algebraic relations between the m_k come from studying boundaries of high-dimensional moduli spaces.

EXAMPLE 6.2. (a) The first relation says that $m_1^2 = 0$, i.e. m_1 is a differential. We get this as follows. Take two Lagrangians L_0 and L_1. We want the difference of the degrees of the intersection points to be 2 so we get strip-breaking. So m_1 is a differential on

$$\bigoplus_{L,L',d} \operatorname{Hom}(L, L') \ . \tag{6.7}$$

(b) We have the relation

$$m_1 m_2 = m_2 (m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1) \tag{6.8}$$

which means m_1 satisfies the Leibniz rule with respect to m_2.

(c) Unfortunately m_2 is not really associative, but m_3 is some kind of associator. In particular, we get the relation:

$$m_2 (\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 m_3 + m_3 (m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1) \tag{6.9}$$

Notice that the first term is a coboundary, and the second vanishes on cocycles with respect to m_1.

(d) Generally we have the formula

$$\sum_{\substack{p,q \\ p\geq 1, q\geq 0 \\ p+q\leq k}} (-1)^{|a_1|+\ldots+|a_q|-q} \, m_{k-p+1} \left(a_k, \ldots, a_{p+q+1}, m_p \left(a_{p+q}, \ldots, a_{q+1}\right), a_q, \ldots, a_1\right) = 0 \ . \tag{6.10}$$

These are called the A_∞ relations.

1.3. The Fukaya category. We can define the products m_k for mutually transverse Lagrangians $L_0, \dots, L_k$, assuming $c_1(X) = 0$, X is compact, and all L_i are oriented, spin, graded, and unobstructed.[6.4] There is a natural ground field called the Novikov field

$$\Lambda_{\text{nov}} = \left\{ \sum_{i \geq 0} a_i q^{\lambda_i} \,\middle|\, \lambda_i \in \mathbb{R}, \lambda_i \to \infty, a_i \in \mathbb{Z} \right\} . \tag{6.11}$$

We could also take the $a_i \in \mathbb{Q}, \mathbb{C}$. The point is that the a_i are the number of holomorphic disks, and the λ_i are the symplectic area of our disks:

$$\int_{\Delta} u^* \omega . \tag{6.12}$$

A priori this is only an A_∞ pre-category since $\operatorname{Hom}(L, L)$ doesn't make sense. But we can use Hamiltonian perturbations to fix this. Given L, L' choose $H_t : X \to \mathbb{R}$, $t \in [0,1]$ giving a Hamiltonian flow φ^t such that $\varphi^1(L') \pitchfork L$. One shows that the resulting A_∞ category $\mathcal{F}(X)$ is well-defined up to canonical A_∞ quasi-isomorphism. Now we finally actually define the following.

Definition 6.1. A *(non-unital) A_∞-algebra* is a $\mathbb{Z}$-graded K vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A_p \tag{6.13}$$

and graded (degree 0) k-linear morphisms

$$m_k : A^{\otimes k} \to A[2-k] \tag{6.14}$$

for any $k \geq 1$ satisfying the A_∞ relations.

Definition 6.2. A *(non-unital) A_∞-category* $\mathcal{A}$ consists of some objects $\operatorname{Obj}(\mathcal{A})$ and for all $X_0, X_1 \in \operatorname{Obj}(\mathcal{A})$ a $\mathbb{Z}$-graded K-vector space

$$\operatorname{Hom}(X_0, X_1) \tag{6.15}$$

and for all $k \geq 1$

$$m_k : \operatorname{Hom}(X_k, X_{k-1}) \otimes \ldots \otimes \operatorname{Hom}(X_1, X_0) \to \operatorname{Hom}(X_0, X_k)[2-k] \tag{6.16}$$

fulfilling the A_∞ relations.

Note that we get an honest category from an A_∞-category by taking the associated homological category $H(\mathcal{F}(X))$ with $m_1 = 0$. This has the same objects, but when we take Hom we pass to cohomology.

Fact 9. *$H(\mathcal{F}(X))$ is unital, i.e. for all L there exists some $e_L \in \operatorname{Hom}^0(L, L)$ such that $m_1(e_L) = 0$, it behaves as a unit wrt m_2, and annihilates all m_k for $k > 2$.*

One calls such a category cohomologically unital (or c-unital). This is a bit annoying, but there is a trick where our c-unital $\mathcal{F}(X)$ can be shown to be A_∞ quasi-equivalent to a unital A_∞ category.

