# Moduli spaces and tropical geometry

Lectures by Sam Payne

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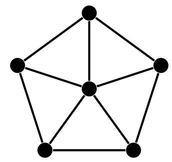

FIGURE 1. The 5-wheel.

## 1. Overview

Our goal is to understand the proof of the following theorem from [**CGP1**]:

Lecture 1; Wednesday January 22, 2020

THEOREM 0.1. $\dim_{\mathbb{Q}} H^{4g-6}(\mathcal{M}_g, \mathbb{Q})$ *grows exponentially with* $g$.

REMARK 0.1. $\mathcal{M}_g$ has complex dimension $3g-3$.

This theorem defied previous expectations.

CONJECTURE 1 (Kontsevich [**K1**], Church-Farb-Putman [**CFP2**]). *For fixed* $k > 0$, $H^{4g-4-k}(\mathcal{M}_j, \mathbb{Q}) = 0$ *for* $g \gg 0$.

The structure of the course is as follows.

- Constructing the moduli space
  (1) Nodal curves and stable reduction theorem
  (2) Deformation theory of nodal curves
  (3) The Deligne-Mumford moduli space of stable curves (1969)
- Cohomology
  (1) Mixed Hodge structure on the cohomology of a smooth variety (early 1970s)
  (2) Dual complexes of normal crossings divisors (tropical geometry)
  (3) Boundary complex of $\mathcal{M}_g$ (tropical moduli space)
- Cohomology of $\mathcal{M}_g$
  (1) Stable cohomology (Madsen-Weiss [**MW**])
  (2) Virtual cohomological dimension of $\mathcal{M}_g$ (Harer [**H2**]) (Vanishing of $H^{4g-5}$ (Church-Farb-Putman [**CFP2**], Morita-Sakasai-Suzuki))
  (3) Euler characteristic of $\mathcal{M}_g$ (Harer-Zagier [**HZ**])
- Graph complexes (Kontsevich [**K1**])
  (1) Feynman amplitudes and wheel classes. See fig. 1 for the 5-wheel.
  (2) Grothendieck-Teichmüller Lie algebra
  (3) Willwacher's theorem [**W**]
- Mixed Tate motives (MTM) over $\mathbb{Z}$
  (1) Mixed Tate motives
  (2) Brown's theorem (conjecture of Deligne-Ihara): "Soulé elements (closely related to Drinfeld's associators) generate a free Lie subalgebra."
  (3) Proof of exponential growth of $H^{4g-6}$.

CHAPTER 1

# Nodal curves and stable reduction theorem

Lecture 2; January 24, 2020

## 1. Nodal curves

We will work over $\mathbb{C}$. We want to show that nodal curves, and families thereof, can be written in a normal form in local coordinates. We will follow chapter X of [**ACG**].

Definition 1.1. A *nodal curve* is a complete curve such that every singular point has a neighborhood isomorphic (analytically over $\mathbb{C}$) to a neighborhood of 0 in $(xy = 0) \subset \mathbb{C}^2$.

Definition 1.2. A *family of nodal curves* over a base $S$ is a flat proper surjective morphism $f\colon \mathcal{C} \to S$ such that every geometric fiber is a nodal curve.

Recall that a flat morphism is the agreed upon notion of a map for which the fibers form a continuously varying family of schemes (or complex analytic spaces, varieties, etc.). Properness is a relative notion of compactness; it ensures that if $\{c_i\}$ is a sequence of points with no limit in $\mathcal{C}$ then $\{f(c_i)\}$ has no limit in $S$.

**Proposition 1.1.** *Let $\pi\colon X \to S$ be a proper surjective morphism of $\mathbb{C}$-analytic spaces. This is a family of nodal curves if and only if at every point $p \in X$ either $\pi$ is smooth at $p$ with one-dimensional fiber, or there is a neighborhood of $p$ that is isomorphic (over $S$) to a neighborhood of $(0, s)$ in $(xy = F) \subseteq \mathbb{C}^2 \times S$ where $s = \pi(p)$ and $F \in \mathfrak{m}_S \subseteq \mathcal{O}_{S,s}$.*

**Lemma 1.2.** *Let $f$ be holomorphic at $0 \in \mathbb{C}^2$. Then $(f = 0)$ has a node at 0 if and only if*

$$0 = f = \frac{\partial f}{\partial x} = \frac{pf}{\partial y} \tag{1.1}$$

*at 0, and the Hessian of $f$ at 0 is non-singular.*

This tells us that these nodes are the "simplest" possible singularities.

Proof. ($\Longrightarrow$): This direction is immediate.
($\Longleftarrow$): Suppose $0 = f = \partial_x f = \partial_y f$ at 0. Then

$$f = a - x^2 + 2bxy + cy^2 \tag{1.2}$$

where $a$, $b$, and $c$ are holomorphic functions. The Hessian is

$$\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix} \tag{1.3}$$

so being non-singular means exactly that

$$b^2 - ac \neq 0 \ . \tag{1.4}$$

After a generic linear change of coordinates, we can assume $a \neq 0$. We can then change coordinates to

$$x_1 = x + \frac{b}{a}y \qquad\qquad y_1 = y\ . \tag{1.5}$$

Then we can write

$$f = a_1 x_1^2 + c_1 y_1^2 \tag{1.6}$$

where $a_1(0), c_1(0) \neq 0$. Choose square roots[1.1] $\alpha$ and $\gamma$ of $a_1$ and $c_1$. Now replace $x_1$ and $y_1$ by $x_2 = \alpha x_1$ and $y_2 = \gamma y_1$ so that

$$f = x_2^2 + y_2^2\ . \tag{1.7}$$

Now for $x_3 = x_2 + iy_2$ and $y_3 = x_2 - iy_2$, we have $f = x_3 y_3$. □

Proof of Proposition 1.1. Let $\pi\colon X \to S$ be proper and surjective. Consider $x \in X$. Then either $\pi$ is smooth with 1-dimensional fiber at $x$ (nothing to show) or $x$ is a node in $\pi^{-1}(s)$, $s = \pi(x)$. Locally near $x$, we have a locally closed embedding $X \subseteq \mathbb{C}^r \times S$ (working over $S$). Then we get a left exact sequence of tangent spaces:

$$0 \longrightarrow T_x X_s \longrightarrow T_x X \longrightarrow T_s S \tag{1.8}$$

where $\dim T_x X_s = 2$. Choose a linear projection $\mathbb{C}^r \to \mathbb{C}^2$ which is an isomorphism on $T_x X_s$. Using this projection we get:

$$\begin{tikzcd} T_x X \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & \mathbb{C}^r \times T_s S \arrow[r] & \mathbb{C}^2 \times T_s S \end{tikzcd} \tag{1.9}$$

and the composition $T_x X \to \mathbb{C}^2 \times T_s S$ is injective. The implicit function theorem then tells us that there is a neighborhood of $x$ which embeds in $\mathbb{C}^2 \times S$ (over $S$). We should think of this as a family of plane curves: each fiber has a single defining equation. More specifically we have the following.

Fact 1 (Lemma 31.18.9 (Stacks project)). *If $\mathcal{Y} \to S$ is a smooth morphism and $D \subseteq \mathcal{Y}$ is flat over $S$, codimension 1 in $\mathcal{Y}$, then $D$ is a Cartier divisor.*

In particular, $X \subseteq \mathbb{C}^2 \times S$ is locally defined by a single equation $F = 0$. Now consider $\partial_x F$, $\partial_y F$, and the Hessian of $F$ with respect to $x$ and $y$. Then the proof of Lemma 1.2 shows

$$F = x_3 y_3 - f \tag{1.10}$$

where $f$ is a function on $S$ which vanishes at $s$. □

Lecture 3; January 27, 2020

## 2. Stability of nodal curves

The following is a corollary of Proposition 1.1.

**Corollary 1.3.** *A family of nodal curves $\pi\colon \mathcal{C} \to S$ is a local complete intersection (lci) morphism.*

This implies that there is a relative dualizing sheaf $\omega_{\mathcal{C}/S}$ which is locally free of rank 1.

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[1.1] There is some subtly here since these are functions rather than scalars. Because $a_1$ and $c_1$ are nonzero at 0, we can ensure that the image of $a_1$ and $c_1$ are, say, contained in an open half space. Now we can choose a branch of log which is defined on this half space. Then multiply by $1/2$ and exponentiate.

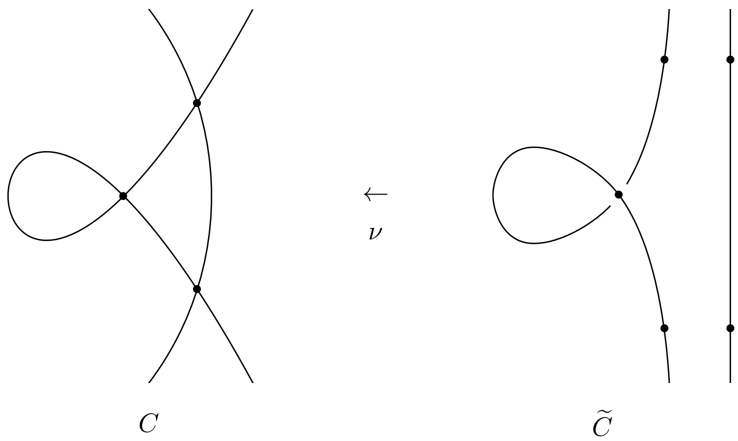



FIGURE 1. The normalization of a nodal curve. The nodal points of $C$ each have two preimages under the normalization $\nu$.

**2.1. Serre duality.** The point here is that the duality properties that we already know about for smooth curves extend naturally to nodal ones.

Let $C$ be a nodal curve (over a point). There is a (natural) isomorphism $H^1(C, \omega_C) \cong \mathbb{C}$. Then Serre duality tells us that for any coherent sheaf $\mathcal{F}$ on $C$,

$$H^1(C, \mathcal{F}) \times \operatorname{Hom}(\mathcal{F}, \omega_C) \to H^1(C, \omega_C) \cong \mathbb{C} \tag{1.11}$$

is a perfect pairing, i.e.,

$$H^1(C, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, \omega_C)^{\vee} \ . \tag{1.12}$$

In particular, if $\mathcal{F}$ is a vector bundle, then

$$H^1(C, \mathcal{F}) \cong H^0(C, \mathcal{F}^{\vee} \otimes \omega_C)^{\vee} \ . \tag{1.13}$$

We can form the normalization[1.2] of a nodal curve as in fig. 1.

Suppose $C$ is nodal with components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Let $\widetilde{C} \xrightarrow{\nu} C$ be the normalization. Write $\widetilde{C}_i$ for the normalization of $C_i$ and

$$\{p_j, q_j\} = \nu^{-1}(x_j) \tag{1.14}$$

(for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$).

A line bundle $L$ on $C$ has *multi-degree* $\underline{\deg}(L)$ to be

$$\underline{\deg}(L) = \left(\deg\left(L|_{C_1}\right), \ldots, \deg\left(L|_{C_s}\right)\right) \tag{1.15}$$

$$= \left(\deg\left(\nu^* L|_{\widetilde{C}_1}\right), \ldots, \deg \nu^* L|_{\widetilde{C}_s}\right) \ . \tag{1.16}$$

The following is a corollary to Serre duality.

**Corollary 1.4.** *If $C$ is connected, and $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ then $H^1(C, L) = 0$.*

By $\underline{\deg}(L) > \underline{\deg}(\omega_C)$ we mean $\deg\left(L|_{C_i}\right) \geq \deg\left(\omega_C|_{C_i}\right)$ for all $i$ and $\underline{\deg}(L) \neq \underline{\deg}(\omega_C)$.

[1.2] Locally, the corresponding algebraic construction is taking the integral closure of the coordinate ring.

PROOF. First note

$$H^1(C,L) \cong H^0\left(C, \omega_C \otimes L^{-1}\right) \ . \tag{1.17}$$

and $\underline{\deg}\left(\omega_C \otimes L^{-1}\right) < 0$.

On any connected component $C_i$ such that $\deg\left(\omega_L \otimes L^{-1}\right)\big|_{C_i} < 0$ all sections vanish. And all sections vanish on components that meet $C_i$, etc. ☐

**Corollary 1.5.** *$L$ is ample if and only if* $\deg\left(L|_{C_i}\right) > 0$ *for all $i$.*

PROOF. ($\Longrightarrow$): This direction is clear. The restriction of ample $L$ to any component is still ample.

($\Longleftarrow$): Suppose $\deg\left(L|_{C_i}\right) > 0$. It is enough to show that $L^{\otimes N}$ is very ample for some $N$. Choose $N$ sufficiently large so that

$$\deg\left(L^{\otimes N}\big|_{C_i}\right) > \deg\left(\omega_C|_{C_i}\right) + 2 \ . \tag{1.18}$$

Let $S \subseteq C$ be the union of two distinct smooth points. Then we have a short exact sequence

$$0 \to I_S \to \mathcal{O}_C \to \mathcal{O}_S \to 0 \tag{1.19}$$

which we can tensor with $L^{\otimes N}$ to get a sequence which is still exact, which gives us a long exact sequence

(1.20)

$$0 \longrightarrow H^0\left(L^{\otimes N}(-s)\right) \longrightarrow H^0\left(L^{\otimes N}\right) \longrightarrow H^0\left(L^{\otimes N}\big|_S\right) \longrightarrow H^1\left(L^{\otimes N}(-s)\right) \longrightarrow \dots$$

but $H^1(L(-s)) = 0$, so we have a surjection

$$H^0(L) \twoheadrightarrow H^0\left(L|_S\right) \ . \tag{1.21}$$

This shows that sections of $L^{\otimes N}$ separate the two points in $S$. Similar arguments show that sections of high tensor powers of $L$ separate arbitrary pairs of points and tangent vectors. Therefore, high tensor powers of $L$ are very ample, and so $L$ is ample. ☐

## 3. Description of $\omega_C$

We now describe the canonical sheaf of a nodal curve in terms of meromorphic differential forms. See [**L2**, Chapter 6] or [**HM**, Chapter 3, Section A] for proofs and further details.

**Proposition 1.6.** *Let $C$ be a nodal curve with nodes $x_1, \dots, x_r$, write $(p_i, q_i) = \nu^{-1}(x_i)$. Then*

$$\omega_C \cong \nu_*\left(\omega'_{\widetilde{C}}\left(p_1 + q_1 + \dots + p_r + q_r\right)\right) \tag{1.22}$$

*where $\omega'_{\widetilde{C}}(p_1 + \dots + q_r) \subseteq \omega_{\widetilde{C}}(p_1, \dots, q_r)$ is the subsheaf where*

$$\mathrm{res}_{p_i}(\omega) + \mathrm{res}_{q_i}(\omega) = 0 \ . \tag{1.23}$$

REMARK 1.1 (Rosenlicht differentials). There is a related explicit description of $\omega_{X/S}$ for a family of nodal curves. Near a point where $X/S \cong (xf = F) \subseteq \mathbb{C}^2 \times S$ $\omega_{C/S}$ is generated by $dx/x$ and $dy/y$ which satisfy

$$\frac{dx}{x} + \frac{dy}{y} = 0 \ . \tag{1.24}$$

DEFINITION 1.3. A nodal curve is *stable* if $\omega_C$ is ample.

Lecture 4; January 29, 2020

**Proposition 1.7.** *Let $X \to S$ be a family of nodal curves. Then*

$$\{s \in S : X_s \text{ is stable}\}$$

*is Zariski open.*

Proof. Let $L$ be any line bundle on $X$. Then

$$\left\{s \in S : L|_{X_s} \text{ is ample}\right\}$$

is Zariski-open. This is Theorem 1.2.17 of [**L1**]. □

Theorem 1.8. *A nodal curve $C$ is stable if and only if* $\mathrm{Aut}\,(C)$ *is finite.*

Proof. Say $C$ has components $C_1, \ldots, C_s$ and nodes $x_1, \ldots, x_r$. Write $\{p_i, q_i\} = \nu^{-1}(x_i)$ for the preimage of the nodes under the normalization $\nu$. Write $Q = \{p_1, q_1, \ldots, p_r, q_r\}$. Notice that $\mathrm{Aut}\,(C)$ is finite if and only if

$$\{\sigma \in \mathrm{Aut}\,(C) : \sigma \text{ acts by 1 on } \{C_1, \ldots, C_s\}\}$$

is finite.

Fix $C_i$. Note that $\mathrm{Aut}\,(C_i)$ is finite if and only if there are only finitely many automorphisms of $\tilde{C}_i$ that fix $Q \cap \tilde{C}_i$. This is the case exactly when

(1) $g\left(\tilde{C}_i\right) \geq 2$;
(2) $g\left(\tilde{C}_i\right) = 1$, and $Q \cap \tilde{C}_i \neq \emptyset$; or
(3) $g\left(\tilde{C}_i\right) = 0$ and $Q \cap \tilde{C}_i \geq 3$.

By direct computation, these are precisely the cases where

$$2g\left(\tilde{C}_i\right) - 2 + \#\left(Q \cap \tilde{C}_i\right) > 0.$$

The left hand side is $\deg\left(\omega_C|_{C_i}\right)$, by our description of the dualizing sheaf in terms of meromorphic differentials.

So we have shown that $\mathrm{Aut}\,(C)$ is finite if and only if the degree of the dualizing sheaf is positive on every component, which is equivalent to $\omega_C$ being ample, i.e., to $C$ being stable. □

Definition 1.4. A *graph* $G$ is a set $X(G)$ together with an involution $i : X(G) \circlearrowleft$ and a retraction $r\colon X(G) \to X(G)^i$. The vertices $V(G)$, half edges $H(G)$, and edges $E(G)$ are defined as:

$$\begin{aligned} V(G) &= X(G)^i \\ H(G) &= X(G) \setminus V(G) \\ E(G) &= H(G)/i\ . \end{aligned}$$

We say $r(h)$ is the vertex incident to $h \in H(G)$.

The *dual graph* $G(C)$ of a nodal curve $C$ is as follows. The vertices $\{v_1, \ldots, v_s\}$ correspond to the components $C_1, \ldots, C_s$; and the half-edges incident to $v_i$ are given by the points of $\tilde{C}_i \cap Q$. An edge is made from a pair of half-edges corresponding to a pair $\{p_i, q_i\}$. The "genus function" assigns the genus of $\tilde{C}_i$ to the corresponding vertex $v_i$. See fig. 2 for examples.

We can read the stability off from the dual graph. Every vertex labelled with a 1 should have at least one incident edge, and all unlabelled vertices should have valence at least 3.

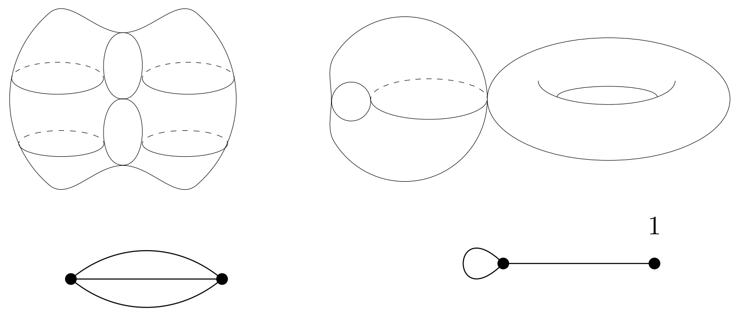



FIGURE 2. Two examples of genus 2 stable curves with their dual graphs below them. Notice we can read their stability off from the graphs. All unlabelled vertices have at least three incident edges, and the labelled one has one incident edge.

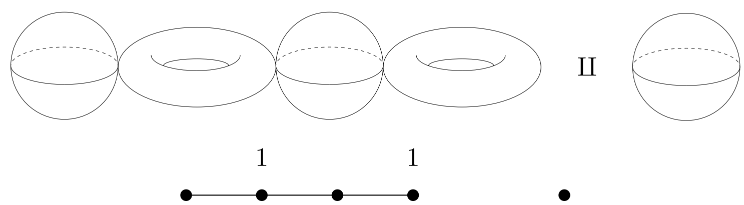



FIGURE 3. An example of an unstable genus 2 curve with its dual graph below it. Notice we can read the fact that it is unstable off of the graph. All three unlabelled vertices of valence less than 3.

Recall that the arithmetic genus of a curve $C$ is

$$p_a(C) = 1 - \chi(\mathcal{O}_C) \ .$$

In particular, if $C$ is connected then $p_a(C) = h^1(\mathcal{O}_C)$. Recall the Euler characteristic of a graph $G$ is

$$\chi(G) = \#V(G) - \#E(G) \ .$$

Note if $G$ is connected, then $h^1(G) = 1 - \chi(G)$. Also note that $C$ is connected if and only if $G(C)$ is connected. The dual graph also detects the arithmetic genus in the following sense.

THEOREM 1.9. *Let $C$ be a nodal curve. Then*

$$p_a(C) = 1 - \chi(G(C)) + \sum_v g(v) \ . \tag{1.25}$$

**Corollary 1.10.** *If $C$ is connected then*

$$p_a(C) = \sum_v g(v) + h^1(G) \ . \tag{1.26}$$

PROOF OF THEOREM 1.9. Proceed by induction on the number of nodes $\#E(G) = \#C^{\text{sing}}$. The base case is when $E(G) = \emptyset$, so the graph is just $s$ vertices $v_i$ with genus

$g(v_i)$. Then

$$1-\chi(\mathcal{O}_C)=1-s+\sum_i g(v_i) \tag{1.27}$$

as desired.