[6.4] This means there are no holomorphic disks with boundary on the L_i.

1.4. The "derived" Fukaya category. We might be confused now, because we are supposed to match this A_∞ category to this derived category which is triangulated. Kontsevich offers a solution by making $\mathcal{A} = \mathcal{F}(X)$ into a "triangulated A_∞-category" by adding artificial complexes to get $\operatorname{Tw}\mathcal{F}(X)$. In particular, for $L_1, \ldots, L_k$ we have $\Delta = (\delta_{ij}) \in \operatorname{Hom}^1(L_j, L_i)$ such that

- (strictly lower triangular) $\delta_{ij} = 0$ for all $j \geq i$,
- (Maurer-Cartan)

$$\sum_{k\geq 1} m_k(\Delta, \ldots, \Delta) = \sum_{i_1 < \ldots, i_k; k \geq 1} m_k\left(\delta_{i_k, i_{k-1}}, \ldots, \delta_{i_2, i_1}\right) \ . \tag{6.17}$$

Lecture 27; December 5, 2019

1.5. Homological mirror symmetry. Now we offer a precise statement of HMS. Recall the Fukaya category had coefficients in this Novikov field Λ_{nov}. Take complex coefficients $\mathbb{K} = \Lambda_{\text{nov}} \otimes \mathbb{C}$. For (X, ω) compact CY, we get a mirror (maximal) degeneration $\mathcal{Y}/S$. In particular we will work over $S = \operatorname{Spec}\mathbb{K}$. Then HMS asserts that:

CONJECTURE 1. *There exists a canonical equivalence of A_∞ categories over $\mathbb{K}$:*

$$\psi : D^b\mathcal{F}(X) = H\left(\operatorname{Tw}^\Pi \mathcal{F}(X)\right) \to D^b(\mathcal{Y}) \ . \tag{6.18}$$

This $\operatorname{Tw}^\Pi \mathcal{F}(X)$ is called the split closure. Said differently, this is the completion of the twisted Fukaya category by idempotents.

2. $D^b_{dg}(X)$ and the HPL

2.1. Čech model of $D^b(X)$. A dg category is an A_∞ category where $m_k = 0$ for $k > 2$. We want to describe the A_∞ structure on $D^b(X)$. The point is that when we start with something which is A_∞ and complete, and pass to cohomology, we get something which is A_∞. The homological perturbation lemma (HPL) tells us that when we start with something which is only dg, we still get something which is only A_∞.

Assume X is smooth and projective. This means that for any coherent $\mathcal{O}_X$-module has a finite locally free resolution. This also means that

$$D^b(X) = D^b(\mathbf{Coh}X) = D^b(\mathbf{Perf}X) \ . \tag{6.19}$$

In general, the perfect complexes are the compact objects. But if we are projective from the beginning, the objects are compact anyway.

Then the dg enhancement is as follows. The objects of $D^b_{dg}(X)$ are bounded complexes of locally free sheaves as before. The morphisms are the interesting part. Write

$$\mathfrak{U} = \{U_i\} \tag{6.20}$$

for an affine cover of X. Then

$$\operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \bigoplus \operatorname{Hom}^n_{D^b_{dg}(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \tag{6.21}$$

where

$$\operatorname{Hom}^n_{D^b_{dg}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_{p+q=n} \check{C}^p\left(\mathfrak{U}, \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[q])\right) \ . \tag{6.22}$$

We write

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[q]) = \mathcal{H}\mathrm{om}^q_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \tag{6.23}$$

and $\check{C}^p$ denotes the the Čech p-cochains:

$$\bigoplus_{i_0<\ldots<i_p} \Gamma\left(U_{i_0}\cap\ldots\cap U_{i_p},\cdot\right) \ . \tag{6.24}$$

This forms a dg-category with differential m_1 given by the Čech differential. For

$$f = \left(f_{n,i_1,\ldots,i_p}\right) \in \check{C}^p\left(\mathfrak{U}, \mathcal{H}\mathrm{om}^q\left(\mathcal{E}^\bullet, \mathcal{F}^\bullet\right)\right) \tag{6.25}$$

$$f = \left(f_{n',i_1,\ldots,i_{p'}}\right) \in \check{C}^{p'}\left(\mathfrak{U}, \mathcal{H}\mathrm{om}^{q'}\left(\mathcal{E}^\bullet, \mathcal{F}^\bullet\right)\right) \ . \tag{6.26}$$

composition is

$$m_2(g,f) = (-1)^{\deg f} g\circ f \tag{6.27}$$

where

$$g\circ f = \left((-1)^{q'p} g_{n+q,i_0,\ldots,i_{p'}} \circ f_{n,i_{p'},\ldots,i_{p+p'}}\right)\Big|_{U_{i_0\ldots,i_{p+p'}}} \ . \tag{6.28}$$

2.2. The HPL.

THEOREM 6.1. *For any A_∞ category $\mathcal{A}$ there exists an A_∞ category structure on $H(\mathcal{A})$ with $m_1=0$ (this is what is called a* minimal A_∞ category*) and an A_∞ quasi-equivalence $H(\mathcal{A})\to\mathcal{A}$.*

PROOF. The construction depends on a choice, for any $x,y\in\mathrm{Obj}(\mathcal{A})$ of a projector Π and a homotopy H:

$$\Pi : \mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,Y) \tag{6.29}$$

$$H : \mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,Y)[-1] \tag{6.30}$$

(where Π is of degree 0 and $\Pi^2=\Pi$) such that

$$\Pi - \mathrm{id} = m_1\circ H + H\circ m_1 \ . \tag{6.31}$$

This will give φ. Here we want φ to be a quasi-equivalence. Choose Π as a projection to representations of cohomology classes.

Then we get a new A_∞ category $\Pi\mathcal{A}$. The objects are the same, and the morphisms are:

$$\mathrm{Hom}_{\Pi\mathcal{A}}(X,Y) = \Pi\left(\mathrm{Hom}_{\mathcal{A}}(X,Y)\right)\subset\mathrm{Hom}_{\mathcal{A}}(X,Y) \tag{6.32}$$

which implies $H(\Pi\mathcal{A}) = H(\mathcal{A})$ as a category. Now define the m_i as follows:

$$m_1^\Pi \coloneqq m_1 \tag{6.33}$$

$$m_2^\Pi \coloneqq \Pi\circ m_2 \tag{6.34}$$

$$m_k^\Pi \coloneqq \sum_{\substack{T \text{ rooted} \\ \text{ribbon tree} \\ k \text{ leaves}}} m_{k,T} \tag{6.35}$$

We should think of these as Feynman rules. In particular, $m_{k,T}$ is defined as m_k at each interior vertex of valency $k+1$, an $H\circ$ at an interior edge, and a $\Pi\circ$ at a root vertex.

EXAMPLE 6.3. Let $k=6$. Then one of the rooted ribbon trees being summer over is pictured in Fig. 7. In particular this contributes:

$$m_{k,T}(a_1,\ldots,a_6) = \Pi\circ m_3\left(a_1, H\circ m_2\left(H\circ m_3\left(a_2,a_3,a_4\right),a_5\right),a_6\right) \ . \tag{6.36}$$

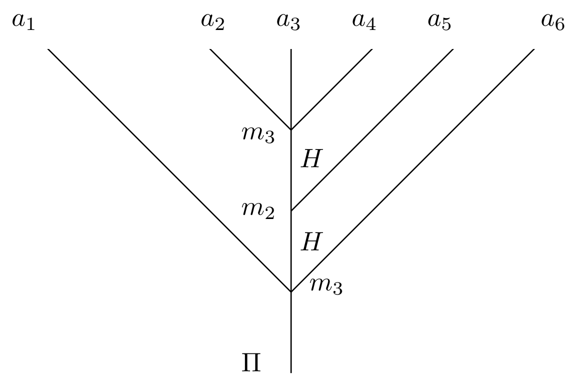

FIGURE 7. A rooted ribbon tree contributing to the definition of m_6^{Π}.

Lemma 6.2. *The m_k^{Π} fulfill the A_∞ relations.*

REMARK 6.1. There exists an A_∞-functor

$$(i_k) : \Pi\mathcal{A} \to \mathcal{A} \tag{6.37}$$

with $H\left(m_k^{\Pi}\right)$ the A_∞ structure on $H(\mathcal{A})$. This is defined by a similar construction:

$$i_k := \sum_{\substack{T \text{ rooted} \\ \text{ribbon tree} \\ k \text{ leaves}}} i_{k,T} \tag{6.38}$$

where the $i_{k,T}$ are defined as the $m_{k,T}$ were, but with the Πs at the root vertices replaced by Hs.