Now suppose $C'$ is obtained from $C$ by gluing two smooth points $p,q$ to $x$. Write $\pi\colon C\to C'$. Then we have an exact sequence of sheaves

$$0\longrightarrow \mathcal{O}_{C'}\longrightarrow \pi_*\mathcal{O}_C\longrightarrow \mathcal{O}_X\longrightarrow 0 \tag{1.28}$$

which implies the Euler characteristic of the middle term is the sum of the Euler characteristics of the other two terms. Now since $\pi$ is proper and finite, $\chi(\pi_*\mathcal{O}_C)=\chi(\mathcal{O}_C)$. Altogether this gives us:

$$\chi(\mathcal{O}_C)=\chi(\pi_*\mathcal{O}_C)=\chi(\mathcal{O}_{C'})+\chi(\mathcal{O}_X)=\chi(\mathcal{O}_{C'})+1\ , \tag{1.29}$$

and the theorem follows. □

Lecture 5; January 31, 2020

## 4. Stable reduction

There are two statements. The first is the nodal reduction theorem (which does not involve stability) and the second is stabilization, which adds uniqueness. The reference is [**ACG**] Chapter X, Section 4. Write

$$\Delta=\{z\in\mathbb{C}:|z|<\epsilon\} \tag{1.30}$$

for a small disk. Write $\Delta^\times=\Delta\setminus\{0\}$ for the punctured disk, both viewed as having one complex dimension.

Consider a flat proper surjective map $\pi\colon X\to\Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Write $X^\times$ for the complement of the fiber over 0. Let $k>0$ be an integer. Consider the map $\varphi_k\colon\Delta'\to\Delta$ from the disk to itself given by $z\mapsto z^k$. Note that $\varphi_k$ is *not* a smooth map. Now we can construct a base change

$$\begin{tikzcd} X_k^\times := X^\times\times_{\varphi_k}\Delta'^\times \arrow[r] \arrow[d, "\pi'"] & X^\times \arrow[d, "\pi"] \\ \Delta'^\times \arrow[r, "\varphi_k"] & \Delta^\times \end{tikzcd} \ . \tag{1.31}$$

Theorem 1.11 (Nodal reduction theorem). *Let $\pi\colon X\to\Delta$ be a flat proper surjective map such that $\pi|_{\Delta^\times}$ is a family of nodal curves. Then there exists an integer $k>0$ such that after a base change as above, the map $\pi'$ extends to a family of nodal curves over $\Delta$.*

Theorem 1.12 (Stable reduction). *If $\pi|_{\Delta^\times}$* is stable*, then this extension can be chosen to be stable, and the fiber over 0 depends* only *on $\pi|_{\Delta^\times}$ up to isomorphism.*

Remark 1.2. Uniqueness is related to separatedness for moduli of stable curves; existence and uniqueness is related to properness.

Remark 1.3. The intuition is as follows. Let $\Sigma$ be a class of objects with a moduli space (or stack) $\mathcal{M}$, i.e., there is a universal family $\mathcal{I}\to\mathcal{M}$ of objects in $\Sigma$ such that any family $X\to S$ of objects in $\Sigma$ is the pullback of the universal family under a unique morphism $S\to\mathcal{M}$. In other words,

$$\operatorname{Hom}(-,\mathcal{M})\cong\{\text{families of }\Sigma\text{ objects over }-\}\ . \tag{1.32}$$

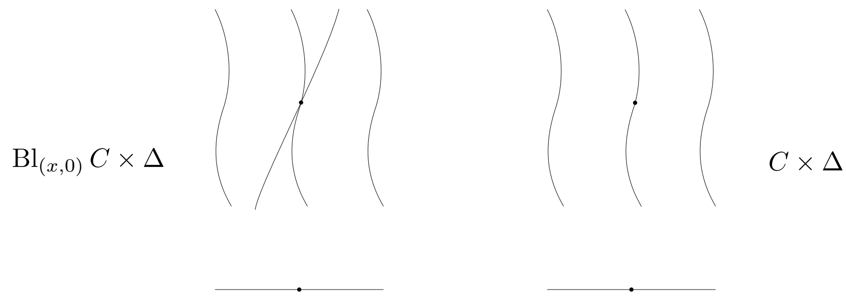



FIGURE 4. The constant family $\pi\colon C \times \Delta \to \Delta$ as well as the blowup $\mathrm{Bl}_{(x,0)} C \times \Delta \to \Delta$.

If $\mathcal{M}$ is separated, i.e., Hausdorff, then for $\Delta^\times \to \mathcal{M}$ there exists at most one extension $\Delta \to \mathcal{M}$. If $\mathcal{M}$ is proper, then each map $\Delta^\times \to \mathcal{M}$ extends uniquely to $\Delta \to \mathcal{M}$. Roughly speaking, when one has a large class of objects with a moduli space $\mathcal{M}'$ such that maps $\Delta^\times \to \mathcal{M}'$ extend in many different ways to $\Delta \to \mathcal{M}'$ then one naturally looks for a stability condition on the parametrized objects, so that the subspace $\mathcal{M} \subset \mathcal{M}'$ parametrizing stable objects is open and proper.

The notion of stability for nodal curves is a prototypical example. Indeed, if we don't impose stability, given a family of nodal curves $X^\times \to \Delta^\times$, then it may extend in many different ways to a nodal family $X' \to \Delta$ (and will always extend in many different ways, after a totally ramified base chance $\Delta \to \Delta$, given by $z \mapsto z^k$). So the existence and uniqueness of the special fiber in the theorem above is a special consequence of our specified stability condition.

EXAMPLE 1.1. Consider a smooth curve $C = (f = 0) \subseteq \mathbb{P}^2$. Then $C \times \Delta^\times \to \Delta^\times$ is a constant family which extends to $C \times \Delta \to \Delta$. Now for any $x \in C$, $C \times \Delta^\times$ also extends to $\mathrm{Bl}_{(x,0)} C \times \Delta$. We can picture this as in fig. 4.

The upshot is that moduli of nodal curves are not separated/Hausdorff. Lecture 6; February 3, 2020

**Interlude: Some motivating examples.**

*Degeneration of a smooth curve to a nodal curve.* We should think of the total space as being a surface. Consider the surface in fig. 5. This has two different rulings, as pictured in fig. 5. As in fig. 5, we can project this surface to a line by taking the intersection with parallel planes at different points of the line. Generically this gives us hyperbolas, but for two special values we get the union of two lines from the two different rulings. In particular this is given by the equation $xy = t^2 - t$. The node is exactly the point of tangency. So when we have a non-reduced curve, this is singular at every point on the curve.

*Degeneration of a smooth curve to a non-reduced curve.* Consider the surface defined by the equation $x^3 + t(x + y + 1) = 0$. At $t = 0$ we just get a line with multiplicity 3. This looks something like fig. 6.

*Understanding the base change and its fibers.* Again we consider a flat proper surjective map $\pi\colon X \to \Delta$ such that $\pi|_{\Delta^\times}$ is a family of nodal curves. For simplicity assume that in fact $X = \mathbb{P}^1 \times \Delta$. Consider the map $\varphi_k\colon \Delta' \to \Delta$ from the disk to itself given by $z \mapsto z^k$. Note that $\varphi_k$ is *not* a smooth map. In particular:

$$\varphi_k^{-1}(0) = \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) . \tag{1.33}$$

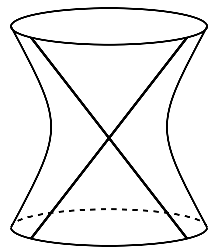


FIGURE 5. The surface given by $xy = t^2 - t$. Projection to the $t$-line has fibers which generically look like hyperbolas, but when the plane is tangent to the surface we get the union of two lines.

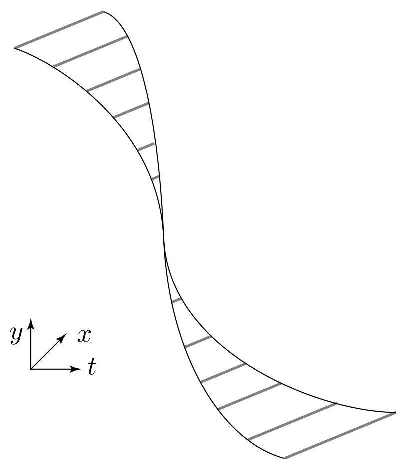



FIGURE 6. The surface $x^3 + t\,(x + y + 1) = 0$. Projecting to the $t$-line gives us smooth fibers which degenerate to a line with multiplicity 3 at $t = 0$.

Consider a base change for a family of curves

$$\begin{tikzcd} \left(\mathbb{P}^1 \times \Delta\right) \times_{\varphi_k} \Delta' \arrow[d, "\pi'"] & \mathbb{P}^1 \times \Delta \arrow[d, "\pi"] \\ \Delta' \arrow[r, "\varphi_k"] & \Delta \end{tikzcd} . \tag{1.34}$$

If we think of the preimage under $\varphi_k \circ \pi'$ we have actually made things worse, since the preimage of 0 is:

$$(\varphi_k \circ \pi)^{-1}(0) \simeq \mathbb{P}^1 \times \operatorname{Spec}\left(\mathbb{C}[\epsilon]/\epsilon^k\right) . \tag{1.35}$$

But in our construction we are replacing $\pi$ by $\pi'$, not $\varphi_k \circ \pi'$. The moral is that (at least for specific $k$) this replacement makes things better.

**Proof of the nodal reduction theorem.**

PROOF OF THEOREM 1.11. We will operate under the simplifying assumption that $X^\times \to \Delta^\times$ is smooth. The first step is to resolve the singularities of $X$. This is easy since $\dim X = 2$. First we normalize to get something regular in codimension 1. Then we blowup the finitely many singular points. Then repeat, i.e., normalize and then blowup the finitely many singular points. It is a theorem (not especially difficult) that this process terminates, giving us $X' \xrightarrow{\pi'} \Delta$ where $X'$ is smooth. However, the central fiber $(\pi')^{-1}(0) = X'_0$ might have arbitrary singularities. To deal with this, we first resolve the non-nodal singularities of $X_0'^{\mathrm{red}}$. The process is again straightforward; the reduced curve $X_0'^{\mathrm{red}}$ has finitely many singular points. We blow up the singular points that are not nodes. The resulting total space is still smooth, and we repeat, blowing up the finitely many singular points of the reduced special fiber that are not nodes. It is again a theorem (and not particularly difficult) that this process terminates. Hence we may assume that $X'$ is smooth and $X_0'^{\mathrm{red}}$ has only nodal singularities.

Locally near each node of $X_0'^{\mathrm{red}}$, the surface $X'$ is isomorphic (over $\Delta$) to $z = x^a y^b$ in $\mathbb{C}^2 \times \Delta$, where $x$ and $y$ are the coordinates on $\mathbb{C}^2$ and $z$ is the coordinate on $\Delta$. Similarly, near each smooth point of ${X'}_0^{\mathrm{red}}$, the surface $X'$ is isomorphic (over $\Delta$) to $z = x^c$. We can then cover ${X'}_0^{\mathrm{red}}$ by finitely many open sets where we have such local charts, and set

$$k = \operatorname{lcm}\{ab, c\} \ . \tag{1.36}$$

The rough idea is that this choice of $k$ will ensure that base change along $\varphi_k \colon \Delta \to \Delta$, given by $z \mapsto z^k$, will unwind the multiplicities of the components of $X'_0$.

In fact, the base change

$$X'' = \varphi_k^* X' \tag{1.37}$$

is not necessarily normal, but we claim that

CLAIM 1.1. $(X'')^\nu \xrightarrow{\pi'} \Delta'$ is a nodal family, where $(X'')^\nu \to X''$ is the normalization.

To prove the claim, we first consider $\pi'$ near a point where $X' \cong (z = x^c)$. Write $z = \zeta^k$ and $k = ch$ so that

$$x^c - z = x^c - \zeta^{ch} = \prod_{\omega^c = 1} \left(x - \omega\zeta^k\right) \ . \tag{1.38}$$

Note that this product gives rise to $c$ different smooth and irreducible components, which are disjoint in the general fiber but intersect in the special fiber. Normalizing pulls apart the intersections in the special fiber, giving rise to the disjoint union $\amalg_{\omega^c=1}\left(x - \omega\zeta^h\right)$, which is smooth over $\Delta$.

It remains to consider $\pi'$ near a point where $X' \cong \left(z = x^a y^b\right)$. Write $k = rsuv$ where $a = ru$, $b = su$, and $(r, s) = 1$. Write $\zeta$ for the coordinate on $\Delta'$. Then $X''$ is locally given by

$$0 = x^a y^b - \zeta^k \ . \tag{1.39}$$

This need not be normal. Indeed, if $u > 1$ then $x^r y^s$ obviously satisfies a nontrivial monic polynomial. Choose $\omega$ a primitive $u$th root of unity, so we have a factorization

$$\left(x^a y^b - \zeta^k\right) = \prod_{i=1}^{u} \left(x^r y^s - \omega^i \zeta^{rsv}\right) \ . \tag{1.40}$$

We can again pass to the disjoint union of surfaces with local defining equations $x^r y^2 - \omega^i z^{rsv}$, but this is only a partial normalization. Indeed, these surfaces are all isomorphic, but $\zeta^{vrs} = x^r y^s$ need not be normal. Then we claim the following.

CLAIM 1.2. The normalization is locally isomorphic to the surface defined by $\zeta^v = \alpha\beta$, where $\zeta$, $\alpha$, $\beta$ are coordinates on $\mathbb{C}^3$, with the normalization map given by $x = \alpha^s$, $y = \beta^r$.

To check that this is the normalization we need to check that

(1) this surface is normal,
(2) the map is generically one-to-one, and
(3) the map is surjective.

To see that this surface normal, notice that $\zeta^v = \alpha\beta$ is the toric surface corresponding to the cone spanned by $(1,0)$ and $(v,1)$ in $\mathbb{R}^2$, with respect to the standard lattice $\mathbb{Z}^2$. It is well-known and easy to prove that toric varieties are normal (see [**F**, §2.1]). We now show that the map is generically one-to-one. Given $(\alpha,\beta,\zeta)$ and $(\alpha',\beta',\zeta')$ so that

$$\alpha^s = (\alpha')^s \qquad \beta^r = (\beta')^r \qquad \zeta = \zeta' \ . \tag{1.41}$$

This means $\alpha' = \sigma\alpha$ for $\sigma$ an $s$th root of unity, and similarly $\beta' = \tau\beta$ for $\tau$ an $r$th root of unity. But if $\alpha$ and $\beta$ are nonzero, then $\alpha\beta = \alpha'\beta'$ implies $\sigma\tau = 1$, so $\sigma = \tau = 1$, so

$$(\alpha,\beta,\zeta) = (\alpha',\beta',\zeta') \ . \tag{1.42}$$

Since the points where $\alpha$ and $\beta$ are nonzero form an open dense set we are done.

Now consider $(x,y,\zeta)$ such that $x^r y^s = \zeta^{vrs}$. Then we must find points $(\alpha,\beta,\zeta)$ such that $\alpha\beta = \zeta^v$ and $x = \alpha^s$, and $y = \beta^r$. Choose $\alpha_0$, $\beta_0$ such that $\alpha_0^s = x$ and $\beta_0^r = y$. The point being that $\alpha_0 \cdot \beta_0 = \xi\zeta^v$ where $\xi^{rs} = 1$. Now write

$$1 = mr + ns \tag{1.43}$$

so the coordinates are

$$\alpha = \alpha_0\xi^{-mr} \qquad \beta = \beta_0\xi^{-ns} \ . \tag{1.44}$$

□

Now we claim that $X'$ in the nodal reduction theorem can be chosen to be stable if $X|_{\Delta^\times}$ is stable. Lecture 7; February 5, 2020

THEOREM 1.13 (Stabilization theorem). *Let $X \xrightarrow{\pi} \Delta$ be a family of nodal curves such that $\pi|_{\Delta^\times}$ is stable. Then there is*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr] & & X' \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{1.45}$$

*such that*

*(1) $\psi\colon X|_{\Delta^\times} \to X'|_{\Delta^\times}$ is an isomorphism;*
*(2) for each component $C_i$ of the central fiber $C = X_0$, $\psi$ maps $C_i$ either to a point, or birationally onto its image; and*
*(3) $X'$ is a family of stable curves.*[1.3]

*Moreover, $X' \to \Delta$ is unique.*

REMARK 1.4. The moral of the story is that

$$X' = \operatorname{Proj}_\Delta\left(\bigoplus_{n\geq 0} \pi_*\left(\omega_{X/\Delta}^{\otimes n}\right)\right) \ . \tag{1.46}$$

[1.3] This means that $X' \to \Delta$ is flat and proper, and its geometric fibers are stable nodal curves.

Recall that when we take this big direct sum we get a sheaf of graded $\mathcal{O}_\Delta$-algebras, so it makes sense to take relative $\operatorname{Proj}_\Delta$, provided that these graded $\mathcal{O}_\Delta$-algebras are finitely generated. The minimal model program deals with finite generation of things like this.

Proof. Suppose $C = X_0$ with components $C_1, \ldots, C_s$. Consider

$$\left\{C_i : \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i : \deg\left(\omega_C|_{C_i}\right) \leq 0\right\} . \tag{1.47}$$

Call this set the set of unstable components. We continue with our simplifying assumption that $X|_{\Delta^\times}$ is smooth and with connected fibers. Note that stability of the general fiber (in the absence of marked points) implies that $p_a(C) \geq 2$. Then the set of unstable components is:

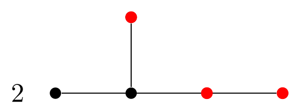
$$\left\{C_i : \omega_C|_{C_i} \text{ is not ample}\right\} = \left\{C_i \cong \mathbb{P}^1 : \#\left\{C_i \cap \operatorname{Cl}\left((C \setminus C_i)\right)\right\} \leq 2\right\} . \tag{1.48}$$

Then we have the following observation from [**ACG**]. Each connected component in the union of the unstable components is a chain of rational curves that intersects the union of the stable components at one or two points, on either or both ends of the chain. Let $C'$ be the curve obtained by contracting all unstable chains. Note that $p_a(C') = p_a(C)$.

Warning 1.1. Now we encounter a minor error in [**ACG**] (page 112, second sentence), where it is claimed that $C'$ is stable. The following is a counterexample to that claim.

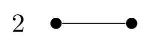
Counterexample 1. Suppose $C$ has the following dual graph:

(1.49)




with three unstable components (in red), that form two chains. After contracting both unstable chains, we get $C'$ with dual graph

(1.50)




which is not stable.

Nevertheless, the argument in [**ACG**] is easily salvaged. By iterating the procedure of contracting chains of unstable rational curves, one eventually does obtain a map $\varphi\colon C \to C'$ such that

(i) $\varphi|_{C_i}$ is either constant or birational onto its image (and an isomorphism on $C_i \cap C^{\text{smooth}}$).
(ii) $p_a(C') = p_a(C)$, and
(iii) $C'$ is stable.

Now, given $\varphi\colon C \to C'$ as above, with $C'$ stable, we follow the arguments in [**ACG**] to construct Lecture 8; February 7, 2020

$$\begin{tikzcd} X \arrow[rr, "\varphi'"] \arrow[dr, "\varphi"'] & \circlearrowleft & X' \arrow[dl, "\pi'"] \\ & \Delta & \end{tikzcd} \tag{1.51}$$

such that

(i) $\pi'\colon X' \to \Delta$ is a family of stable nodal curves,
(ii) $\varphi'$ is an isomorphism over $\Delta^\times$,
(iii) $X'_0 \cong C'$, and
(iv) $\varphi'|_{X_0} = \varphi$.

Let $L_0$ be $\varphi^* \omega_{C'}$ and $\underline{d} = \underline{\deg}\,(L_0)$, i.e., $\underline{d} = (d_1, \ldots, d_s)$, where $d_i = \deg\left(L_0|_{C_i}\right)$. Note all $d_i \geq 0$. Choose $d_i$ sections of $\pi$ that meet $C_i$ at distinct smooth points of $C$. (We can find a section through an arbitrary smooth point of $C$, using Hensel's lemma.) Let $D$ be the divisor on $X$ given by the union of these sections. Then $L = \mathcal{O}\,(D)$ is a line bundle on $X$, and

$$\underline{\deg}\left(L|_{X_0}\right) = \underline{\deg}\,(L_0)\ . \tag{1.52}$$

Then we make the following observations:

- $L$ is relatively ample over $\Delta^\times$,
- $L|_{X_0}$ is the pullback of an ample line bundle $L'$ on $C'$.

**Lemma 1.14.** *For any line bundle $M'$ on $C'$,*

$$H^i\,(C', M') = H^i\,(C, \varphi^* M') \tag{1.53}$$

*(for $i \in \{0,1\}$).*

Proof. The pullback induces an isomorphism on $H^0$, and

$$\begin{aligned} \chi\,(M') &= \chi\,(\mathcal{O}_{C'}) + \deg\,(M') \\ &= \chi\,(\mathcal{O}_C) + \deg\,(\varphi^* M') \\ &= \chi\,(\varphi^* M')\ . \end{aligned}$$

□

The consequences are as follows. For large $n$, $H^1\,(X_0, L^{\otimes n}) = 0$ (vanishing on $C'$ by ampleness and Lemma 1.14). This implies $h^0\,(X_s, L^{\otimes n})$ is a constant function of $s \in \Delta$. Therefore $\pi_* L^{\otimes n}$ is locally free by Grauert's theorem.[1.4]

Now we choose $n$ sufficiently large such that $L^{\otimes n}$ is very ample on fibers over $\Delta^\times$, and the restriction of $L^{\otimes n}$ to $C$ is the pullbacks of a very ample line bundle on $C'$. Then $\pi_* L^{\otimes n}$ induces $\psi\colon X \to \Delta \times \mathbb{P}^N$, and $\psi|_C$ agrees with $\varphi\colon C \to C'$. Take $X' = \mathrm{im}\,(\psi)$. Note that $X' \to \Delta$ is flat by the Hilbert polynomial criterion, and hence is the required family of stable nodal curves. ■

Definition 1.5. An *$n$-pointed nodal curve* is a pair $(X; p_1, \ldots, p_n)$ such that $X$ is a nodal curve, and $p_1, \ldots, p_n$ are distinct smooth points of $X$.

Definition 1.6. We say $(X; p_1, \ldots, p_n)$ is *stable* if and only if $\omega_X\,(p_1 + \ldots + p_n)$ is ample.

Theorem 1.15. *$(X; p_1, \ldots, p_n)$ is stable if and only if*

$$\mathrm{Aut}\,(X; p_1, \ldots, p_n) = \{\sigma \in \mathrm{Aut}\,(X) : \sigma\,(p_i) = p_i\} \tag{1.54}$$

*is finite.*

Definition 1.7. A *family of pointed nodal curves* is a family of nodal curves $\pi\colon X \to S$ with sections $\sigma_1, \ldots, \sigma_n$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1"'] \arrow[u, bend right=70, "\sigma_n"'] \end{tikzcd} \tag{1.55}$$

such that $\{\sigma_i\,(S)\}$ are disjoint and contained in $\pi^{\text{smooth}}$.

[1.4] Recall this says that if the dimension of $H^i$ is constant, the sheaf is coherent, and the morphism is proper, the $R^i \pi_*$ is locally free. See Chapter III of [**H4**].

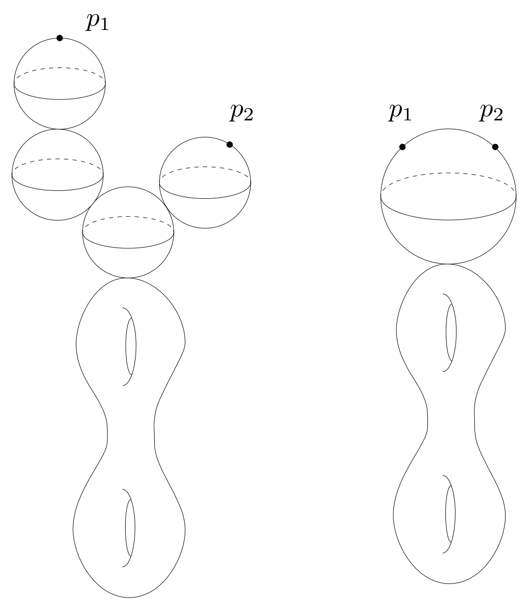



FIGURE 7. The left curve is unstable. When we stabilize, we contract to get a stable curve as on the right. Note that the marked points follow the contraction.

Then there are generalizations of nodal reduction, stabilization, and stable reduction for pointed curves as well. Note that, when we contract during the stabilization process, the marked points follow the contraction. See fig. 7.

An argument similar to the construction of $X'$ above shows that stabilization of nodal curves behaves well in families, i.e., given a family of nodal curves $\mathcal{C} \to S$ there is morphism of families of nodal curves $\mathcal{C} \to \mathcal{C}'$ over $S$ such that $\mathcal{C}'$ is a family of stable nodal curves and the restriction to a fiber $C$ is the stabilization map $\varphi\colon C \to C'$ obtained by contracting chains of unstable rational curves, and then iterating.

Lecture 9; February 10, 2020

CHAPTER 2

# Deformation theory

The reference for today's material is [**ACG**, Chapter XI, section 2].

DEFINITION 2.1. A *deformation* of a proper (connected) scheme $X$ is a flat and proper morphism $\mathcal{X} \xrightarrow{\varphi} S$ to a pointed scheme $(S, s)$ together with an isomorphism $\mathcal{X}_s \xrightarrow{\sim} X$.

An *infinitesimal deformation* is a deformation over $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$.

Sometimes these infinitesimal deformations are referred to as *first order deformations.*

A morphism of deformations is a cartesian square

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ (S,s) \arrow[r] & (S',s') \end{tikzcd} \tag{2.1}$$

such that the induced map

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rrr, bend right] & \mathcal{X}_s \arrow[r] & \mathcal{X}'_{s'} \arrow[r, "\sim"] & X \end{tikzcd} \tag{2.2}$$

is the identity.

THEOREM 2.1. *If $X$ is smooth then the isomorphism classes of infinitesimal deformations of $X$ are in natural bijection with $H^1(X, T_X)$.*

PROOF. The first step is to find a natural map from the isomorphism classes of infinitesimal deformations to $H^1(X, T_X)$. Let $\mathcal{X} \to S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ be an infinitesimal deformation. Since $X$ is smooth and smoothness is open in families, the morphism $\mathcal{X} \to S$ is smooth, and gives rise to the short exact sequence

$$0 \to T_X \to T_{\mathcal{X}} \to \varphi^* T_S \to 0 \ . \tag{2.3}$$

This induces a long exact sequence on cohomology:

(2.4)

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, T_X) \arrow[r] & H^0(X, T_{\mathcal{X}}) \arrow[r] & H^0(X, \varphi^* T_S) \arrow[r, "\delta"] \arrow[d, equal] & H^1(X, T_X) \arrow[r] & \cdots \\ & & & \mathbb{C} \cdot d\epsilon & & \end{tikzcd}$$

so $d\epsilon \in H^0(X, \varphi^* T_S)$ maps to some class $\delta(d\epsilon) \in H^1(X, T_X)$. We claim that the map taking an infinitesimal deformation $\mathcal{X} \to S$ to $\delta(d\epsilon)$ gives the required bijection.

Let $\mathcal{X} \to S$ be an infinitesimal deformation, $\mathcal{X}_0 \xrightarrow{\sim} X$. Note that $\mathcal{O}_{\mathcal{X}}$ is locally free (of rank 2) as an $\mathcal{O}_X$-module. Now we cover $\mathcal{X}$ by finitely many open $U_\alpha$ such that $\mathcal{O}_{\mathcal{X}}|_{U_\alpha}$ is free. Let $z_{\alpha_1}, \ldots, z_{\alpha_n}$ be local coordinates on these $U_{\alpha_i} \subseteq X$, and let $f_{\alpha\beta}$ be the transition functions, i.e., $z_\alpha = f_{\alpha\beta} z_\beta$. These functions satisfy

$$f_{\alpha\beta}\left(f_{\beta\gamma}\left(z_\gamma\right)\right) = f_{\alpha\gamma}\left(z_\gamma\right) \ . \tag{2.5}$$

Now consider $\mathcal{X}$ as being glued from the $U_\alpha \times S$. In particular $U_\alpha \times S$ is glued to $U_\beta \times S$ along $(U_\alpha \cap U_\beta) \times S$. So we have $z_\alpha$ and $\epsilon z_\alpha$, and

$$z_\alpha = \underbrace{f_{\alpha\beta}\left(z_\beta\right) + \epsilon g_{\alpha\beta}\left(z_\beta\right)}_{\tilde{f}_{\alpha b}(z_\beta)} \ , \tag{2.6}$$

i.e., we write $\tilde{f}_{\alpha\beta}$ for the new transition functions, and moreover, the new transition functions agree with the old ones modulo $\epsilon$. This is the gluing data describing the construction of $\mathcal{X}$ from the charts $U_\alpha \times S$.

Remark 2.1. The geometric picture is that we start with some $X$, then we spread this out into a higher-dimensional fibration. So assuming we've shrunk $U_\alpha$ sufficiently, it has no interesting topology, and if we look at it inside of the fibers all at once, this is just a cylinder $U_\alpha \times S$. So then the total space is glued out of these cylinders.

These transition functions satisfy the gluing condition

$$\tilde{f}_\alpha\left(\tilde{f}_{\beta\gamma}\left(z_\gamma\right)\right) = \tilde{f}_{\alpha\gamma}\left(z_\gamma\right) \tag{2.7}$$

$$= \underbrace{f_{\alpha\beta}\left(f_{\beta\gamma}\right)}_{f_{\alpha}\gamma} + f_{\alpha\beta}\left(\epsilon g_{\beta\gamma}\right) + \epsilon g_{\alpha\beta}\left(f_{\beta\gamma}\right) \ . \tag{2.8}$$

The first term just comes from gluing on $X$, and the second term can be thought of as a version of Leibniz's rule:

$$\frac{\partial f_{\alpha\beta}}{\partial z_\beta} g_{\beta\gamma} + g_{\alpha\beta} = g_{\alpha\gamma} \ . \tag{2.9}$$

Another way of writing this is that:

$$\Theta_{\alpha\beta} = \left(g_{\alpha_i\beta_i}\right) \begin{pmatrix} \partial/\partial z_{\alpha_1} \\ \vdots \\ \partial/\partial z_{\alpha_n} \end{pmatrix} \in H^0\left(U_{\alpha\beta}, T_X|_{U_{\alpha\beta}}\right) \tag{2.10}$$

is a cocycle, so it defines a class:

$$[\Theta_{\alpha\beta}] \in H^1\left(X, T_X\right) \tag{2.11}$$

which is the image of 1 in $H^1(X, T_X)$.

The point of this is that the deformation $\varphi\colon X \to S$ goes to the coboundary $\delta\left(\partial/\partial\epsilon\right)$ where we regard $\partial/\partial\epsilon \in H^0\left(\varphi^* T_S\right)$.

Moreover, we can reverse engineer the argument above, i.e., given a 1-cocycle with coefficients in $T_X$ we can construct a deformation $\mathcal{X} \to S$. One also checks, by direct computation with cocycles, that cohomologous cocycles give rise to isomorphic deformations, and hence one gets a well-defined inverse to the map

$$\{\text{isomorphism classes of deformations}\} \twoheadrightarrow H^1\left(X, T_X\right) \ . \tag{2.12}$$

□

Let $\mathcal{X} \to (B, b_0)$ be a deformation. Recall that elements of $T_{B,b_0}$ (i.e., "tangent vectors") correspond to morphisms $S \to B$ that send the underlying point of $S = \operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$ to $b_0$. Pulling back $\mathcal{X}$ along such a tangent vector $S \to B$ gives an infinitesimal deformation of the special fiber $X = \mathcal{X}_{b_0}$. The natural bijection between isomorphism classes of infinitesimal deformations of $X$ and $H^1(X, T_X)$ therefore gives rise to the *Kodaira-Spencer map* $\rho\colon T_{B,b_0} \to H^1(X, T_X)$. (We have constructed this map set-theoretically, but it is a linear map of vector spaces.) Lecture 10; February 12, 2020

Let us now consider the case where $X = C$ is a smooth and stable curve, i.e., a smooth curve of genus $g(C) \geq 2$. By Serre duality, we have a canonical isomorphism

$$H^1(C, T_C) \cong H^0(C, T_C^\vee \otimes \omega_C)^\vee \cong H^0\left(C, \omega_C^{\otimes 2}\right)^\vee \ . \tag{2.13}$$

Note that sections of $\omega^{\otimes 2}$ are sometimes referred to as the *quadratic differentials*. Since $\deg\left(\omega_C^{\otimes 2}\right) = 4g - 4$ and $g \geq 2$, Riemann-Roch tells us that

$$h^0\left(\omega_C^{\otimes 2}\right) = 3g - 3 \ . \tag{2.14}$$

Hence the space of infinitesimal deformations of $C$ has dimension $3g - 3$.

The ideal sheaf of a point $p \in C$ is locally free,[2.1], so we have a short exact sequence

$$0 \longrightarrow I_p \cong \mathcal{O}(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0 \quad . \tag{2.15}$$

Tensoring with $T_C(p)$ gives us the short exact sequence

$$0 \to T_C \to T_C(p) \to T_C(p)|_p \to 0 \ . \tag{2.16}$$

This induces a long exact sequence

$$H^0(C, T_C) \longrightarrow H^0(C, T_C(p)) \longrightarrow H^0(p, T_C(p)) \xrightarrow{\delta} H^1(C, T_C) \longrightarrow \cdots \quad . \tag{2.17}$$

Note that $H^0(C, T_C(p))$ vanishes, since $g(C) \geq 2$, and the vector space $H^0(p, T_C(p))$ is 1-dimensional. Hence the choice of $p$ gives rise to a 1-dimensional subspace $\delta_p \subseteq H^1(C, T_C)$ which is an infinitesimal deformation well-defined up to $\mathbb{C}^\times$. These are called *Schiffer deformations*.

An alternative construction is as follows. The complete linear series of quadratic differentials gives a map

$$C \to \mathbb{P}\left(H^0\left(C, \omega_C^{\otimes 2}\right)^\vee\right) \ , \tag{2.18}$$

and $p \in C$ maps the point $\delta_p$ in this projective space.

Fact 2 (Important fact). *Schiffer deformations are integrable, i.e., they come from deformations over a small disk $\Delta = \{z : |z| < b\}$.*

The idea is as follows. Let $p \in C$ be a point in our curve. Then let $U$ be a neighborhood of $p$ with a local coordinate $z\colon U \xrightarrow{\sim} \Delta$ which maps $U$ isomorphically to the disk $\Delta$. Then define:

$$U' = \{z \in U : |z| < b/3\} \qquad\qquad U'' = \{w \in U : |w| < 2b/3\} \ . \tag{2.19}$$

That is $U' \subset U'' \subset U$. Then we can think of $C$ as being obtained by gluing

$$C = (C \setminus U') \cup U'' \ . \tag{2.20}$$

[2.1] This is using the fact that $C$ is smooth and 1-dimensional. If instead $p$ were a point on a smooth surface, or a node on a singular curve, for example, then the ideal sheaf of $p$ will have rank 1 everywhere away from the point, but the fiber over $p$ has rank 2.

In particular, for $t$ sufficiently small, consider the space $C_t$ obtained by gluing $C \setminus U'$ to $U''$ along $w = z + t/z$.

Claim 2.1 ([**ACG**, XI, §2]). $\delta_p$ is the infinitesimal deformation associated to the family $\{C_t\}$.

Moreover, if we choose multiple distinct points $p_1, \ldots, p_s \in C$, then we get multiple Schiffer deformations that are simultaneously integrable. Indeed, by choosing disjoint coordinate patches at the points and performing the construction above on each patch, we can simultaneously integrate all $\delta_{p_i}$ to get

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \Delta^s \end{array} . \tag{2.21}$$

Note that

$$f = f_{\left|\omega_C^{\otimes 2}\right|} \colon C \otimes \mathbb{P}\left(H^0\left(C, \omega^{\otimes 2}\right)^\vee\right) \tag{2.22}$$

is nondegenerate, i.e., the image is not contained in a hyperplane, so the Schiffer deformations span $H^1(C, T_C)$. In particular, if we choose $s = 3g - 3$ general points $p_1, \ldots, p_s$ in $C$, then representatives of $\{\delta_{p_1}, \ldots, \delta_{p_s}\}$ form a basis for $H^1(C, T_C)$. Hence the Kodaira-Spencer map for $\varphi \colon \mathcal{C} \to \Delta^s$

$$\rho \colon T_{\Delta^s, 0} \xrightarrow{\sim} H^1(C, T_C) \tag{2.23}$$

is an isomorphism.

The existence of such a family, over a smooth base, for which the Kodaira-Spencer map is an isomorphism is a very special feature of the geometry and deformation theory of curves. It is related to the existence of Kuranishi families and smoothness of moduli spaces (stacks) of curves, as we will discuss in the coming lectures. The paper [**V2**] shows that moduli spaces (stacks) of smooth projective surfaces with very ample canonical bundle exhibit arbitrarily bad singularities, so the pleasantness of this situation for curves must not be taken for granted.

Definition 2.2. A deformation

$$\begin{array}{c} \mathcal{C} \\ \downarrow{\scriptstyle\varphi} \\ (B, b_0) \end{array} \tag{2.24}$$

$(C_{b_0} \xrightarrow{\sim} C)$ is a Kuranishi family if for any deformation $\mathcal{D} \xrightarrow{\varphi} (E, e_0)$ of $C$, and any sufficiently small neighborhood $U$ of $e_0$, there is a unique morphism of deformations

$$\varphi'|_U \to \varphi . \tag{2.25}$$

These can be thought of as *local moduli spaces*. We will study Kuranishi families not only for smooth curves, but also for nodal curves. Lecture 11; February 14, 2020

## 1. Deformations of nodal curves

Happy Valentine's Day. Let $C$ be a nodal curve.

Theorem 2.2. *There is a natural bijection between isomorphism classes of infinitesimal deformations of $C$ and* $\operatorname{Ext}^1\left(\Omega_C^1, \mathcal{O}_C\right)$.

REMARK 2.2. If $C$ is in fact smooth, then the sheaf of Kähler differentials $\Omega_C^1$ is the dualizing sheaf $\Omega_C^1 \cong \omega_C$. So

$$\operatorname{Ext}^1(\omega_C, \mathcal{O}_C) \cong \operatorname{Ext}^1\left(\omega_C^{\otimes 2}, \omega_C\right) \tag{2.26}$$

$$\cong H^0\left(C, \omega_C^{\otimes 2}\right) \tag{2.27}$$

$$\cong H^1\left(C, \left(\omega_C^{\otimes 2}\right)^\vee \otimes \omega_C\right) \tag{2.28}$$

$$\cong H^1(C, T_C) \tag{2.29}$$

where the second and third isomorphisms come from (appropriate versions of) Serre duality. So, in the special case where $C$ is smooth, we recover our previous identification of infinitesimal deformations with $H^1(C, T_C)$.

PROOF. Let $S = \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ and let $\mathcal{C} \to S$ be an infinitesimal deformation of $C$. Then we get an exact sequence of sheaves on $\mathcal{C}$:

$$\varphi^*\Omega_S^1 \to \Omega_{\mathcal{C}}^1 \to \Omega_{\mathcal{C}/S}^1 \to 0\ . \tag{2.30}$$

Now tensoring is right-exact, so we can tensor with $\mathcal{O}_C$ to get:

$$\varphi^*\Omega_S^1 \otimes \mathcal{O}_C \to \Omega_{\mathcal{C}}^1 \otimes \mathcal{O}_C \to \Omega_C^1 \to 0\ . \tag{2.31}$$

CLAIM 2.2.1. $\varphi^*\Omega_S^1 \otimes \mathcal{O}_C \to \Omega_{\mathcal{C}}^1 \otimes \mathcal{O}_C$ is injective.

PROOF. The sheaf $\varphi^*\Omega_S^1 \otimes \mathcal{O}_C$ is trivial of rank 1, generated by $d\epsilon \otimes 1$. At a smooth point of $C$, $\mathcal{C}$ is locally $S \times \mathcal{C}$, and hence the image of $d\epsilon \otimes 1$ is nonzero near this point. Since the smooth points are open and dense in $C$, this is enough to prove the claimed injectivity. □

The claim shows that (2.31) is short exact. Using the identification $\mathcal{O}_C \xrightarrow{\cong} \varphi^*\Omega_S^1 \otimes \mathcal{O}_C$ given by $1 \mapsto d\epsilon \otimes 1$, we can then view $\Omega_{\mathcal{C}}^1 \otimes \mathcal{O}_C$ as an extension of $\Omega_C^1$ by $\mathcal{O}_C$. We thus get a map from isomorphism classes of infinitesimal deformations of $C$ to extension classes in $\operatorname{Ext}^1\left(\Omega_C^1, \mathcal{O}_C\right)$.

CLAIM 2.2.2. This assignment of extension classes to isomorphism classes of infinitesimal deformations of $C$ is injective.

PROOF. Suppose $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are infinitesimal deformations giving rise to the same extension class. We must show that these infinitesimal deformations are isomorphic.

Since the induced extensions of $\Omega_C^1$ by $\mathcal{O}_C$ are isomorphic, we have a sheaf isomorphism $\gamma$ such that the following diagram commutes:

$$\begin{tikzcd} & \Omega_{\mathcal{C}}^1 \otimes \mathcal{O}_C \arrow[dd, "\cong"', "\gamma"] \arrow[dr] & \\ \mathcal{O}_C \arrow[ur] \arrow[dr] & & \Omega_C^1 \\ & \Omega_{\mathcal{C}'}^1 \otimes \mathcal{O}_C \arrow[ur] & \end{tikzcd}\ . \tag{2.32}$$

To prove the claim, we must show that there is an isomorphism $\beta\colon \mathcal{O}_{\mathcal{C}} \xrightarrow{\sim} \mathcal{O}_{\mathcal{C}'}$ (over $S$) which restricts to the identity on $\mathcal{O}_C$. □

Claim 2.2.2'. For each $h \in \mathcal{O}_{\mathcal{C}}$, there is a unique $\beta(h) \in \mathcal{O}_{\mathcal{C}'}$ such that

$$\beta(h)|_C = h|_C \tag{2.33}$$

and

$$d\beta(h)|_C = \gamma(\, dh|_C) \ . \tag{2.34}$$

Here, we write $d\beta(h)|_C$ for the image of $d\beta(h)$ in $\Omega^1_{\mathcal{C}} \otimes \mathcal{O}_C$.

Proof. First we show that uniqueness holds, even locally. If $f|_C = 0$, then locally $f = \epsilon g$. This implies $df = g\, d\epsilon|_C$. If, in addition, $df|_C = 0$ then $f = 0$. Local uniqueness follows.