Lemma 6.3. *(i_k) is an A_∞ functor.*

□

SKETCH PROOF OF LEMMA 6.2. Consider

$$\mu_k = \sum_{\substack{T \text{ ribbon tree} \\ e \text{ edge}}} \mu_{k,(T,e)} \tag{6.39}$$

where the $\mu_{k,(T,e)}$ are defined as $m_{k,T}$ but with H at the edge e replaced by $\Pi = \mathrm{id} = m_1 \circ H - H \circ m_1$ and m_1 at each external edge.

Then either

(i) using $\Pi - \mathrm{id}$, order by edges e:

$$\mu_k = \underbrace{\mu_k^{m_1}}_{e \text{ external}} + \underbrace{\mu_k^{\Pi} - \mu_k^{\mathrm{id}}}_{e \text{ internal}} \tag{6.40}$$

or

(ii) using $m_1 \circ H + H \circ m_1$, order by internal vertices adjacent to e.

Now apply A_∞ for m_l to obtain a new sum over trees with each summand appearing in μ_k^{id}. The signs work out, so we get $\mu_k = -\mu_k^{\mathrm{id}}$. Then (6.40) implies

$$\mu_k^{m_1} + \mu_k^{\Pi} = 0 \ . \tag{6.41}$$

Then these turn out to be the A_∞ relations for m_k^{Π}. □

3. Polishchuk's rigidiy theorem

3.1. Reducing to the algebra A_L. Let $L \in D^b(X)$ be an ample line bundle on a projective variety X. Then we get a full subcategory $\langle L \rangle$ of $D^b_{dg}(X)$ with objects L^n for $n \in \mathbb{Z}$. Then m_2 on $D^b(X) = H\left(D^b_{dg}(X)\right)$ given by the cup product on

$$A_L = \bigoplus_{p,q\in\mathbb{Z}} H^p(X, L^q) \ . \tag{6.42}$$

Then we observe that A_L contains the homogeneous coordinate ring:

$$A_L \supseteq R_L = \bigoplus_{n\geq 0} H_0(X, L^n) \ , \tag{6.43}$$

is a minimal[6.5] A_∞ algebra, and that $\langle L \rangle$ split-generates $D^b(X)$. Therefore $D^b(X)$ as an A_∞-category is characterized by A_L as an A_∞ algebra.

Lecture 28; December 10, 2019

3.2. A_∞ structures on A_L. Then we have the following question:

QUESTION 1. How much information does the A_∞ structure on A_L (beyond m_2) hold?

REMARK 6.2. For X CY and $\dim X = d$ we have

$$A_L = \bigoplus_{p\geq 0, q\in\mathbb{Z}} H^p\left(X, L^{\otimes q}\right) = \bigoplus_{q\geq 0} H^0\left(X, L^{\otimes q}\right) \oplus \bigoplus_{q<0} \underbrace{H^d\left(X, L^{\otimes q}\right)}_{\simeq H^0\left(X, L^{\otimes -q}\right)^*} \ . \tag{6.44}$$

REMARK 6.3. This has an internal grading given by powers of L.

THEOREM 6.4 (Polishchuk). *Let X be a projective variety, $\dim X = d$, and L very ample. Let $H^q(X, L^{\otimes p}) = 0$ for all p and all $q \neq 0$. Then:*

- *(i) Up to strict A_∞-isomorphisms $((f_n), f_1 = \mathrm{id})$ and scaling, there exists a unique non-trivial ($m_k \neq 0$ for some $k > 2$) A_∞ structure on A_L preserving the internal grading.*
- *(ii) The strict A_∞ isomorphism is unique up to homotopy.*

REMARK 6.4. This says that we can somehow localize the content of HMS. The point of this rigidity is that to prove HMS, we just need to equate the rings and show this little extra information is the same.

3.3. A_∞-structures and Hochschild cocycles. Let $A = \bigoplus_{k\in\mathbb{Z}} A_k$ be a graded algebra and (m_n), (m'_n) be two A_∞ refinements (where $m_1 = m'_1 = 0$ and $m_2 = m'_2$ are the given multiplication). Then we can observe that if $m_i = m'_i$ for all $i < n$ then

(i) $c = m_n - m'_n$ is a Hochschild n-cocycle with co-differential

$$\begin{aligned}\delta_c(a_1, \dots, a_{n+1}) &= (-1)^{n \deg a_1} a_1 c(a_2, \dots, a_{n+1}) \\ &\quad + \sum (-1)^j c(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} c(a_1, \dots, a_n) a_{n+1}\end{aligned} \tag{6.45}$$

This is true because somehow this is the only thing left in the A_∞ relations.