Given local uniqueness and the basic properties of sheaves, it is enough to prove the existence of $\beta(h)$ locally. First, extend $h|_C$ to some $\tilde{h}$ on $\mathcal{C}'$. The difference between $d\tilde{h}\Big|_C$ and $\gamma(\, dh|_C)$ is of the form $g\, d\epsilon$. Set

$$\beta(h) = \tilde{h} - \epsilon g \ . \tag{2.35}$$

This gives rise to a canonical set theoretic map

$$\beta \colon \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}'} \ . \tag{2.36}$$

This is a priori only a map of sheaves of sets, but in fact it is a map of sheaves of rings, as can be seen using the Leibniz rule. This proves claim 2.2.2', which implies claim 2.2.2. $\square$

Claim 2.2.3. The map from deformations to extensions is surjective.

Proof. Now we have the following local-to-global exact sequence for Ext:

$$0 \to H^1\left(C, \mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to \mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right) \to H^0\left(C, \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_C}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \to 0 \ . \tag{2.37}$$

Recall that the sheaf $\mathcal{E}\mathrm{xt}$ encodes information about local extensions, i.e., the stalk of $\mathcal{E}\mathrm{xt}^1$ at $p$ classifies extensions of $\Omega^1_{\mathcal{C},p}$ by $\mathcal{O}_{\mathcal{C},p}$. In particular, it vanishes at points where $\Omega^1_{\mathcal{C}}$ is locally free, and hence is supported on $C^{\mathrm{sing}}$:

$$H^0\left(\mathcal{E}\mathrm{xt}^1\left(\Omega^1_C, \mathcal{O}_C\right)\right) = \bigoplus_{p \in C^{\mathrm{sing}}} \mathrm{Ext}^1\left(\Omega^1_{C,p}, \mathcal{O}_{C,p}\right) \ . \tag{2.38}$$

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The local-to-global exact sequence is a consequence of the local-global Ext spectral sequence:

$$E_2^{pq} = H^p\left(\mathcal{E}\mathrm{xt}^q\right) \Rightarrow \mathrm{Ext}^{p+q} \ . \tag{2.39}$$

This is an example of a Grothendieck spectral sequence for the composition of two functors. See Wikipedia page and this Stack Exchange post for a more detailed discussion and further references.

The lefthand term in (2.37) is naturally identified with the set of isomorphism classes of locally trivial extensions of $\Omega^1_C$ by $\mathcal{O}_C$, as follows.

An extension $\mathcal{O}_C \to \mathcal{F} \to \Omega^1_C$ is locally trivial if there is an open cover $\{U_\alpha\}$ of $C$, such that the extension splits on each $U_\alpha$ via isomorphisms

$$\mathcal{F}|_{U_\alpha} \xrightarrow{\varphi_\alpha} \mathcal{O}_C \oplus \Omega^1_C \ . \tag{2.40}$$

On $U_\alpha \cap U_\beta$, we then have transition functions

$$\mathcal{O}_C \oplus \Omega^1 \xrightarrow{\varphi_\alpha^{-1}} \mathcal{F}\left(U_\alpha \cap U_\beta\right) \xrightarrow{\varphi_\beta} \mathcal{O}_C \oplus \Omega^1_C \quad . \tag{2.41}$$

The maps $\mathcal{O}_C \to \mathcal{O}_C$ and $\Omega^1_C \to \Omega^1_C$ induced by $\varphi_\beta \circ \varphi_\alpha^{-1}$ are the identity, and the map $\mathcal{O}_C \to \Omega^1_C$ is zero. Let $f_{\alpha\beta}$ be the induced map $\Omega^1_C(U_\alpha \cap U_\beta) \to \mathcal{O}_C(U_\alpha \cap U_\beta)$. Then $\{f_{\alpha\beta}\}$ is a 1-cocycle for $\mathcal{H}\mathrm{om}\left(\Omega^1, \mathcal{O}\right)$. Different choices of trivialization give rise to cohomologous cocycles. Conversely, a 1-cocycle for $\mathcal{H}\mathrm{om}\left(\Omega^1, \mathcal{O}\right)$ gives rise to a locally trivial extension, and cohomologous cocycles give rise to isomorphic extensions. In this way $H^1\left(C, \mathcal{H}\mathrm{om}\left(\mathcal{O}^1_C, \mathcal{O}_C\right)\right)$ classifies locally trivial extensions of $\Omega^1_C$ by $\mathcal{O}_C$.

Let us now turn attention to extensions of $\Omega^1_C$ by $\mathcal{O}_C$ that are not locally trivial. Roughly speaking, such extensions correspond geometrically to "smoothings of nodes". Near a node $p \in C^{\mathrm{sing}}$, the curve $C$ is locally isomorphic to $(xy = 0) \subset \mathbb{C}^2$. The conormal exact sequence for this inclusion is

$$I_C/I_C^2 \to \Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_C \to \Omega^1_C \to 0\ . \tag{2.42}$$

Note that $I_C/I_C^2$ is locally free of rank 1; it is the line bundle $\mathcal{O}_{\mathbb{C}^2}(-C)|_C$.

Localizing the conormal exact sequence at $p$ and deriving the functor $\mathrm{Hom}\left(-, \mathcal{O}_{C,p}\right)$ gives us the long-exact sequence:

$$\mathrm{Hom}\left(\Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}_{C,p}, \mathcal{O}_{C,p}\right) \xrightarrow{\eta} \mathrm{Hom}\left((I_C/I_C^2)_p, \mathcal{O}_{C,p}\right) \longrightarrow \mathrm{Ext}^1\left(\Omega^1_{C,p}, \mathcal{O}_{C,p}\right) \longrightarrow 0\ . \tag{2.43}$$

The last term is 0 because

$$\mathrm{Ext}^1\left(\Omega^1_{\mathbb{C}^2}\big|_{C,p}, \mathcal{O}_{C,p}\right) \simeq \mathrm{Ext}^1\left(\mathcal{O}^{\oplus 2}_{C,p}, \mathcal{O}_{C,p}\right) = 0\ . \tag{2.44}$$

The image of $\eta$ is

$$\mathfrak{m}_p \,\mathrm{Hom}\left((I_C/I_C^2)_p, \mathcal{O}_{C,p}\right) \cong \mathfrak{m}_p \tag{2.45}$$

so we get a non-canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right) \cong \mathcal{O}_{C,p}/\mathfrak{m}_p \cong \mathbb{C}\ . \tag{2.46}$$

Carrying through the computations more carefully, we would get a canonical isomorphism

$$\mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right) \cong T_{\widetilde{C},p_1} \otimes T_{\widetilde{C},q_1} \tag{2.47}$$

where $\{p_1, q_1\} = \nu^{-1}(p) \subseteq \widetilde{C}$. See [**ACG**, XI, §3].

Example 2.1. For $C = (xy = 0)$ and $a \in \mathbb{C}$, we have the deformation $xy = a\epsilon$. So we get a Kodaira-Spencer class

$$\rho\left(xy = a\epsilon\right) \in \mathrm{Ext}^1\left(\mathcal{O}^1_{C,p}, \mathcal{O}_{C,p}\right)\ . \tag{2.48}$$

A direct computation/diagram chase yields

$$\rho\left(xy = a\epsilon\right) = a\rho\left(xy = \epsilon\right),\ . \tag{2.49}$$

and $\rho(xy = \epsilon) \neq 0$. Putting this together with the calculation showing $\mathrm{Ext}^1$ is 1-dimensional, we see that all isomorphism classes of infinitesimal deformations are of this form.

This concludes the proof of claim 2.2.3, □

which completes the proof of Theorem 2.2. ■

**Proposition 2.3.** *$H^1\left(C, \mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right)\right) \cong H^1\left(\widetilde{C}, T_{\widetilde{C}}\left(-p_1 - q_1 - \ldots - p_r - q_r\right)\right)$ where the $p_i, q_i$ are the preimages of the node $x_i \in C^{sing} = \{x_1, \ldots, x_r\}$. The RHS classifies deformations of $\left(\widetilde{C}, p_1, q_1, \ldots, p_r, q_r\right)$.*

PROOF. It is enough to show that

$$\mathcal{H}\mathrm{om}\left(\Omega^1_C, \mathcal{O}_C\right) \cong T_{\widetilde{C}}\left(-p_1 - \ldots - q_r\right) \ . \tag{2.50}$$

The idea is that $\Omega^1_C = \mathcal{I}\omega_C$ where $\mathcal{I}$ is the ideal sheaf of $C^{\mathrm{sing}}$. Locally near $x_j$,

$$\mathcal{I}\omega \cong \mathcal{I}\omega_{\widetilde{C}_1}\left(-p_j\right) \oplus \mathcal{I}_{\widetilde{C}_2}\left(-q_j\right) \tag{2.51}$$

and

$$\mathcal{I}_{x_j} = \nu_* \mathcal{I}_{(p_j \cup q_j)} \ . \tag{2.52}$$

□

THEOREM 2.4. *Let $C$ be a nodal curve. Then there is a deformation $\mathcal{C} \to (\Delta^s, 0)$ such that the Kodaira-Spencer map $\rho\colon T_0(\Delta^s) \to \mathrm{Ext}^1\left(\Omega^1_C, \mathcal{O}_C\right)$ is an isomorphism. From our short exact sequences we explicitly get that*

$$s = 3g - 3 + \dim \mathrm{Hom}\left(\Omega^1_C, \mathcal{O}_C\right) \tag{2.53}$$

$$= 3g - 3 + h^0\left(\widetilde{C}, T_{\widetilde{C}}\left(-p_1 - \ldots - q_r\right)\right) \tag{2.54}$$

$$= 3g - 3 \tag{2.55}$$

*where the $h^0$ vanishes since $C$ is stable.*

PROOF. Glue Schiffer deformations at smooth points to the $(xy = a\epsilon)$ deformations at the nodes. □

Lecture 13;
February 19, 2020

## 2. Kuranishi families

We will follow [**ACG**, XI, §§4-6]. Recall the following definition.

DEFINITION 2.3. A deformation $\mathcal{X} \to (B, b_0)$ of $X$ is a *Kuranishi family* if for any other deformation $\mathcal{X}' \to (B', b_0')$ and any sufficiently small neighborhood $U$ of $b_0'$, there is a unique morphism of deformations:

$$\begin{array}{ccc} \mathcal{X}'_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ (U, b_0') & \longrightarrow & (B, b_0) \end{array} \ . \tag{2.56}$$

By definition, a morphism of deformations is a cartesian square, so $\mathcal{X}'_U$ is the fiber product of $\mathcal{X}$ and $U$ over $B$, i.e., the deformation $\mathcal{X}'_U \to (U, b_0')$ is just $\mathcal{X} \to (B, b_0)$ pulled back along the map $(U, b_0) \to (U', b_0')$. In this sense, a Kuranishi family is a moduli space for deformations.

We can then make the following observations.

1. When a Kuranishi family exists, it is locally unique up to unique isomorphism, i.e., if $\begin{array}{c}\mathcal{X}\\ \downarrow \\ (B, b_0)\end{array}$ and $\begin{array}{c}\mathcal{X}'\\ \downarrow \\ (B', b_0')\end{array}$ are Kuranishi families, then for every sufficiently

small neighborhood $U$ of $b_0$ there is a unique neighborhood $U'$ of $b_0'$ and a unique isomorphism of deformations:

$$\begin{tikzcd} \mathcal{X}_U \arrow[r, "\simeq"] \arrow[d] & \mathcal{X}_{U'} \arrow[d] \\ (U, b_0) \arrow[r, "\simeq"] & (U', b_0') \end{tikzcd} . \tag{2.57}$$

2. The Kodaira Spencer map of any Kuranishi family

$$\rho \colon T_{B,b_0} \xrightarrow{\cong} \{\text{isomorphism classes of infinitesimal deformations of } X\} \tag{2.58}$$

is an isomorphism.

3. Suppose a Kuranishi family exists. Let $\mathcal{X} \to (B, b_0)$ be a deformation such that $B$ is smooth at $b_0$, and such that the Kodaira Spencer map $\rho$ is an isomorphism, then $\mathcal{X} \to (B, b_0)$ is Kuranishi. This follows from the universal property and some version of the implicit function theorem.
4. If $\mathcal{X} \to (B, b_0)$ is Kuranishi family for $X$ and $\mathrm{Aut}\,(X)$ is finite, then $\mathrm{Aut}\,(X)$ acts on $\mathcal{X}_U \to (U, b_0)$ for a basis of neighborhoods $U$ of $b_0$.

Theorem 2.5. *Let $C$ be a nodal curve. Then a Kuranishi family for $C$ exists if and only if $C$ is stable.*

Remark 2.3. The analogous statement holds for nodal curves with marked points, but we will just go through the construction for unmarked curves.

**Corollary 2.6.** *The base of a Kuranishi family for a stable curve $C$ of genus $p_a(C) = g$ has local dimension $3g - 3$.*

**Corollary 2.7.** *If $\mathcal{C} \to (B, b_0)$ is Kuranishi for a nodal curve $C$ then there is a neighborhood of $b_0$ such that $\mathcal{C}_U \to (U, x)$ is Kuranishi for all $x \in U$.*

The picture to have in mind here is that $B/\,\mathrm{Aut}(C)$ looks like an open patch in the moduli space of curves.

One key technical input in the proof of Theorem 2.5 is the existence and projectivity of the Hilbert scheme, which is is the moduli space of subschemes of $\mathbb{P}^N$ with fixed Hilbert polynomial. This is one small piece of the important foundational work of Grothendieck [**G**]. See [**FGI**$^+$], especially Part 2 (by Nitsure) and Part 3, §6 (by Fantechi) for further reading.

Proof of Theorem 2.5. First choose $N$ such that $\omega_C^{\otimes N}$ is very ample for all stable curves $C$ of genus $g$, (e.g. $N \geq 3$). Then notice that $\left|\omega_C^{\otimes N}\right|$ embeds $C$ in $\mathbb{P}^{N'}$ with Hilbert polynomial $p$ independent of $C$. Then we have open $U \subset \mathrm{Hilb}\left(\mathbb{P}^{N'}, p\right)$ parametrizing stable curves embedded by $\left(\omega_C^{\otimes N}\right)$. Notice that the group $\mathrm{PGL} = \mathrm{PGL}_{N'+1}$ acts on $U$.

Fact 3. *The stabilizer of a point $x$ corresponding to the $N$-canonical embedding of a stable curve $C$ is canonically isomorphic to* $\mathrm{Aut}\,(C)$.

Consider the PGL orbit through $x$. This is smooth of the same dimension as PGL. Write $G = \mathrm{Aut}\,(C)$. Then $G \subseteq \mathrm{PGL}$ acts as $\mathrm{Stab}\,(C)$, and $T_X\,(\mathrm{PGL} \cdot X)$ is $G$-invariant. Let $L \subseteq \mathbb{P}^K$ be a complementary $G$ invariant linear space (where $K$ is the dimension of the projective space which $\mathrm{Hilb}\left(\mathbb{P}^{N'}, p\right)$ lives).

The universal family of subschemes of $\mathbb{P}^{N'}$ over $U \cap L$ is Kuranishi for $C$. The picture is that Hilb might have some additional pieces (including higher-dimensional pieces) that parameterize unstable curves, but we just want to intersect with $U$. The fact that this has the Kuranishi property is deduced from the universal property of the Hilbert scheme. □

Lecture 14; February 21, 2020

A consequence of the above is the following. Let $C$ be any stable curve. Then there is an algebraic deformation $\mathcal{C} \to (X, x)$ such that

(1) $X$ is affine;
(2) $\mathcal{C} \to X$ is Kuranishi at every point $x' \in X$;
(3) $G = \mathrm{Aut}\,(C)$ acts on $(\mathcal{C} \to X)$, and the induced map

$$\{g \in G : gx' = x'\} \xrightarrow{\sim} \mathrm{Aut}\,(\mathcal{C}_{x'}) \tag{2.59}$$

is an isomorphism
(4) any isomorphism $\mathcal{C}_{x_1} \xrightarrow{\sim} \mathcal{C}_{x_2}$ is induced by some $g$ such that $gx_1 = x_2$.

Remark 2.4. $X/G$ (at least set-theoretically) parameterizes

$$\{\mathcal{C}_x : x \in X\} \,/ \cong \ . \tag{2.60}$$

**Lemma 2.8.** *Let $X = \mathrm{Spec}\,A$ be an affine variety (scheme of finite type over $\mathbb{C}$) with the action of a finite group $G$. Then the ring of invariants*

$$A^G = \{a \in A : ga = a \text{ for all } g \in G\} \tag{2.61}$$

*is finitely generated and*

$$X/G = \mathrm{Spec}\,A^G \ . \tag{2.62}$$

*Moreover, if $X$ is normal then so is $X/G$.*

We omit the proof, which is give in [**ACG**].

Definition 2.4. Write $\overline{M}_g$ for the collection of isomorphism classes of stable curves of genus $g$. For each curve $C$, we can build a Kuranishi family $X_C$, with the action of $G_C = \mathrm{Aut}\,(C)$, so we have a cover of this set by algebraic varieties:

$$\overline{M}_g = \bigcup_C X_C/G_C \ . \tag{2.63}$$

Claim 2.2. The "gluing" maps are holomorphic, so $\overline{M}_g$ is a complex analytic space.

Suppose

$$U = X_C/G_C \qquad\qquad U' = X_{C'}/G_{C'} \ . \tag{2.64}$$

The universal property of Kuranishi families implies that $U \cap U'$ is open in both $U$ and $U'$. Indeed, if we have a point in the intersection, then we lift it to $x \in X$ and $x' \in X'$, and a sufficiently small neighborhood of $x'$ is uniquely biholomorphic to a unique neighborhood of $x$ in $X$. It follows that the inclusion $U' \to U$ is holomorphic away from the branch locus $B'$ of $X' \to U'$. Covering $U$ by bounded domains, using the normality of $U'$, and applying Riemann Existence Theorem, it follows that the holomorphic inclusion $U' \smallsetminus B' \to U$ extends to a holomorphic map $U' \to U$, as required.

Modulo the definitions of orbifolds and Deligne-Mumford stacks, which are technical and omitted, the construction above has the following consequences:

THEOREM 2.9. *$\overline{M}_g$ is the coarse space of a smooth complex-analytic orbifold $\overline{\mathcal{M}}_g$ that represents the moduli functor for stable curves of genus $g$:*

$$\mathfrak{M}_g \colon \textbf{Spaces} \to \textbf{Sets} \tag{2.65}$$

*which maps a space $S$ to families of stable nodal curves over $S$ up to isomorphism.*

With more care, we can get the following:

THEOREM 2.10. *$\overline{\mathcal{M}}_g$ is a smooth algebraic (Deligne-Mumford) stack with coarse space $\overline{M}_g$. Moreover, $\overline{M}_g$ is an irreducible projective algebraic variety.*

The analogous statements also hold with marked points, i.e., for $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$.

We now briefly sketch a proof of the irreducibility of $\overline{M}_{g,n}$ over $\mathbb{C}$, since this fact (and especially Corollary 2.11, below), will be important for our approach to studying the top weight cohomology of $M_g$. We begin by considering the case where there are no marked points. From our study of deformation theory of stable curves, we know that the subspace $M_g$ parameterizing smooth curves is open and dense, so it is enough to show that this is irreducible. Moreover, since $\mathcal{M}_g$ is smooth, it is enough to show that $M_g$ connected. Now $M_g$ is the quotient of Teichmüller space (a contractible domain) by the mapping class group:

$$M_g = \mathcal{T}_g / \operatorname{Mod}(S_g) \ . \tag{2.66}$$

In particular, as a quotient of a connected space, it is connected.

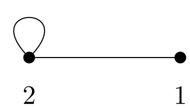
For the more general statement with marked points, we proceed by induction on the number of marked points (using irreducibility of $M_g$ as the base case. Consider the forgetful map $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ given by forgetting the $n$th marked point and stabilizing if necessary, Then this map is the universal curve $\mathcal{C}_{g,n-1} \to \mathcal{M}_{g,n-1}$. Since it is a fiber bundle whose base is irreducible, by the induction hypothesis, and whose fiber is irreducible, its total space is irreducible. And therefore the coarse space $M_{g,n}$ is irreducible as well.

**Corollary 2.11.** *There is a stratification*

$$\overline{M}_g = \coprod_G M_G \tag{2.67}$$

*where $M_G$ is the space of stable curves with dual graph $G$. In particular, each $M_G$ is irreducible.*

For example, let $G$ be the following graph.




Then

$$M_G = M_{2,3} \times M_{1,1} / \operatorname{Aut}(G) \tag{2.68}$$

so it is the quotient of an irreducible space by a finite group, and hence irreducible. Furthermore, $M_G \subset \overline{M}_{G'}$ if and only if $G'$ is obtained from $G$ by (weighted) edge contractions. Note that the codimension of $M_G$ is the number of edges:

$$\operatorname{Codim} M_G = \#E(G) \ . \tag{2.69}$$

So we have a combinatorial stratification of $\overline{M}_g$ into irreducible pieces indexed by dual graphs of stable curves, with containments encoded by weighted edge contractions. This will be essential input when we study the top weight cohomology of $M_g$.

Lecture 15; February 24, 2020

## 3. Boundary complexes and weight filtrations

See [**D2**, **D3**, **D4**], [**PS**], and [**V3**, **V4**] for references.

Let $X$ be an algebraic variety of dimension $\dim_{\mathbb{C}} X = n$. We will study the singular cohomology with coefficients in some ring $H^*(X, A)$. Most often we will consider $A = \mathbb{Q}$ or $\mathbb{C}$. The rational cohomology $H^*(X,\mathbb{Q})$ carries a canonical increasing weight filtration $W_\bullet$. By extending scalars, we also get a weight filtration on $H^*(X,\mathbb{C})$ which carries, in addition, a decreasing Hodge filtration $F^\bullet$. Together, these two filtrations form a mixed Hodge structure, in the sense of Deligne.

We will focus primarily on the weight filtration from Deligne's theory, listing some of its essential properties that we will use repeatedly (in the spirit of Grothendieck's "yoga of weights"). Note that the proofs of these properties (which we omit) rely on properties of the Hodge filtration.

The weight filtration on $H^k(X,\mathbb{Q})$ is an increasing filtration

$$0 \subseteq W_0 H^k(X,\mathbb{Q}) \subseteq W_1 H^k(X,\mathbb{Q}) \subseteq \ldots \subseteq W_{2k} H^k(X,\mathbb{Q}) = H^k(X,\mathbb{Q}), \tag{2.70}$$

whose associated graded pieces are

$$\operatorname{Gr}_j^W H^k(X,\mathbb{Q}) = W_j H^k(X,\mathbb{Q}) / W_{j-1} H^k(X,\mathbb{Q}) \tag{2.71}$$

Note that $\operatorname{Gr}_j^W H^*(X,\mathbb{Q})$ is sometimes informally referred to as "weight $j$ cohomology" or the "weight $j$ part of cohomology," even though it is a subquotient, not a subspace. We say that $H^k(X,\mathbb{Q})$ *has weights in* $I \subseteq \{0,\ldots,2k\}$ if

$$\operatorname{Gr}_j^W H^k(X,\mathbb{Q}) = 0 \tag{2.72}$$

for $j \notin I$.

The weight filtration satisfies the following properties:

- If $X$ is compact, then $H^k(X,\mathbb{Q})$ has weight in $\{0,\ldots,k\}$.
- If $X$ is smooth, then $H^k(X,\mathbb{Q})$ has weights in $\{k,\ldots,2k\}$.
- For all $k$, $H^k(X,\mathbb{Q})$ has weights in $\{0,\ldots,2n\}$.

The last condition is meaningful when $k > n$. A key special case is when $X$ is smooth and compact (e.g., smooth and projective). In this case $\operatorname{Gr}_j^W H^k(X,\mathbb{Q})$ vanishes for $j \neq k$, and we say that $H^k(X,\mathbb{Q})$ has (pure) weight $k$.

Remark 2.5. There are similar filtrations on $H_k(X,\mathbb{Q})$ and $H_c^k(X,\mathbb{Q})$.

Important. *All* natural maps between cohomology groups of algebraic varieties strictly respect weight filtrations.

Example 2.2. If $f\colon X \to Y$ is a morphism, then

$$f^* W_j H^k(Y,\mathbb{Q}) \subseteq W_j H^k(X,\mathbb{Q}) \ . \tag{2.73}$$

Moreover,

$$f^* H^k(Y,\mathbb{Q}) \cap W_j H^k(X,\mathbb{Q}) = f^* W_j H^k(Y,\mathbb{Q}) \ . \tag{2.74}$$

I.e., $f^*$ induces

$$\operatorname{Gr}_j^W H^k(Y,\mathbb{Q}) \to \operatorname{Gr}_j^W H^k(X,\mathbb{Q}) \tag{2.75}$$

and

$$\operatorname{rank} f^*|_{H^k} = \sum_j \operatorname{rank} f^*|_{\operatorname{Gr}_j^W H^k} \ . \tag{2.76}$$

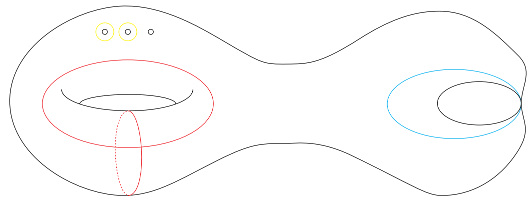


FIGURE 1. A triple punctured curve $C$ of geometric genus 1 with 1 node.

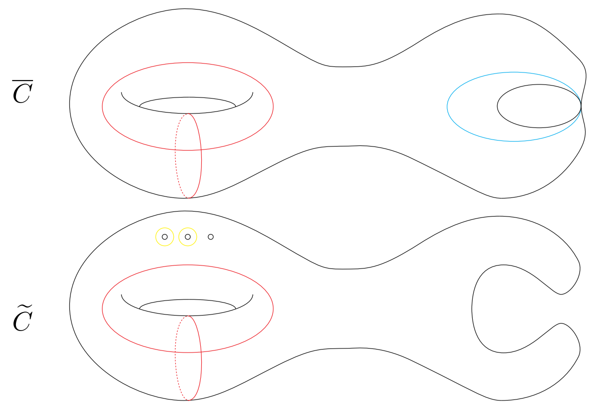



FIGURE 2. A compactification of our curve on top, and the normalization on the bottom.

EXAMPLE 2.3. Consider the nodal curve in fig. 1. Note that $H_1(C, \mathbb{Q}) \cong \mathbb{Q}^5$ has a basis given by the classes of the red, yellow, and blue curves. We consider also the dual basis for $H^1(C, \mathbb{Q})$.

We consider the compactification $i \colon C \hookrightarrow \bar{C}$, obtained by adding three smooth points (this makes $\bar{C}$ unique), as shown in fig. 2. This has a basis for homology given by the classes of the blue and red curves; the yellow curves are homologous to zero in $\bar{C}$.

Dualizing to cohomology we get

$$H^1(C, \mathbb{Q}) / i^* H^1(\bar{C}, \mathbb{Q}) = \mathrm{Gr}_2^W H^1(C, \mathbb{Q}) \tag{2.77}$$

Hence the dual basis elements corresponding to the yellow curves freely generate $\mathrm{Gr}_2^W H^1(C, \mathbb{Q})$.

On the other hand, we can normalize to get $\nu \colon \widetilde{C} \to C$. See fig. 2. On $\widetilde{C}$, the classes of the yellow and red curves give a basis for $H_1$. Passing to cohomology, we have

$$\ker\left(\nu^* \colon H^1(C, \mathbb{Q}) \to H^1\left(\widetilde{C}, \mathbb{Q}\right)\right) = W_0 H^1(C, \mathbb{Q}) \ . \tag{2.78}$$

So, the dual basis element corresponding to the blue curve generates $W_0 H^1(C, \mathbb{Q})$.

Dual basis elements corresponding to the red curves are generators of $f_* H^1\left(\widetilde{\overline{C}}, \mathbb{Q}\right)$, and these freely generate $\mathrm{Gr}_1^W H^1(C, \mathbb{Q})$. .

Lecture 16;
February 26, 2020

## 4. Poincaré duality

Let $X$ be an irreducible variety of dimension $\dim_{\mathbb{C}} X = n$. If $X$ is smooth then Poincaré duality tells us that the natural map

$$H^k(X,\mathbb{Q}) \times H_c^{2n-k}(X,\mathbb{Q}) \xrightarrow{\smile} H_c^{2n}(X,\mathbb{Q}) \xrightarrow{\int_X} \mathbb{Q} \tag{2.79}$$

is a perfect pairing. The basic properties of the weight filtration (i.e., the facts that weights are additive under tensor product, and that there are no nontrivial natural maps between cohomology groups of different weights, as discussed below) ensure that this induces perfect pairings

$$\mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \times \mathrm{Gr}_{2n-j}^W H_c^{2n-k}(X,\mathbb{Q}) \to \mathrm{Gr}_{2n}^W H_c^{2n}(X,\mathbb{Q}) = H_c^{2n}(X,\mathbb{Q}) \ . \tag{2.80}$$

For arbitrary $X$, the idea behind the weight filtration is that

$$\mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \tag{2.81}$$

carries a (pure) Hodge structure of weight $j$. In other words,

$$\mathrm{Gr}_j^W H^k(X,\mathbb{C}) = \mathrm{Gr}_j^W H^k(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{2.82}$$

$$= \bigoplus_{p+q=j} H^{p,q}\left(\mathrm{Gr}_j^W H^k(X,\mathbb{C})\right) \tag{2.83}$$

and $H^{p,q} = \overline{H^{q,p}}$.

Important. All natural maps between cohomology groups of algebraic varieties respect Hodge structures and the $p,q$ decomposition but not necessarily cohomological degree. In particular, there are no nontrivial maps between $\mathrm{Gr}_j^W$ and $\mathrm{Gr}_{j'}^W$ for $j \neq j'$.

Example 2.4. Let $X$ be an algebraic variety with Zariski closed subset $V \subset X$. Then we have a long exact sequence

$$\cdots \longrightarrow H^k(X,\mathbb{Q}) \longrightarrow H^k(V,\mathbb{Q}) \xrightarrow{\delta} H_c^{k+1}(X \setminus V,\mathbb{Q}) \longrightarrow H^{k+1}(X,\mathbb{Q}) \longrightarrow \cdots \ . \tag{2.84}$$

In particular, let $X$ be $X = \tilde{\tilde{C}}$ from example 2.3. Take $V$ to be the three points which were initially punctures. For $k = 0$, we get:

$$0 \to H^0(X,\mathbb{Q}) \to H^0(V,\mathbb{Q}) \xrightarrow{\delta} W_0 H_c^1(X \setminus V,\mathbb{Q}) \to W_0 H^1(X,\mathbb{Q}) = 0 \ . \tag{2.85}$$

The first term is zero because $X \smallsetminus V$ is not compact, and hence $H_c^0(X \smallsetminus V,\mathbb{Q}) = 0$. The last term is zero because $X$ is smooth and projective, and hence $H^1(X,\mathbb{Q})$ has weight 1.

Then

$$W_0 H_C^1(X \setminus V,\mathbb{Q}) \cong H^0(V,\mathbb{Q}) / H^0(X,\mathbb{Q}) \cong \widetilde{H}^0(V,\mathbb{Q}) \tag{2.86}$$

and applying Poincaré duality gives us

$$\mathrm{Gr}_2^W H^1(X \setminus V,\mathbb{Q}) \cong \widetilde{H}_0(V,\mathbb{Q}) \ . \tag{2.87}$$

**4.1. Mayer-Vietoris.** Recall the Mayer-Vietoris sequence. Let $X = U_1 \cup U_2$ for $U_i$ open. Then we get a long exact sequence

$$\cdots \to H^k(X, \mathbb{Q}) \to H^k(U_1, \mathbb{Q}) \oplus H^k(U_2, \mathbb{Q}) \to H^n(U_1 \cap U_2, \mathbb{Q}) \xrightarrow{\delta} H^{n+1}(X, \mathbb{Q}) \to \cdots . \tag{2.88}$$

We will be especially interested in cases where $U_1$, $U_2$, and $U_1 \cap U_2$ are open subvarieties (or open tubular neighborhoods of closed subvarieties).

Now we have a Mayer-Vietoris spectral sequence. Assume

$$X = U_1 \cup \ldots \cup U_r \tag{2.89}$$

for open $U_i$. This gives us a spectral sequence. The $E_0$ page is.

$$E_0^{p,q} = \bigoplus_{\substack{I \subseteq \{1,\ldots,r\} \\ |I=q+1|}} \left( C^p \left( \bigcap_{i \in I} U_i \right), \mathbb{Q} \right) . \tag{2.90}$$

On this direct sum there are two differentials. One increases $p$ (this is just the ordinary differential on cochains) and the other one is the combinatorial differential which increases the number of open sets being intersected. Then the $E_1$ page is given by:

$$E_1^{p,q} = \bigoplus_{\substack{I \subseteq \{1,\ldots,r\} \\ |I|=q+1}} \left( H^p \left( \bigcap_{i \in I} U_i \right), \mathbb{Q} \right) . \tag{2.91}$$

If each $U_i$ deformation retracts to a smooth projective variety, then the yoga of weights implies the spectral sequence collapses at $E_2$. This is because every differential on the $E_2$ page and beyond is a map between Hodge structures of different weights, and hence must be the zero map.

Lecture 17; February 28, 2020

## 5. Cohomology of simple normal crossing divisors

Let $X$ be an algebraic variety with $V \subseteq X$ a closed subvariety. This gives us a long exact sequence (of MHS)

$$\mathrm{Gr}_j^W \left( \cdots \to H^k(X, \mathbb{Q}) \to H^k(V, \mathbb{Q}) \xrightarrow{\delta} H_c^{k+1}(X \setminus V, \mathbb{Q}) \otimes H^{k+1}(X, \mathbb{Q}) \to \cdots \right) . \tag{2.92}$$

An important special case is $X$ smooth and proper, and is a simple normal crossings divisor.

Definition 2.5. Let $D \subseteq X$ be a divisor (subvariety of pure codimension 1). Then $D$ is a simple normal crossings divisor if each irreducible component of $D$ is smooth, and the components meet transversely. In other words, $(X, D)$ is locally isomorphic to $(\mathbb{C}^n, H)$, where $H$ is a union of coordinate hyperplanes, near each point of $D$.

Counterexample 2. Two divisors which are not simple normal crossings divisors are given in fig. 3.

Example 2.5. Let $X \supseteq V$ be smooth and projective.

Proposition 2.12.

$$\mathrm{Gr}_j^W H_c^k(X \setminus V, \mathbb{Q}) = \begin{cases} 0 & j \neq k-1, k \\ \mathrm{coker}\left(H^{k-1}(X, \mathbb{Q}) \to H^{k-1}(V, \mathbb{Q})\right) & j = k-1 \\ \ker\left(H^k(X, \mathbb{Q}) \to H^k(V, \mathbb{Q})\right) & j = k \end{cases} \tag{2.93}$$

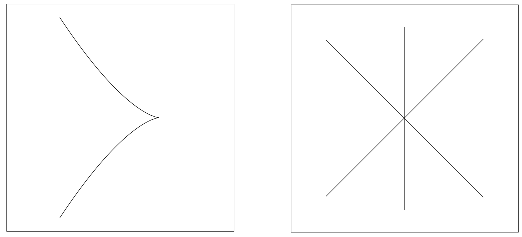


FIGURE 3. Two divisors $D$ which are not simple normal crossings divisors.

Write $D_1, \dots, D_r$ for the smooth components of $D$. For $I \subseteq \{1, \dots, r\}$ let

$$D_I = \bigcap_{i \in I} D_i \ . \tag{2.94}$$

This is a smooth and proper subvariety of codimension $|I|$ in $X$. Now we get a Mayer-Vietoris spectral sequence, with

$$E_1^{pq} = \bigoplus_{|I|=p+1} H^q \left(D_I, \mathbb{Q}\right) \Rightarrow H^{p+q} \left(D, \mathbb{Q}\right). \tag{2.95}$$

Note that the $E_1$ page is supported in the non-negative orthant, and its $j$th row is a complex of Hodge structures of weight $j$.

$$\begin{array}{ll}
\vdots & \\
\text{weight } 2 & 0 \longrightarrow \bigoplus_i H^2\left(D_i, \mathbb{Q}\right) \longrightarrow \bigoplus_{i_0 < i_1} H^2\left(D_{i_0} \cap D_{i_1}, \mathbb{Q}\right) \longrightarrow \cdots \\
\text{weight } 1 & 0 \longrightarrow \bigoplus_i H^1\left(D_i, \mathbb{Q}\right) \longrightarrow \bigoplus_{i_0 < i_1} H^1\left(D_{i_0} \cap D_{i_1}, \mathbb{Q}\right) \longrightarrow \dots \\
\text{weight } 0 & 0 \longrightarrow \bigoplus_i H^0\left(D_i, \mathbb{Q}\right) \longrightarrow \bigoplus_{i_0 < i_1} H^0\left(D_{i_0} \cap D_{i_1}, \mathbb{Q}\right) \longrightarrow \cdots
\end{array}$$

For each weight $j$, the corresponding row is

$$0 \to \bigoplus_{i=1}^{r} H^j\left(D_j, \mathbb{Q}\right) \xrightarrow{d_0} \bigoplus_{0<i_1<i_1\leq r} H^j\left(D_{i_0} \cap D_{i_1}, \mathbb{Q}\right) \xrightarrow{d_1} \cdots \ . \tag{2.96}$$

This spectral sequence collapses at $E_2$, and gives

$$\mathrm{Gr}_j^W H^{i+j}\left(D, \mathbb{Q}\right) = \frac{\ker d_i}{\operatorname{im} d_{i-1}} \tag{2.97}$$

The $j = 0$ row of this spectral sequence is already of considerable interest. We will identify this row with the cellular chain complex of a dual complex $\Delta(D)$ that encodes the combinatorics of the strata of $D$, as follows.

The dual complex $\Delta\left(D\right)$ is the $\Delta$-complex with vertices $v_1, \dots, v_r$ corresponding to the irreducible components $D_1, \dots, D_r$, edges $[v_i, v_j]$ corresponding to the irreducible components of $D_i \cap D_j$, 2-faces $\langle v_i, v_j, v_k\rangle$ corresponding to irreducible components of $D_i \cap D_j \cap D_k$, and so on for higher dimensional faces. The inclusions of faces correspond to containments of closed strata, and vice versa, containments of faces correspond to inclusions of closed strata. In other words, the correspondence between faces of $\Delta(D)$ and strata of $D$ is order reversing with respect to inclusions on both sides.

The $j = 0$ row of the spectral sequence discussed above is canonically isomorphic to the cellular chain complex of $\Delta(D)$, and hence we have

$$W_0 H^i(D, \mathbb{Q}) \cong H_i(\Delta(D), \mathbb{Q}) \ . \tag{2.98}$$

Then the pair sequence for $(X, D)$ in weight 0 is

$$W_0 H^i(X, \mathbb{Q}) \to W_0 H^i(D, \mathbb{Q}) \to W_0 H_c^{i+1}(X \setminus D, \mathbb{Q}) \to W_0 H^{i+1}(X, \mathbb{Q}) \ . \tag{2.99}$$

When $i = 0$ the first term is $\mathbb{Q}$ and the last term is 0, so

$$W_0 H_c^{i+1}(X \setminus D, \mathbb{Q}) = \widetilde{H}^i(\Delta(D), \mathbb{Q}) \ . \tag{2.100}$$

Applying Poincaré duality then gives

$$\mathrm{Gr}_{2n}^W H^{2n-*}(X \setminus D, \mathbb{Q}) \cong \widetilde{H}_{*-1}(\Delta(D), \mathbb{Q}) \tag{2.101}$$

where $n = \dim X$.

**Corollary 2.13.** *$H_*(\Delta(D), \mathbb{Q})$ only depends on $X \setminus D$.*

Lecture 18; March 2, 2020

EXAMPLE 2.6. One compactification of $X = \mathbb{C}^n$ is $\overline{X} = \mathbb{P}^n$. Take $D$ to be the hyperplane at $\infty$, then $\Delta(D) = \mathrm{pt}$. Alternatively, $\overline{X}' = (\mathbb{P}^1)^n$ is a compactification of $X$. Now $D' = \overline{X}' \setminus X$ has $n$ components, and dual complex the simplex $\Delta(D') = \Delta^{(n-1)}$. Notice, as a sanity check, that these have the same rational (even integral) homology.

EXAMPLE 2.7. Let $X = (\mathbb{C}^\times)^n$. Any compactification of $\mathbb{C}^n$ is a compactification of this, so we can take the compactification $\overline{X} = \mathbb{P}^n$. Then $D = \overline{X} \setminus X$ has components $D_0, \ldots, D_n$. This is a simple normal crossings divisor.

$$\Delta(D) = \partial\Delta^{(n)} \simeq S^{n-1} \ . \tag{2.102}$$

As before, take a different compactification $\overline{X}' = (\mathbb{P}^1)^n$. Then

$$D' = (\mathbb{P}^1)^n \setminus (\mathbb{C}^\times)^n \ . \tag{2.103}$$

Now this has $2n$ irreducible components $D_i^0$ and $D_i^\infty$ for all $i \in \{1, \ldots, n\}$. So the faces of $\Delta(D')$ correspond to a subset $I \subseteq \{1, \ldots, n\}$ as well as a function $I \to \{0, \infty\}$. So the number of $k$-faces is given by:

$$2^{k+1}\binom{n}{k+1} \ . \tag{2.104}$$

For $n = 3$, we can picture $D'$ as the cube given by the convex hull of $\{\pm 1, \pm 1, \pm 1\}$. Then $\Delta(D')$ is the polar dual of the unit cube, i.e. the octahedron. In general, $\Delta(D')$ is the boundary of the polar dual of the $n$-cube. This is sometimes called the hyper octahedron. But notice that this always has the homotopy type of the sphere.

**Proposition 2.14** ([**D1**])**.** *The homotopy type of $\Delta(D)$ depends only on $X$.*

We will not give Danilov's original proof. Instead we will provide a modern way of thinking about this, as an application of toroidal weak factorization of birational maps.

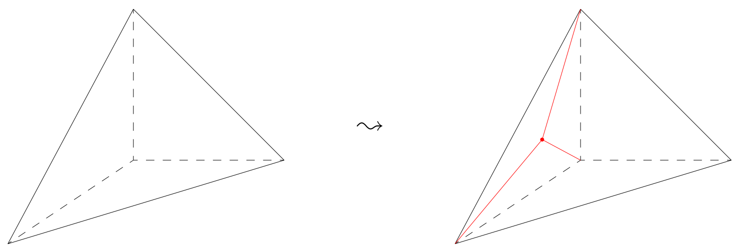


FIGURE 4. On the left we have $\Delta(D_i)$. Aftering blowing up, we get the subdivided complex on the right.