(ii) Changing m_n by a strict A_∞ isomorphism, this leads to a change of c only by Hochschild coboundaries.

[6.5] This means $m_1 = 0$.

3.4. Our situation. We have $A = R \oplus M$ where

$$R = \bigoplus_{q \geq 0} H^0(X, L^q) \qquad \deg = 0 \tag{6.46}$$

$$M = \bigoplus_{q < 0} H^d(X, L^q) \qquad \deg = d \ , \tag{6.47}$$

and

$$m_n = A \otimes \ldots \otimes A \to A[2] \ . \tag{6.48}$$

Let's look at the degrees. We have $a_i \in M$ for k indices, so we get degree kd for $0 \leq k \leq n$. Then this goes to something of degree $kd + 2 - n$.

Since A has only degree 0 and d, we have either $kd + 2 - n = 0$, or d. I.e.

$$kd + 2 - n = 0 \qquad \text{or} \qquad (k-1)d + 2 - n = 0 \ . \tag{6.49}$$

So write $l = k$ or $k - 1 \in \{0, \ldots, d\}$. In any case, we obtain $c = m_n - m'_n$ a Hochschild cocycle of degree $2 - n$ and internal degree 0, which we write as:

$$c^n_{0,n-2} = c^{ld+2}_{0,ld} \tag{6.50}$$

i.e. in the subscript we write the internal degree followed by the cohomological degree.

3.5. The standard (co-)bar resolution and HH^***.** Let A be an associative k-algebra (where k is a commutative unital ring) with augmentation map (surjective) $\epsilon : A \twoheadrightarrow k = A/A_+$.

Now we can consider the complex

$$\ldots \to A_+ \otimes A_+ \otimes A \to A_+ \otimes A \to A \xrightarrow{\epsilon} \otimes k \to 0 \ . \tag{6.51}$$

The differential is given by

$$d(a_0 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \ . \tag{6.52}$$

FACT 10. *This is a complex* ($d^2 = 0$)*.*

When A is unital, this complex is exact. This is called the *bar construction.*[6.6]

For M an A-bimodule, then

$$C^n(A, M) \coloneqq \mathrm{Hom}_k\left(A_+^{\otimes n}, M\right) \tag{6.53}$$

with dc is a complex, called the *cobar complex.* This has cohomology:

$$\mathrm{HH}^n(A, M) \coloneqq H^n(C^\bullet(A, M), \delta) \tag{6.54}$$

$$\mathrm{HH}^n(A) \coloneqq \mathrm{HH}^n(A, A) \ . \tag{6.55}$$

called the Hochschild cohomology.

The internal degree and cohomological degree give us a bigrading on HH, written

$$\mathrm{HH}^n_{0,n-2}(A) \ . \tag{6.56}$$

REMARK 6.5. There are also Hochschild (co-)homology of general A_∞ algebras, but we won't need this.

[6.6] When this was first written down by Eilenberg and Mac Lane they used a vertical bar | as shorthand for the tensor product. This is where the name comes from.

3.6. Proof of the rigidity theorem.

Proposition 6.5. *For $A = R \oplus M$ as above, the following hold:*

(i) for all $i < l\,(d+2)$

$$\mathrm{HH}^{i}_{0,ld}\,(A) = 0\ . \tag{6.57}$$

(ii) $\dim \mathrm{HH}^{d+1}_{0,d}\,(A) \leq 1$.

Corollary 6.6. *Any two A_∞ refinements (m_n), (m'_n) are (strict A_∞) isomorphic iff*

$$\left[m_{d+2} - m'_{d+2}\right] = 0 \in \mathrm{HH}^{d+2}_{0,d}\,(A)\ . \tag{6.58}$$

Proof. $[m'_n - m_n] \in \mathrm{HH}^{ld+2}_{0,ld}$ for $0 \leq l \leq n$. Proposition 6.5 implies this is 0 if $ld + 2 < l\,(d+2)$. This is true automatically if $l > 1$. So we just have to consider $l = 1$. In this case we are looking at $m'_{d+2} - m_{d+2}$. □

Proof of Theorem 6.4. By (ii) from Proposition 6.5, either $m_n = 0$ for all $n > 2$ or unique up to scale. □

Remark 6.6. For X smooth, we have $m_{d+2} \neq 0$. This leaves us with the second possibility. This is based on looking at the ideal sheaf of a point, taking the Koszul resolution, then $m_d \neq 0$.