THEOREM 2.15 ([**AKMW**]). *Let $\overline{X}$ and $\overline{X}'$ be two snc compactifications of $X$. Then the birational map $\overline{X} \dashrightarrow \overline{X}'$ factors as*

$$\overline{X} = \overline{X}_0 \dashrightarrow \overline{X}_1 \dashrightarrow \overline{X}_1 \dashrightarrow \cdots \dashrightarrow \overline{X}_i = \overline{X}' \tag{2.105}$$

*where $\overline{X}_i \dashrightarrow \overline{X}_{i+1}$ is either*

*(1) the blowup along a smooth subvariety $\overline{Z}_{i+1}$ which only intersects the strata of $D_{i+1} = \overline{X}_{i+1} \setminus X_{i+1}$ transversely, or*
*(2) the inverse of such a blowup.*

EXAMPLE 2.8. Let $X = \mathbb{C}^n$ and take $\overline{X}_i = \mathbb{P}^n$, $\overline{X}_{i+1} = \mathrm{Bl}_p \mathbb{P}^n$ for $p \in \mathbb{P}^n \setminus \mathbb{C}^n$. Then $D = H_\infty$ and $\Delta(D) = \mathrm{pt}$, so

$$\Delta(D_{i+1}) = \begin{array}{ccc} \mathrm{Bl}_p H_\infty & & E \\ \bullet & \text{------} & \bullet \end{array} \tag{2.106}$$

which is the mapping cone of a null-homotopic map, and so a homotopy equivalence.

EXAMPLE 2.9. Let $X = (\mathbb{C}^\times)^n$, $\overline{X}_i = \mathbb{P}^n$, and

$$\overline{X}_{i+1} = \mathrm{Bl}_{(0:\cdots:0:1)} \mathbb{P}^n\ . \tag{2.107}$$

So we start with $\Delta(D_i) = \partial\Delta^{(n-1)}$ and then when we blowup at a point, we get a new vertex corresponding to the exceptional divisor of that blowup. Then we get new edges corresponding to linear subspaces which contain the point. So this corresponds to stellar subdivision of the face which was blown up, to give us the complex in fig. 4.

In general, recall $k$-faces of $\Delta(D_i)$ correspond to codimension $k+1$ strata. So blowing up a 1-dimensional stratum of $D_i$ corresponds to stellar subdivision along the barycenter of a codimension 1 face of $\Delta(D_i)$. The point being that we get a homeomorphic simplex.

Let $D_1, \ldots, D_r$ be the irreducible components of a simple normal crossings divisor $D$. Lecture 19; March 4, 2020

DEFINITION 2.6. A codimension $j$ stratum is an irreducible component of

$$D_{i_1} \cap \cdots \cap D_{i_j} \tag{2.108}$$

for some $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, r\}$.

REMARK 2.6. This agrees with the usual notion of a closed stratification.

Sketch of proof of Proposition 2.14. Let $D \subseteq \overline{X}$ be a simple normal crossings divisor. Let $Z \subseteq D$ be a smooth irreducible subvariety transverse to the strata of $D$. Since $Z$ is irreducible, there is a smallest stratum $Y_Z \subseteq D$ which contains $Z$. Write $D_Z \subseteq Z$ for the intersection of $Z$ with components of $D$ that do not contain $Z$ (or equivalently $Y_Z$). This means $D_Z$ is simple normal crossings in $Z$.

By Theorem 2.15 it is sufficient to consider

$$\mathrm{Bl}_Z\left(\overline{X}\right) = \overline{X}' \xrightarrow{\pi} \overline{X} \ . \tag{2.109}$$

Write $D' = \pi^{-1}(0) = \overline{X}' \setminus X$. This is a simple normal crossings divisor.

If $Z = Y_Z$, then $\Delta(D')$ is the stellar subdivision of $\Delta(D)$ along a face $\sigma_{Y_Z}$ corresponding to $Y_Z$.

Now assume $Z \subsetneq Y_Z$. Codimension $j$ strata of $D'$ are either strict transforms of codimension $j$ strata of $D$, or new strata. These are the strata of the exceptional divisor. These correspond to pairs $(Y, W)$ such that $Y$ is a stratum of $D$, $W$ is a stratum of $Z$, and $W \subseteq Y$. The correspondence works as follows. Consider irreducible components $\widetilde{D}_1, \ldots, \widetilde{D}_r$ and $E = \pi^{-1}(Z)$. Then the $\widetilde{D}_i$ are strict transforms of the $D_1, \ldots, D_r$. The new strata are irreducible components of $E \cap \widetilde{D}_I$. $E$ is the projectivized normal bundle of $Z$ in $X$. So a stratum in $E$ is the projectivized normal bundle of a stratum of $Z$ in a stratum of $X$. So now the correspondence sends a pair $(Y, W)$ to the projectivized normal bundle of $W$ in $Y$.

Form the join of $\Delta(D_Z)$ with the face $\sigma_{Y_Z}$ corresponding to $Y_Z$. Then we can map:

$$\begin{tikzcd} \Delta(D_Z) \star \sigma_{Y_Z} \arrow[d, "f"] \\ \Delta(D) \end{tikzcd} \tag{2.110}$$

where vertices correspond to irreducible components of the intersection of $Z$ with irreducible components of $D$ that do not contain $Y_Z$ (but *do* intersect $Y_Z$). Then

$$\Delta(D') = \mathrm{cone}(f) \ , \tag{2.111}$$

and $f$ is in fact null-homotopic as follows. Every maximal face in $\mathrm{im}(f)$ contains $\sigma_{Y_Z}$, so choose a vertex $v$ of $\sigma_{Y_Z}$, and $\mathrm{im}(f)$ is star-shaped around $v$. $\square$

## 6. Normal crossings divisors

We now consider a mild generalization of simple normal crossings divisors that will be essential for our applications.

Let $X$ be a smooth irreducible variety of $\dim_{\mathbb{C}} X = n$.

Definition 2.7. A divisor $D \subseteq X$ has *normal crossings* if it is locally analytically isomorphic to

$$(x_1 \ldots x_k = 0) \subset \mathbb{C}^n \ , \tag{2.112}$$

for some $k$.

Example 2.10. For $X$ a smooth surface, a curve $C \subseteq X$ is normal crossings if and only if $C$ is nodal.

The difference between normal crossings and simple normal crossings is that the irreducible components of a normal crossings divisor are not required to be smooth.

Lecture 20; March 6, 2020

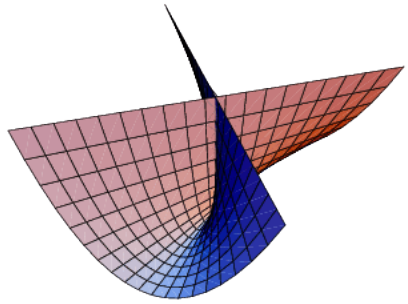


FIGURE 5. The Whitney umbrella. This figure is from Wolfram Mathworld.

The normalization $\nu\colon \widetilde{D} \to D$ of a normal crossings divisor $D$ is a resolution of singularities and has the following interpretation:

$$\widetilde{D} = \{(x, b) : x \in D, b \text{ branch of } D \text{ s.t. } x \in b\} \ . \tag{2.113}$$

The point being that singularities of normal crossings divisors are easy to resolve: normalization can be constructed locally analytically, and the normalization of a union of coordinate hyperplanes is just the disjoint union of those hyperplanes. In particular, the preimage of every stratum is smooth.

EXAMPLE 2.11 (Whitney umbrella). Consider

$$D = \left(x^2 y = z^2\right) \subseteq \mathbb{C}^3 \ . \tag{2.114}$$

See fig. 5. If we stay away from the $y = 0$ line we have two surfaces crossing transversely. At the origin it is not normal crossings, but along the $z$ axis, it is. So restrict to the divisor

$$D \subseteq \mathbb{C}^3 \setminus (y = 0) \ . \tag{2.115}$$

In $\mathbb{C}^3 \setminus (y = 0)$, $D$ is normal crossings, but not simple normal crossings.

The normalization is

$$\begin{array}{ccc} \widetilde{D} = \mathbb{A}^2_{(x,u)} \setminus (u = 0) & \xrightarrow{\quad \pi \quad} & D \\ & & \\ (x, u) & \longmapsto & \left(x, u^2, xu\right) \end{array} \ . \tag{2.116}$$

This is well-defined, $\widetilde{D}$ is normal, and it is finite (since $u^2 = y$). This is also birational since for $x \neq 0$ we have $u = z/x$.

Now define $Z$ to be the singular locus of $D$:

$$Z = \left\{(0, y, 0) : y \in \mathbb{C}^\times\right\} = (0, 0) \times \mathbb{C}^\times \ . \tag{2.117}$$

Notice that

$$\pi^{-1}(Z) = \left\{(0, u) : u \in \mathbb{C}^\times\right\} \tag{2.118}$$

so explicitly

$$\pi\colon \mathbb{C}^\times_u \to \mathbb{C}^\times_y \ . \tag{2.119}$$

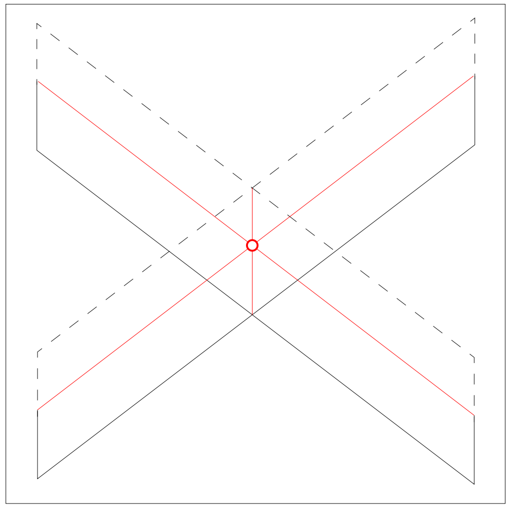


FIGURE 6. A locally closed stratum given by pairwise intersection of the planes minus the triple intersection.

The monodromy is the action

$$\pi_1(Z,x) \circlearrowright \{\text{branches of } D \text{ at } x\} \tag{2.120}$$

for any point $x \in Z$.

Let $\widetilde{D} \to D$ be the normalization of a normal crossings divisor. Write $Z \subseteq D$ for a (closed) stratum, then write $Z^\circ$ for the locally closed stratum. For example, if we have three planes meeting, we would take a pairwise intersection minus the triple intersection to get $Z^\circ$ as in fig. 6.

DEFINITION 2.8. The *monodromy of a stratum* $Z^\circ \subseteq D$ is the action of $\pi_1(Z^\circ, z)$ on the branches of $Z^\circ$ at $z$. The orbits correspond to irreducible components of $\nu^{-1}(Z^\circ)$.

Observe that if $D$ has simple normal crossings, then the monodromy of every stratum is trivial. This is not the case for normal crossings.

EXAMPLE 2.12. Consider the boundary divisor $D_g = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$. This is normal crossings as a stack/orbifold. The irreducible components and codimension 2 strata are as in fig. 7 for $g = 2$.

Let $G$ be a stable dual graph. Then we get a locally closed stratum $Z^\circ = \mathcal{M}_G$ given by:

$$\mathcal{M}_G = \prod_{v \in V(G)} \mathcal{M}_{g(v),n(v)} / \operatorname{Aut}(G) \ . \tag{2.121}$$

Given a curve $C$, this defines a class $[C] \in \mathcal{M}_G$. Then given an identification $\Delta(C) \xrightarrow{\sim} G$ gives an identification with the branches of $D_g$ at $[C]$ with the nodes of $C$, which by definition are the edges of $G$.

Now we want to describe the monodromy action on $\mathcal{M}_G$ at the curve $[C]$.

PROPOSITION 2.16. *The monodromy* $\pi_1(\mathcal{M}_G, [C]) \circlearrowright E(G)$ *has image equal to* $\operatorname{Im}(\operatorname{Aut}(G))$.

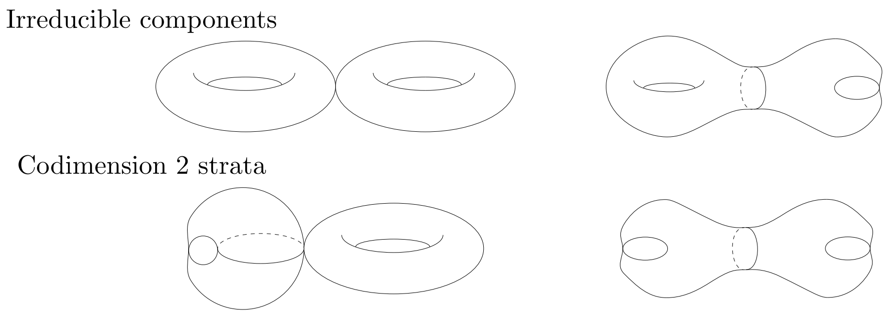



FIGURE 7. The irreducible components and codimension 2 strata of $D_2$.

PROOF. The idea is to use the correspondence between quotients of $\pi_1$ and covering spaces. So consider the covering space (as stacks/orbifolds):

$$\prod_{v \in V(G)} \mathcal{M}_{g(v),n(v)} \to \mathcal{M}_G \ . \tag{2.122}$$

This gives rise to $\pi_1 \twoheadrightarrow \mathrm{Aut}\,(G) \circlearrowright E\,(G)$. □

Lecture 21; March 9, 2020

**6.1. Combinatorial topology.** Now we will take some time to consider the combinatorial topology of dual complexes. First we consider the notion of a $\Delta$-complex. Roughly, this is a space obtained by gluing standard simplices along inclusions of faces that *respect vertex ordering*.[2.2] We will also consider the notion of an augmented $\Delta$-complex, which is a $\Delta$-complex along with a continuous map to a discrete set.

Now we want to make this description both formal and combinatorial. So we need to specify some sets, which are the sets of $p$-simplices, and then we need to specify the inclusions of faces in a way which will respect vertex ordering. Let $[p] = \{0, \dots, p\}$ for $p \geq -1$. Now define a category $\Delta_{\text{inj}}$ as follows. The objects are $\{[p] : p \geq 0\}$. The morphisms are given by order preserving injections $[p] \to [q]$.

DEFINITION 2.9. A $\Delta$-complex is a functor $\Delta_{\text{inj}}^{\text{op}} \to \mathbf{Sets}$.

In other words, a $\Delta$-complex a sheaf of sets on $\Delta_{\text{inj}}$.

A $\Delta$-complex $Y$ has a geometric realization $|Y|$. This is

$$|Y| = \amalg_{p \geq 0} \left(Y_p \times \Delta^p\right) / \sim \tag{2.123}$$

where $\Delta^p$ is the standard $p$-simplex, and the equivalence relation $\sim$ is as follows. Let $y \in Y_q$, $\theta\colon\ [p] \to [q]$, and $a \in \Delta^p$. Then we identify:

$$(y, \theta_* a) \sim (\theta^* y, a) \ . \tag{2.124}$$

The point is that the inclusion tells us which $p$-simplex is a $p$-face of a $q$-simplex corresponding to a given inclusion $\theta$. So this equivalence relation tells us that we should identify these simplices with their image.

When we want to talk about augmented complexes, we do the same construction, only we include the empty set as one of our objects, i.e. we consider $\Delta_{\text{inj}} \cup \{[-1]\}$. There is a

[2.2] The point is that in order to have a good cellular homology theory (i.e. to define a differential map) we need the simplices to be oriented.

unique map from $[-1]$ to any other object, so this is an initial object in the category. So when we take op of the category, we exactly get a continuous map to the discrete set $Y_{-1}$.

An augmented $\Delta$-complex is determined by the sets $Y_p$ for $\partial \geq -1$, and then there are the maps $d_i \colon Y_p \to Y_{p-1}$ corresponding to the unique inclusion $[p-1] \to [p]$ whose image does not contain $i$. These satisfy the relation

$$d_i d_j = d_{j-1} d_i \tag{2.125}$$

for $i < j$. Conversely, given such sets and maps satisfying this relation, they can be extended to a $\Delta$-complex.

For $D$ a simple normal crossings divisor, $\Delta(D)$ is not quite a $\Delta$-complex. It becomes a $\Delta$-complex after ordering the irreducible components $D_1, \ldots, D_s$.

This leads to a natural generalization, which we will call a symmetric $\Delta$-complex.[2.3] Let $I$ be the category with the same objects as $\Delta^{\mathrm{inj}}$, i.e. $\{[p] : p \geq -1\}$, but the morphisms are now all injective maps.

Definition 2.10. A symmetric $\Delta$-complex is a functor

$$Y : I^{\mathrm{op}} \to \mathbf{Sets} \; . \tag{2.126}$$

The same definition for the geometric realization works, only now we have many more arrows $[p] \to [q]$, so we are doing a lot more gluing.

As above, there is an analogous characterization of symmetric $\Delta$-complexes. We have sets $Y_p$, the action of the symmetric group $S_{p+1} \circlearrowright Y_p$, and $d_i : Y_p \to Y_{p-1}$ satisfying the same relation (2.125).

Remark 2.7. Note that an element of $Y_p$ should be thought of as a $p$-simplex of $|Y|$ together with an ordering of its vertices.

Then we get a functor from $\Delta$-complexes to symmetric $\Delta$-complexes sending $Y \mapsto Y'$. Explicitly,

$$Y'_p = Y_p \times S_{p+1} \; . \tag{2.127}$$

Note however that not every symmetric $\Delta$-complex occurs in this way.

Example 2.13 (Half-interval). Let $Y_{-1}$, $Y_0$, and $Y_1$ each have one element, and every other $Y_p$ is empty. The geometric realization $|Y|$ is the interval mod $S_2$, which looks like:

 $/S_2 \quad = $  .

So it is homeomorphic to the interval, has one vertex and one edge, and that edge has a stabilizer. This is a symmetric $\Delta$-complex but not a $\Delta$-complex.

Lecture 22; March 11, 2020

**6.2. Dual complexes of normal crossings divisors.** We now describe the dual complex of a normal crossings divisor as a symmetric $\Delta$-complex. Let $D \subseteq X$ be a normal crossings divisor. Write $\widetilde{D}$ for the normalization:

$$\widetilde{D} = \{(z, b) : z \in D, b \text{ branch of } D \text{ at } z\} \; . \tag{2.128}$$

Now define:

$$\widetilde{D}_p = \widetilde{D} \times_X \ldots \times_X \widetilde{D} \setminus \{(z_0, \ldots, z_p) : z_i = z_j \text{ for some } i \neq j\} \; . \tag{2.129}$$

[2.3] The terminology is based on symmetric semi-simplicial sets.

So we have removed all diagonals from the fiber product, and what is left has pure dimension

$$\dim \widetilde{D}_p = \dim X - p - 1 \ . \tag{2.130}$$

Note that by definition we have $\widetilde{D}_0 = \widetilde{D}$, $\widetilde{D}_{-1} = X$. Then

$$\widetilde{D}_1 = \{(z; b_0, b_1) : z \in D, b_0 \neq b_1 \text{branches of } D \text{ at } z\} \tag{2.131}$$

and in general

$$\widetilde{D}_p = \{(z; b_0, \ldots, b_p) : z \in D, b_0, \ldots, b_p \text{ distinct branches of } D \text{ at } z\} \ . \tag{2.132}$$

Note that this is smooth.

Now we will describe the dual complex $\Delta(D)$ as a symmetric $\Delta$-complex. Recall our notation $\Delta(D)_p \coloneqq \Delta(D)([p])$.

Then we define the dual complex by setting

$$\Delta(D)_p \coloneqq \left\{ \text{irreducible components of } \widetilde{D}_p \right\} \ .$$

An injection $\theta \colon [p] \hookrightarrow [q]$ induces a forgetful map

$$\widetilde{D}_q \to \widetilde{D}_p \ . \tag{2.133}$$

Since $\widetilde{D}_p$ is smooth, irreducible components do not intersect. It follows that each irreducible component of $\widetilde{D}_q$ maps into a unique irreducible component of $\widetilde{D}_p$, giving a well-defined map

$$\Delta(D)_q \to \Delta(D)_p \ . \tag{2.134}$$

Remark 2.8. If $D$ is simple normal crossings with irreducible components $D_1, \ldots, D_s$. Then this new $\Delta(D) \in$ **Sym$\Delta$-Cx** is (canonically isomorphic to) the image of the old $\Delta(D) \in$ **$\Delta$-Cx** under the functor **$\Delta$-Cx** $\to$ **Sym$\Delta$-Cx**.

**Proposition 2.17.** *Let $X$ be smooth, $\overline{X}$ smooth and compact such that $D = \overline{X} \setminus X$ is a normal crossings divisor. Then the homotopy type of $|\Delta(D)|$ only depends on $X$.*

Proof. The proof is via toroidal weak factorization of birational maps, just as in the simple normal crossings case, only now we are working with symmetric $\Delta$-complexes instead of $\Delta$-complexes. $\square$

Recall a stable tropical curve of genus $g$ is the dual graph $G$ of a stable algebraic curve of genus $g$ along with a length function $\ell \colon E(G) \to \mathbb{R}_{>0}$. Write $M_G^{\mathrm{trop}}$ for the isomorphism classes of stable tropical curves of genus $g$. Then, set-theoretically, we have

$$M_G^{\mathrm{trop}} = \coprod_G \mathbb{R}_{>0}^{E(G)} / \operatorname{Aut}(G) \ . \tag{2.135}$$

Now we need to understand the topology on $M_g^{\mathrm{trop}}$. Consider shrinking an edge as in fig. 8. As the length goes to 0, this corresponds to smoothing the corresponding node. Note that if the edge is a loop, corresponding to a self-intersection point of a component, then smoothing the self-intersection increases the geometric genus of that component by 1. Similarly, if the edge connects vertices corresponding to two distinct components of genus $g_1$ and $g_2$, then smoothing the node gives rise to a single component of geometric genus $g_1 + g_2$. The resulting operations on the dual graph are called *weighted edge contractions*, as we now describe.

Recall that the dual graph $G$ of a nodal curve comes with a weight function $g \colon V(G) \to \mathbb{Z}_{\geq 0}$ taking a vertex to the geometric genus of the corresponding irreducible component. If $e \in E(G)$ is an edge, then the ordinary edge contraction $G' = G/e$ is the underlying graph of the weighted edge contraction. The weights change as follows:

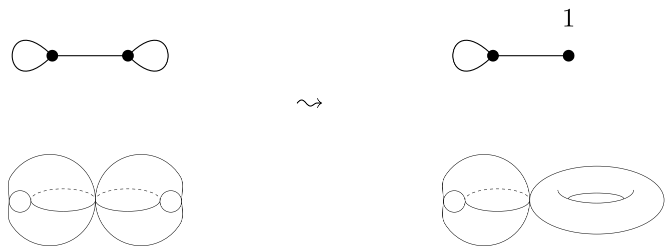



FIGURE 8. As we shrink the edge length to 0, the corresponding operation on curves is smoothing the node. So it becomes a connected component of genus 1, so we need to label the vertex.

- If $e$ is a loop edge at $v \in V(G)$ and $v'$ is the corresponding edge in $G'$, the $g(v') = g(v) + 1$.
- If $e$ connected two vertices $v_1$ and $v_2$ and $v'$ is their image in $G/e$, then

$$g(v') = g(v_1) + g(v_2) \ . \tag{2.136}$$

Note that

$$g(G') = h^1(|G'|) + \sum_{v' \in V(G')} g(v') = g(G) \ . \tag{2.137}$$

We return our attention to the boundary divisor $D_g = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. To understand its dual complex, we must study the geometry of Lecture 23; March 30, 2020

$$\left(\widetilde{D}_g\right)_p = \{X, z_0, \dots, z_p : X \text{ stable curve, } z_0, \dots, z_p \text{ distinct nodes of } X\} \tag{2.138}$$

$$= \coprod_G \overline{\mathcal{M}}_G / \ker(\operatorname{Aut} G \to \operatorname{Aut} E(G)) \tag{2.139}$$

where this is a disjoint union over isomorphism classes of stable graphs of genus $g$ with $p+1$ edges, and

$$\overline{\mathcal{M}}_g = \prod_v \overline{\mathcal{M}}_{g_v, n_v} \ . \tag{2.140}$$

Then we have

$$\Delta(D_g)_p = \left\{\begin{matrix}\text{isomorphism classes of stable graphs of genus } g \\ \text{with } p+1 \text{ ordered edges}\end{matrix}\right\} \ . \tag{2.141}$$

A point in the relative interior of a $p$-face of $|\Delta(D_g)|$ corresponds to the isomorphism classes of stable tropical curves $G$ with

$$\# E(G) = p + 1 \tag{2.142}$$

and

$$\sum_{e \in E(G)} \ell(e) = 1 \ . \tag{2.143}$$

This means

$$|\Delta(D_g)| = \Delta_g \tag{2.144}$$

is the moduli space of stable tropical curves of genus $g$ and volume 1.

**Corollary 2.18.**

$$\widetilde{H}_{i-1}\left(\Delta_g, \mathbb{Q}\right) \cong \mathrm{Gr}^W_{6g-6} H^{6g-6-i}\left(\mathcal{M}_g, \mathbb{Q}\right) \ . \tag{2.145}$$

We can use this in both directions. If we know somehow that the cohomology of $\mathcal{M}_g$ vanishes in some degree, this means the rational homology of $\Delta_g$ vanishes in the corresponding degree. Similarly, if we can prove nonvanishing of rational homology of $\Delta_g$, we get a corresponding nonvanishing statement for the cohomology of $\mathcal{M}_g$. Moreover, if we can show that the rational homology $\Delta_g$ is large in some degree, this means the cohomology of $\mathcal{M}_g$ is at least as large, in the corresponding degree.

**6.3. Rational homology for symmetric $\Delta$-complexes.** We now begin a discussion of rational cellular homology for symmetric $\Delta$-complexes and its comparison to rational singular homology. Consider

$$C_p\left(Y\right) = \left(\mathbb{Q}^{Y_p}\right)_{S_{p+1}} \ . \tag{2.146}$$

These are the $S_{p+1}$ coinvariants of

$$S_{p+1} \circlearrowright \mathbb{Q}^{Y_p} \otimes \mathrm{sgn} \ . \tag{2.147}$$

Recall that $Y_p$ is, roughly speaking, the set of pairs consisting of a $p$-simplex in $Y$ together with a total ordering of its vertices. Then $\mathbb{Q}^{Y_p} \otimes \mathrm{sgn}$ is freely generated by oriented simplices in $Y$, together with a total ordering of the vertices that is compatible with the orientation. Taking $S_{p+1}$-coinvariants identifies two such pairs whenever they correspond to the “same” oriented $p$-face, in the colimit construction of the dual complex. In this way, it is one thinks of $C_p(Y)$ as the group of cellular $p$-chains.

The point is that we want to think of acting by an even permutation as gluing two simplices in an orientation preserving way, and acting by an odd permutation as orientation reversing.

Note that

$$d_i \colon Y_p \to Y_{p-1} \tag{2.148}$$

induces

$$\sum_i \left(-1\right)^i d_{i*} \colon \mathbb{Q}^{Y_p} \to \mathbb{Q}^{Y_{p-1}} \tag{2.149}$$

which descends to

$$d \colon C_p\left(Y\right) \to C_{p-1}\left(Y\right) \tag{2.150}$$

and because of how the signs work, $d^2 = 0$. So this is a chain complex, and if we take the homology of this complex, we recover the rational, reduced, singular homology of $|Y|$.

Theorem 2.19.

$$\frac{\ker d}{\operatorname{im} d} \cong \widetilde{H}_*\left(\left|Y\right|, \mathbb{Q}\right) \ . \tag{2.151}$$

Proof. The proof is the same as the proof that cellular homology agrees with singular homology for $\Delta$-complexes. For example, one can use the spectral sequence associated to the filtration of $|Y|$ by its $p$-skeletons. □

Next we will study $H^*\left(\mathcal{M}_g, \mathbb{Q}\right)$, $C_*\left(\Delta_g\right)$, and $\widetilde{H}_*\left(\Delta_g\right)$.

CHAPTER 3

# Cellular homology of a symmetric $\Delta$-complex

Lecture 24; April 1, 2020

## 1. Cellular homology of a $\Delta$-complex

We now give further details to explain the cellular homology of a symmetric $\Delta$-complex, and explain why it agrees with singular homology with rational coefficients.

We begin by reviewing the cellular homology of a $\Delta$-complex.

**1.1. Attaching a $p$-cell to a $\Delta$-complex.** Given $Y$, we attach a $p$-cell $\Delta^{(p)}$ to $Y$ to get

$$Y' = Y \cup \Delta^{(p)} / \left( x \sim f(x) : x \in \partial\Delta^{(p)} \right) \tag{3.1}$$

where $f$ is the attaching map:

$$f \colon \partial\Delta^{(p)} = S^{p-1} \to Y \ . \tag{3.2}$$

Note that $Y' = \operatorname{cone}(f)$, and $f$ $f$ induces a map on homology:

$$f_* : \widetilde{H}_* \left( S^{p-1} \right) \to \widetilde{H}_* (Y) \ . \tag{3.3}$$

Recall $\widetilde{H}_* \left( S^{p-1} \right) \cong \mathbb{Z}$ in degree $p-1$.

To a pair $(Y', Y)$, we get a pair sequence:

$$H_{i+1}(Y', Y) \longrightarrow H_i(Y) \longrightarrow H_i(Y') \longrightarrow H_i(Y', Y) \xrightarrow{\partial} H_{i-1}(Y) \quad . \tag{3.4}$$

We know

$$H_i(Y', Y) = \begin{cases} \mathbb{Z} & i = p \\ 0 & \text{else} \end{cases} \tag{3.5}$$

which implies that for $i \neq p, p-1$ we have

$$H_i(Y') \cong H_i(Y) \ . \tag{3.6}$$

For $i = p$ and $p-1$ we have a 5-term exact sequence:

$$0 \to H_p(Y) \to H_p(Y') \to \mathbb{Z} \xrightarrow{\partial} H_{p-1}(Y) \to H_{p-1}(Y') \to 0 \tag{3.7}$$

where

$$\partial(1) = f_* \left[ S^{p-1} \right] \ . \tag{3.8}$$

In other words,

$$H_{p-1}(Y') \cong H_{p-1}(Y) / \operatorname{im}(f_*) \tag{3.9}$$

and, with rational coefficients, there is an isomorphism:

$$H_p(Y') \cong H_p(Y) \oplus \ker(f_*) \, . \tag{3.10}$$

This tells us how to calculate the homology of a Δ-complex after adding a $p$-cell. This means that if we have a finite or finite-dimensional Δ-complex, we can describe the homology by iterating this procedure. Roughly speaking, the spectral sequence associated to the filtration by skeleta, which we now describe, is an efficient way of re-packaging the information from all of these long exact sequences.

**1.2. Filtration by skeleta.** Let $Y$ be a finite-dimensional Δ-complex, and $A$ an abelian group. Write $Y^{(p)} \subseteq Y$ for the union of all cells of dimension $\leq p$. So we have a filtration on $Y$

$$Y^{(0)} \subseteq Y^{(1)} \subseteq \cdots \subseteq Y^{(n)} = Y \ . \tag{3.11}$$

From this we get a filtration on the singular chains, given by which chains are supported in these subspaces:

$$C_* \left(Y^{(0)}\right) \subseteq C_* \left(Y^{(1)}\right) \subseteq \cdots \subseteq C_* \left(Y^{(n)}\right) = C_* \left(Y\right) \ . \tag{3.12}$$

When we have a filtered complex, and when the filtration is finite, then we get a spectral sequence which abuts to the associated graded of the homology of the total sequence. In particular, we have

$$E_0^{p,q} = C_{p+q} \left(Y^{(p)}, Y^{(p-1)}; A\right) = C_{p+q} \left(Y^{(p)}, A\right) / C_{p+q} \left(Y^{(p-1)}, A\right) \ . \tag{3.13}$$

The differential on this page is vertical, i.e. it is fixing $p$ and decreasing $q$ by 1. So the $E_1$ page is just the relative homology. Then it is a general fact that this eventually converges to:

$$E_1^{p,q} = H_{p+q} \left(Y^{(p)}, Y^{(p-1)}; A\right) \Rightarrow E_\infty^{p,q} = \frac{\operatorname{im}\left(H_{p+q}\left(Y^{(p)}\right) \to H_{p+q}\left(Y\right)\right)}{\operatorname{im}\left(H_{p+q}\left(Y^{(p-1)}\right) \to H_{p+q}\left(Y\right)\right)} \ . \tag{3.14}$$

Now note

$$H_{p+q} \left(Y^{(p)}, Y^{(p-1)}; A\right) \cong \begin{cases} A^{\oplus p-\text{cells}} & q = 0 \\ 0 & \text{else} \end{cases} \ . \tag{3.15}$$

So this is supported in the row $q = 0$, i.e. it collapses at $E_2$. And this row is the cellular chain complex with coefficients in $A$. So singular homology agrees with cellular homology for Δ-complexes with arbitrary coefficients.

Remark 3.1. Note that we could have considered any filtration by closed subspaces. We would still get a filtration on the singular chain complex, and the induced spectral sequence has relative chains, for successive steps in the filtration of the space, on the $E_0$-page and relative homology for these pairs on the $E_1$-page. The key place where we used this particular filtration was when we noticed it was supported in the $q = 0$ row.

## 2. Cellular homology of a symmetric Δ-complex

**2.1. Attaching a $p$-cell to a symmetric Δ-complex.** Recall $\sigma \in Y_p$ is a $p$-cell, along with an ordering of the vertices, i.e. it is identified with the standard $p$-simplex. Recall $S_{p+1} \circlearrowright Y_p$, so each element $\sigma$ has a stabilizer $G_\sigma \subseteq S_{p+1}$. Attaching a $p$-cell now gives us:

$$Y' = \left(Y \cup \Delta^{(p)}/G_\sigma\right) / \left(x \sim f\left(x\right) : x \in \partial\Delta^{(p)}/G_\sigma\right) \tag{3.16}$$

where $f$ is the attaching map

$$f \colon S^{p-1}/G_\sigma \to Y \tag{3.17}$$

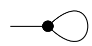
and

$$Y' = \operatorname{cone}(f) \ . \tag{3.18}$$

EXAMPLE 3.1. Consider the boundary of the 3-simplex, and quotient by $\mathbb{Z}/4\mathbb{Z}$, i.e. choose a permutation of our vertices:

$$Y = \partial\Delta^{(3)} / (\mathbb{Z}/4\mathbb{Z}) \ . \tag{3.19}$$

Then $Y^{(0)}$ is a point, and $Y^{(1)}$ has two edges. One has no stabilizer, and one has a $\mathbb{Z}/2\mathbb{Z}$ stabilizer. So our 1-skeleton is



Then $Y^{(2)}$ is $\mathbb{RP}^2$.

As before, we get the pair sequence

$$H_i(Y;A) \to H_i(Y';A) \to H_i(Y',Y;A) \to H_{i-1}(Y;A) \to H_{i-1}(Y;A) \ . \tag{3.20}$$

This relative homology is now

$$H_i(Y',Y;A) \cong \widetilde{H}_{i-1}\left(S^{p-1}/G_\sigma;A\right) \ . \tag{3.21}$$

If $A = \mathbb{Q}$ then

$$\widetilde{H}_{i-1}\left(S^{p-1}/G_\sigma;A\right) = \begin{cases} \mathbb{Q} & i=p, G_\sigma \subset A_{p+1} \\ 0 & \text{else} \end{cases} \ . \tag{3.22}$$

If $A = \mathbb{Z}$ then it can be significantly more complicated.

EXAMPLE 3.2. For $p = 3$, $G_\sigma = \mathbb{Z}/4\mathbb{Z}$, we have

$$S^2/G_\sigma \cong \mathbb{RP}^2 \tag{3.23}$$

and

$$H_1\left(S^2/G_\sigma;\mathbb{Z}\right) = \mathbb{Z}/2\mathbb{Z} \ . \tag{3.24}$$

This example highlights an important difference from ordinary Δ-complexes and more general CW-complexes. In those classical constructions, adding $p$-cells can only add generators to $H_p$ and relations to $H_{p-1}$. This example shows that, for symmetric Δ-complexes if we attach a symmetric 3-cell, then we may alter $H_1$.

**2.2. Filtration by skeleta.** As before, the filtration by $p$-skeleta induces a spectral sequence, and the $E_1$ page will be the relative homology. If we take $\mathbb{Q}$ coefficients, then the same argument as before shows that $E_1^{p,q} = 0$ for $q \neq 0$. So the $q = 0$ row is what we would call "cellular homology". The chains are given by

$$C_p(Y) = \left(\mathbb{Q}^{Y_p} \otimes \operatorname{sgn}\right)_{S_{p+1}} \ . \tag{3.25}$$

So this is $\mathbb{Q}^\alpha$ where $\alpha$ is the number of $S_{p+1}$ orbits in $Y_p$, with stabilizer contained in $A_{p+1}$. And in fact we have

$$H_*(C_*(Y)) \cong H_*(|Y|;\mathbb{Q}) \ . \tag{3.26}$$

Lecture 25; April 3, 2020

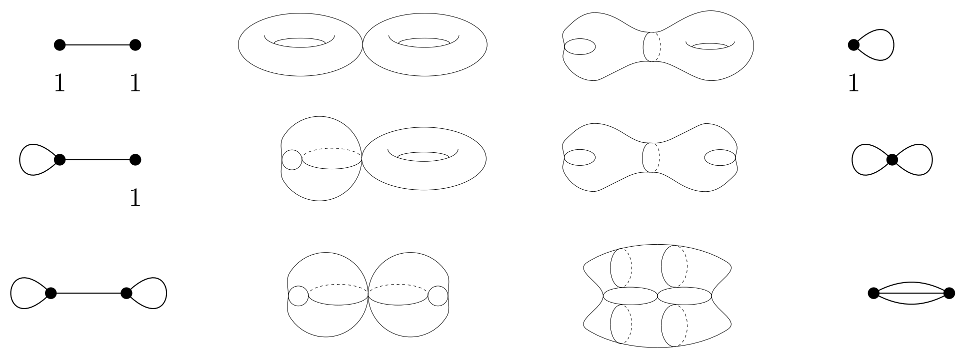
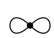



FIGURE 1. Top line: the curves in the codimension 1 strata. Middle line: the curves in the codimension 2 strata. Bottom line: the curves in the codimension 3 (and therefore dimension 0) strata. Collapsing an edge (and increasing its label by 1) tells us that the curve corresponding to the old graph is a degeneration of the curve corresponding to the new graph.

## 3. Dual complex of the boundary divisor of $\overline{\mathcal{M}}_g$

**3.1.** $\Delta_2$**.** We will describe $\Delta_2$, the dual complex of $\overline{\mathcal{M}}_2 \setminus \mathcal{M}_2$. If we have a genus 2 curve, it can degenerate to a stable nodal curve in one of two ways, as in the top line of fig. 1. So these are the curves in the codimension 1 strata of the boundary divisor. Then these curves can further degenerate as indicated in fig. 1, and these are the curves in the codimension 2 strata.

Note the upper left curve is reducible, so it cannot degenerate to an irreducible curve on the right in the middle. In terms of the dual graphs, this is saying that we cannot contract an edge of  to get  whereas the other graph on the middle line can be contracted into this graph.

Note that the stable graphs with three edges (the bottom line of fig. 1) correspond to the curves in the minimal strata, so these should be zero-dimensional. To see this geometrically, note that both curves of this type are determined by three points of $\mathbb{P}^1$. But any three distinct points can be taken to 0, 1, and $\infty$ by a linear change of coordinates. So they do indeed sit in 0-dimensional strata.

The curves on the middle line of fig. 1 sit in one-dimensional strata. For the left example, the $j$ invariant gives us our single parameter, and the right example is determined by four points in $\mathbb{P}^1$, which have a cross ratio which is our single parameter. The curves on the top sit in two-dimensional strata. On the left we have two $j$ invariants, and on the right we have a choice of a single point, so in both cases we have a two-parameter family of curves of these types.

Now we describe $\Delta_2$ as the space of graphs of these types. First, we have a 1-parameter family given by the solid horizontal line in fig. 1. The graphs on this line have one vertex, one loop of length $t$, and one of length $1-t$. The left vertex corresponds to $t=0$ where we have  but this only reaches $t = 1/2$, since we have quotiented out by the $\mathbb{Z}/2$ action.

The vertical 1-parameter family has graphs with two vertices, an edge between them of length $t$, and a self loop on one of the vertices of length $1-t$. The bottom corresponds to

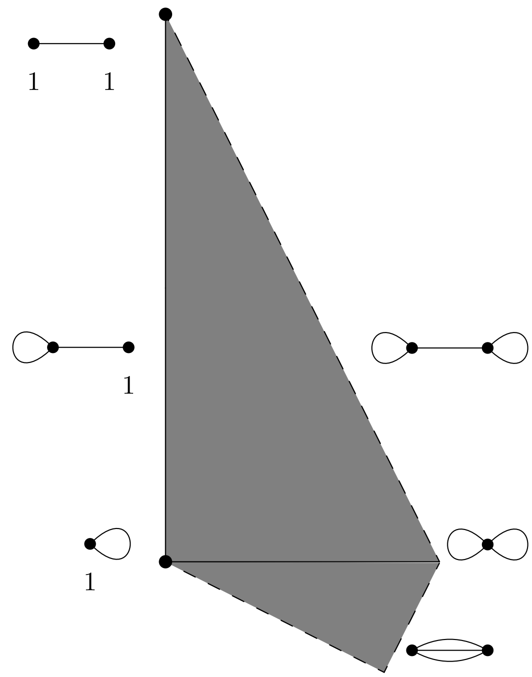



FIGURE 2. The cells of $\Delta_2$. Points are labelled with the dual complex of the corresponding curve.

$t = 0$, where we have still have and this time there is nothing to quotient out by (since one edge has a self-loop and the other doesn't) so we reach all the way to $t = 1$ where we have since it has no self loop, and an edge of length 1.

Graphs such as sit in a triangle with $\mathbb{Z}/2$ action. After quotienting we get the piece which is glued to the two 1-cells in the only way possible. Similarly, graphs such as sit in a triangle with an $S_3$ action. After quotienting out by this we glue this to the existing complex on the bottom.

Now it is clear that $\Delta_2 \simeq \mathrm{pt}$ is contractible. Therefore, by Corollary 2.18, we have

$$\mathrm{Gr}^6_W H^* (\mathcal{M}_2; \mathbb{Q}) = 0 \ . \tag{3.27}$$

**3.2.** $\Delta_3$**.** The example of $\Delta_2$ captured many of the features of $\Delta_g$ for general $g$. But as we increase $g$, we will get much more interesting $\Delta_g$. $\Delta_3$ is already not contractible. We will list the cells of $\Delta_3$ and use this to compute the rational homology. As it turns out, in general, it is much easier to describe the rational homology than the homotopy type.