Proof of Proposition 6.5. We have $A = R \oplus M$. Then we want to write this as $k \oplus R_+ \oplus M$. Then for fixed l we have

$$C^i \coloneqq C^i_{0,ld} = \underbrace{C^i\,(0)}_{\deg=0} \oplus \underbrace{C^i\,(d)}_{\deg=d}\ . \tag{6.59}$$

These are contained in:

$$C^i\,(0) \subset \mathrm{Hom}\left(A_+^{\otimes i}, R\right) \tag{6.60}$$

$$C^i\,(d) \subset \mathrm{Hom}\left(A_+^{\otimes i}, M\right)\ . \tag{6.61}$$

These cochains look as follows. For $c \in C^i\,(0)$

$$c : \left[T\,(R_+) \otimes M \otimes T\,(R_+) \otimes \ldots \otimes M \otimes T\,(R_+)\right]_i \to R\ . \tag{6.62}$$

$T\,(R_+)$ is the tensor algebra over k and i is the number of factors in $H^*\,(L^*)$. The analogous statement holds for the cochains in $C^i\,(d)$.

This implies we have an exact sequence

$$0 \to C^\bullet\,(d) \to C^\bullet \to C^\bullet\,(0) \to 0\ . \tag{6.63}$$

So this reduces the proposition to showing that

(i) $H^i\,(C^\bullet\,(0)) = H^i\,(C^\bullet\,(d)) = 0$ for $i < l\,(d+2)$ and

(ii) $l = 1$ implies $H^{d+2}\,(C^\bullet\,(d)) = 0$, and $\dim H^{d+2}\,(C^\bullet\,(0)) \leq 1$.

Consider the filtration from

$$C^\bullet\,(0) = \bigoplus_{j \geq 0} C^\bullet\,(0)_j \tag{6.64}$$

Then δ decomposes as

$$\delta\,(x_j)_{j \geq 0} = \left(\sum_{j \leq j'}, \delta_{jj}, x_{j'}\right)_{j' \geq 0}\ . \tag{6.65}$$

The factors of M may decrease, never increase.

Now we work inductively by

$$0 \to \mathrm{gr}_i^\bullet C^\bullet(0) \to \tag{6.66}$$

This means we need to show (i) and (ii) for $H^*\left(C^\bullet(0)_j, \delta_{jj}\right)$. We can write this as

$$\left(C^\bullet(0)_j, \delta_{jj}\right) = \mathrm{Hom}\left(K_{l,j}^\bullet, R_j\right)[-m] \tag{6.67}$$

with $K_{l,j}^\bullet$ the jth internal degree graded piece of

$$K_l^\bullet = B^\bullet(k, M, \ldots, M, k) \tag{6.68}$$

some kind of generalized bar complex:

$$B^\bullet(M_1, \ldots, M_n) = M_1 \otimes T(R_+) \otimes \ldots \otimes T(R_+) \otimes M_n \ . \tag{6.69}$$

We prove this by induction on n. This is Proposition 3.7 in Polishchuk. Write $B^\bullet(M_1, \ldots, M_n)$ for the total complex for bicomplex with

$$\deg_{0/1}(x_1 \otimes t_1 \otimes \ldots \otimes t_{n-1} \otimes x_n) = \sum_{i \equiv 0/1 \pmod 2} \deg(t_i) \ . \tag{6.70}$$

Then the spectral sequence E_1 terms are

$$H^*(M_1 \otimes T(R) \otimes M_2) \otimes T(R_+) \otimes H^*(M_2 \otimes T(R_+) \otimes M_3) \otimes T(R_+) \ldots \ . \tag{6.71}$$

Then we have a lemma

Lemma 6.7.

$$H^i(B^\bullet(M_1, M_2)) = \begin{cases} M & i = -d-1 \\ 0 & o/w \end{cases} \ . \tag{6.72}$$

this implies $E_1 = B^\bullet(M_1', \ldots, M_n')$ up to shift $n' < n$. □

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