EXAMPLE 3.3. First we write down all 3-valent graphs of genus 3. These are in fig. 3. These all have six edges and 4 vertices.

Now consider all of the genus 3 graphs with 5 edges. Since these are all given by collapsing an edge of a graph in fig. 3, there at most 30, but there are a lot of redundancies. In particular, we get the graphs in fig. 4. Now we want to consider all of the genus 3 graphs with 4 edges. There at most 40, but there are a lot of redundancies. These are pictured in

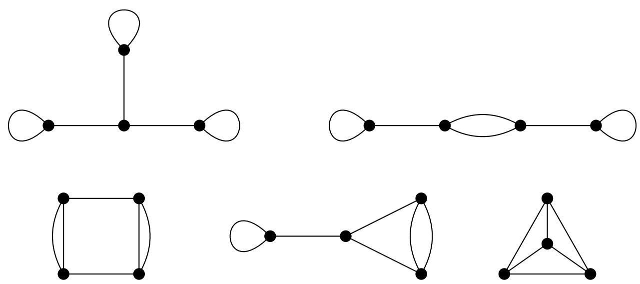


FIGURE 3. The five graphs of genus 3 with valence 3. Notice they all have six edges and four vertices.

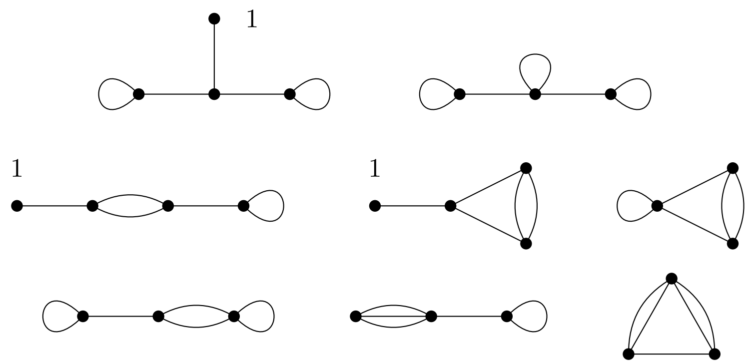



FIGURE 4. Stable graphs of genus 3 with 5 edges.

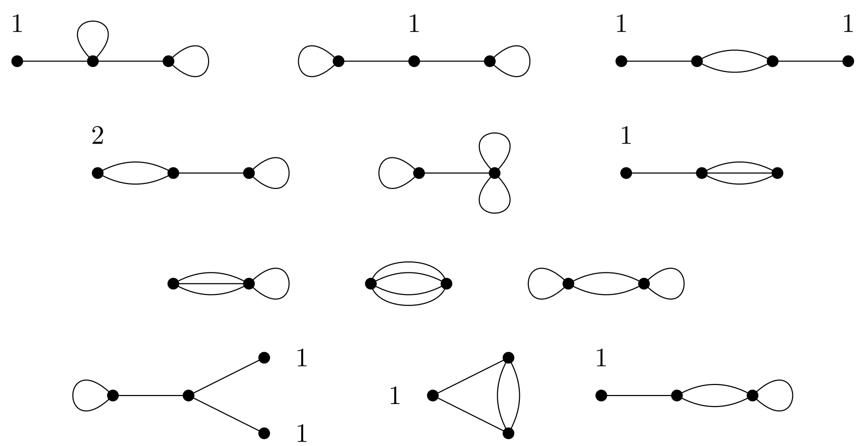



FIGURE 5. Stable graphs of genus 3 with 4 edges.

fig. 5. If we collapse another edge to get graphs with 3 edges, we get the graphs in fig. 6. Finally the graphs with 2 edges are in fig. 7, and there are only two graphs with 1 edge as in fig. 8.

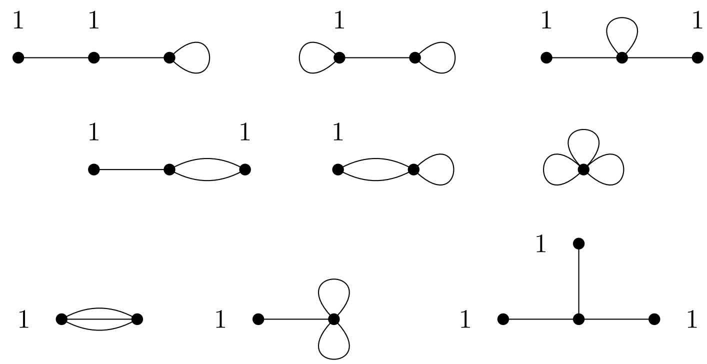


FIGURE 6. Stable graphs of genus 3 with 3 edges.

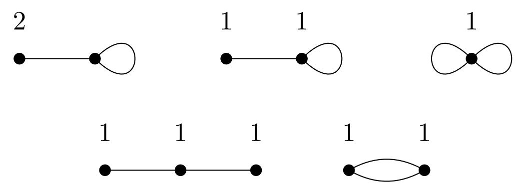


FIGURE 7. Stable graphs of genus 3 with 2 edges.

FIGURE 8. Stable graphs of genus 3 with 1 edge.

Note that $\Delta_3$ is a five-dimensional complex with 41 cells. Many of these cells have automorphisms that act by odd permutations on the corresponding edge sets, and this dramatically simplifies the computation of rational homology. For instance, whenever a graph has multiple edges (meaning two or more edges between the same pair of vertices), there is an automorphism acting by a simple transposition on these edges, and hence the corresponding cell does not contribute to the cellular chain complex for $\Delta_g$. Another useful insight is that, to go from graph that does not have any loop edges or vertices of positive weight to one that does, by a series of edge contractions, one has to pass through a graph with multiple edges. This will lead to a direct sum decomposition of the cellular chain complex of $\Delta_g$, with two summands generated, respectively, by graphs with and without loop edges or vertices of positive weight.

Lecture 26; April 6, 2020

**Proposition 3.1.** *The cellular chain complex $C_\bullet(\Delta_g)$ splits as a direct sum:*

$$C_\bullet(\Delta_g) = C_\bullet\left(\Delta_g^{lw}\right) \oplus C_\bullet\left(\Delta_g, \Delta_g^{lw}\right) \tag{3.28}$$

*where $\Delta_g^{lw}$ is the subcomplex of graphs with a loop edge or a vertex of positive weight, and*

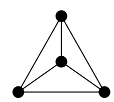
$$C_\bullet\left(\Delta_g, \Delta_g^{lw}\right) = C_\bullet(\Delta_g)/C_\bullet\left(\Delta_g^{lw}\right) \ . \tag{3.29}$$

PROOF. To see this, note that we have a short exact sequence

$$0 \to C_\bullet\left(\Delta_g^{\mathrm{lw}}\right) \to C_\bullet\left(\Delta_g\right) \to C_\bullet\left(\Delta_g, \Delta_g^{\mathrm{lw}}\right) \to 0\ . \tag{3.30}$$

The splitting map $\sigma$ takes $C_\bullet(\Delta_g, \Delta_g^{\mathrm{lw}})$ to the subcomplex of $C_\bullet(\Delta_g)$ spanned by graphs without loops or vertices of positive weight, in the obvious way. This is clearly a splitting at the level of vector spaces. The fact that it is a map of complexes follows from the observation that the only way to get a new vertex of positive weight by contracting an edge is to contract a loop, and the only way to get a new loop by contracting an edge is by contracting an edge between two vertices connected by multiple edges. The latter are zero in $C_\bullet(\Delta_g)$, and it follows that $\sigma$ is a map of complexes, as required. $\square$

Looking at figs. 3, 4, 6 and 7 and fig. 8, one sees that, for $g = 3$, $C_\bullet\left(\Delta_g, \Delta_g^{\mathrm{lw}}\right)$ is spanned by

(3.31) $K_4 =$  .

**Proposition 3.2.** $\mathrm{Aut}\,(K_4)$ *acts by alternating permutations on the set of edges.*

PROOF. The action on vertices induces a canonical identification $\mathrm{Aut}\,(K_4) = S_4$. Recall that $S_4$ is generated by transpositions, and then observe that each transposition of vertices acts by a double transposition on the set of edges. $\square$

So therefore the complex $C_\bullet\left(\Delta_g, \Delta_g^{\mathrm{lw}}\right)$ is just $\mathbb{Q}$ in degree 5. The other summand in $C_\bullet(\Delta_3)$ does not contribute to homology, as shown by the following proposition.

**Proposition 3.3.** $\Delta_g^{lw}$ *is contractible. In particular,* $C_\bullet\left(\Delta_g^{lw}\right)$ *is acyclic.*

PROOF. See [**CGP2**, Theorem 1.1]. $\square$

THEOREM 3.4. $\widetilde{H}_\bullet\left(\Delta_3, \mathbb{Q}\right) \cong \mathbb{Q}$ *in degree* 5*, spanned by* $K_4$*.*

Combining with Corollary 2.18, we have the following.

**Corollary 3.5.** $\mathrm{Gr}_W^{12} H^*\left(\mathcal{M}_3, \mathbb{Q}\right) \cong \mathbb{Q}$ *in degree* 6*.*

This was first proved (in a different way) by Looijenga in [**L3**]; it was the first explicit example of unstable cohomology on $\mathcal{M}_g$. Supposedly Benson Farb calls this the dark matter, because of relations between $H^*\left(\mathcal{M}_g, \mathbb{Q}\right)$ and mathematical physics, and since we know (from Euler characteristic calculations) that there is so much more of it than what is in the subring generated by stable cohomology.

**3.3. Topology of $\Delta_4$.** We begin by identifying a large contractible subcomplex of $\Delta_4$. In the genus 3 case we considered the space of curves with loops and nonzero weights. This is contained in a larger subcomplex:

$$\Delta_g^{\mathrm{lw}} \subset \Delta_g^{\mathrm{br}} \tag{3.32}$$

where

$$\Delta_g^{\mathrm{br}} = \overline{\{\text{stable tropical curves with bridges}\}}\ . \tag{3.33}$$

**Proposition 3.6.** $\Delta_4^{br}$ *is contractible.*

PROOF. See [**CGP2**, Theorem 1.1]. $\square$

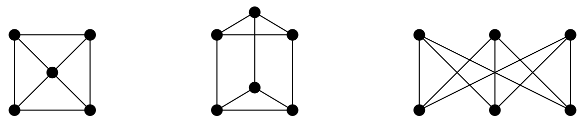


FIGURE 9. The three isomorphism classes of stable graphs of genus 4 without loops, vertices of positive weight, or multiple edges. The first is the 1-skeleton of the square pyramid, the second is the 1-skeleton of the triangular prism, and the third is the graph $K_{3,3}$.

We already know that graphs with multiple edges do not contribute to the cellular chain complex,[3.1] so the next step is to list all isomorphism classes of stable graphs of genus 4 without loops, vertices of positive weight, or multiple edges. There are 3, depicted in fig. 9.

We claim that the class of each of these graphs is zero in $C_\bullet(\Delta_4)$. Let us start with the 1-skeleton of the square pyramid. We can reflect this along the diagonal. This exchanges three pairs of edges, i.e. it is an odd permutation, to the class of this graph is zero in the cellular chain complex. Similarly, the 1-skeleton of the triangular prism has an automorphism interchanging the top triangle with the bottom triangle, which exchanges three pairs of edges. And transposing two non-adjacent vertices in $K_{3,3}$ exchanges three pairs of edges as well. This proves the claim. As a consequence, we have an isomorphism of chain complexes

$$C_\bullet\left(\Delta_4\right) \cong C_\bullet\left(\Delta_4^{\mathrm{br}}\right) \ . \tag{3.34}$$

Since $\left(\Delta_4^{\mathrm{br}}\right)$ is contractible, this complex is acyclic, and

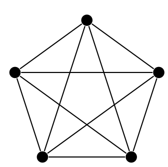
$$\mathrm{Gr}_{18}^W H^*\left(\mathcal{M}_4;\mathbb{Q}\right) = 0 \ . \tag{3.35}$$

We have shown that $\Delta_3$ has rational homology of $S^5$. In fact $\Delta_3$ is homotopy equivalent to $S^5$ [**ACP**, Theorem 1.2]. We have also shown that $\Delta_4$ has rational homology of a point. However, $\Delta_4$ is not contractible. In fact $H_5\left(\Delta_4;\mathbb{Z}\right)$ has 3-torsion, and both $H_6\left(\Delta_4;\mathbb{Z}\right)$ and $H_7\left(\Delta_4;\mathbb{Z}\right)$ have 2-torsion [**ACP**, Theorem 1.3]. One can see this using the spectral sequence associated to the filtration by closed subspaces. Lecture 27; April 8, 2020

Recall we had $d\left(K_4\right) = 0$ because every edge is contained in a triangle. This gives us an easy way to construct cellular cycles in $\Delta_g$. We just write down graphs where every edge is contained in a triangle

Consider, for instance, $K_5$:

(3.36)



.

This is the 1-skeleton of the 4-simplex. As with $K_4$, every edge is contained in a triangle, so $d\left(K_5\right) = 0$. We claim, however, that $[K_5] \in \mathrm{Im}(d)$.

Toi see this, recall

$$\widetilde{H}_9\left(\Delta_6;\mathbb{Q}\right) \cong \mathrm{Gr}_W^{30} H^{20}\left(\mathcal{M}_6;\mathbb{Q}\right) \ . \tag{3.37}$$

Then we have Harer's theorem

[3.1]In face, the closure of the locus of curves with bridges or multiple edges is contractible; see [**ACP**, Theorem 6.1].

THEOREM 3.7 (Harer [**H2**]).

$$\operatorname{vcd}(\mathcal{M}_g) = 4g - 5 \tag{3.38}$$

*where* vcd *is the virtual cohomological dimension.*

In other words, for any local system $\mathcal{E}$ with rational coefficients,

$$H^j(\mathcal{M}_g, \mathcal{E}) = 0 \tag{3.39}$$

for $g > 4g - 5$ and there exists a local system $\mathcal{E}$ such that $H^{4g-5}(\mathcal{M}_g; \mathcal{E}) \neq 0$.

When $g = 6$, this implies

$$H^j(\mathcal{M}_6; \mathbb{Q}) = 0 \tag{3.40}$$

for $j > 19$. So we have the following corollary.

**Corollary 3.8.** $[K_5] \in \operatorname{im}(d)$, *i.e.* $[K_5] = 0$ *in* $\widetilde{H}_9(\Delta_6; \mathbb{Q})$.

Similarly, $d(K_n) = 0$ for $n > 5$, but $[K_n] \in \operatorname{im}(d)$.

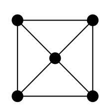
**Corollary 3.9.** $\widetilde{H}_j(\Delta_g; \mathbb{Q}) = 0$ *for* $j \leq 2g - 2$.

The following are open problems.

(1) Provide a combinatorial proof that $\widetilde{H}_j(\Delta_g; \mathbb{Q}) = 0$ for $j \leq 2g - 2$.
(2) Provide a combinatorial proof that $K_n \in \operatorname{im}(d)$ for $n > 4$.

We now consider a different sequence of graphs of increasing genus that naturally generalize $K_4$. Let $W_g$ be the wheel graph of genus $g$, i.e., the 1-skeleton of a $g$-gonal prism. So $W_3 = K_4$.

Next, consider the 4-wheel $W_4$:

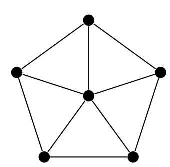
(3.41)  .

Note that the automorphism group is

$$\operatorname{Aut}(W_4) = D_4\ . \tag{3.42}$$

However, reflecting across a diagonal transposes three pairs of edges. Hence $[W_4] = 0$ in $C_\bullet(\Delta_4)$. Similarly, $[W_g] = 0$ in $C_\bullet(\Delta_4)$ whenever $g$ is even. But when $g$ is odd, the automorphisms of $W_g$ act by even permutations on the edge set.

Now consider $W_5$:

(3.43)  .

Again

$$\operatorname{Aut}(W_5) = D_5\ . \tag{3.44}$$

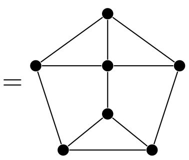
The class $[W_5]$ is nonzero in $C_\bullet(\Delta_5)$, and $d([W_5]) = 0$ because every edge is contained in a triagnle, so it is natural to ask whether $[W_5] \in \operatorname{im}(d)$.

So what are the stable graphs $G$ such that $[W_5]$ appears with nonzero coefficient in $d([G])$?

Any such edge must have an edge connecting a vertex of valence 4 to one of valence 3, such that contracting it gives the central vertex of valence 5 in $W_5$.

One possibility is:

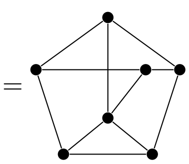
(3.45) $G =$



This graph has exactly one vertex $v$ of valence 4, which must be fixed by any automorphism. Similarly, the top vertex is the unique 3-valent vertex whose star is contained in the star of $v$. And there is a unique vertex adjacent to both vertices not adjacent to $v$. Each of these must be fixed by any automorphism, and it follows that the unique nontrivial automorphism is reflection across the vertical axis. This interchanges four pairs of edges, and hence $[G] \neq 0$ in $C_* (\Delta_5)$.

The other possibility for a graph $G'$ such that $W_5$ appears with nonzero coefficient in $d([G'])$ is:

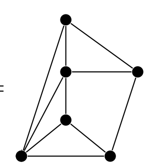
(3.46) $G' =$



However, $G'$ has an automorphism obtained by interchanging the two vertices adjacent to the 4-valent vertex that are not contained in a triangle. This acts by transposing three pairs of edges, and hence $[G'] = 0$ in the cellular chain complex. This proves that $G$ is the unique graph such that $W_5$ appears with nonzero coefficient in $d([G])$.

So, now we compute $d([G])$. Contracting one edge of $G$ takes us back to $W_5$, and there are two other edges not contained in a triangle, but these are swapped by the unique automorphism of $G$. Contracting either of these edges gives:

(3.47) $G'' =$



One checks that any automorphism of $G''$ must act by even permutations on the edges, so $[G''] \neq 0$ in $C_\bullet (\Delta_5)$.

With appropriate orientations, and checking signs carefully, one finds:

$$d(G) = W_5 + 2G'' \ . \tag{3.48}$$

Moreover, there is no other graph $H$ such that $[G'']$ appears with nonzero coefficient in $d([H])$.

As a consequence, we conclude that $W_5 \notin \operatorname{im}(d)$, so

$$0 \neq [W_5] \in \widetilde{H}_9 (\Delta_5; \mathbb{Q}) = \operatorname{Gr}^{24}_W H^{14} (\mathcal{M}_5; \mathbb{Q}) \ . \tag{3.49}$$

## 4. Cohomology of $\mathcal{M}_g$

Recall $\dim_{\mathbb{C}} \mathcal{M}_g = 3g-3$, so $H^j (\mathcal{M}_g; \mathbb{Q}) = 0$ for $j > 6g-6$. We know from Harer's Theorem 3.7 that $\operatorname{vcd}(\mathcal{M}) = 4g-5$, which implies $H^j (\mathcal{M}_g; \mathbb{Q}) = 0$ for $j > 4g-5$. Furthermore, the singular cohomology of $\mathcal{M}_g$ with rational coefficients vanishes in degree equal to the virtual cohomological dimension, i.e. $H^{4g-5}(\mathcal{M}_g; \mathbb{Q}) = 0$ [**CFP1**, **MSS**].

Lecture 28; April 10, 2020

We now discuss what is known about the cohomology of $\mathcal{M}_g$ in low degrees, and the subring generated by low degree cohomology.

**4.1. Low degree cohomology of $\mathcal{M}_g$, stabilization, and the tautological ring.** See [**V1**] for a survey concerning the moduli space of curves and its tautological ring.

The relative cotangent bundle $\mathbb{L}$ of the map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is

$$\mathbb{L} = \left\{(C, p, v) : (C, p) \in \mathcal{M}_{g,1}, v \in T_p^* C\right\} \tag{3.50}$$

and fits into

$$\mathbb{L} \to \mathcal{M}_{g,1} \xrightarrow{\pi} \mathcal{M}_g \ . \tag{3.51}$$

This is the relative dualizing bundle for this smooth map.

Define

$$\psi \coloneqq c_1 (\mathbb{L}) \tag{3.52}$$

to be the first Chern class.

4.1.1. *Chern classes.* See [**MS**] for a reference. There are a few different ways to think about Chern classes. Let $\mathbb{P}^n$ be complex projective space. We have

$$\mathbb{P}^n \subseteq \mathbb{P}^{n+1} \subseteq \cdots \tag{3.53}$$

and we define

$$\mathbb{P}^\infty = \bigcup_n \mathbb{P}^n \ . \tag{3.54}$$

Theorem 3.10. *The collection of isomorphism classes of line bundles on $X$ is equivalent to collection of maps $X \to \mathbb{P}^\infty$.*

Theorem 3.11.

$$H^* (\mathbb{P}^\infty) = \mathbb{Z} [x] \tag{3.55}$$

*where* $\deg (x) = 2$.

This doesn't quite determine $x$. We also insist that

$$\int_{\mathbb{P}^1} x = 1 \ . \tag{3.56}$$

So if $\mathbb{L}$ is a line bundle on $X$, then $X \xrightarrow{f_{\mathbb{L}}} \mathbb{P}^\infty$ (up to homotopy) gives us

$$c_1^{\text{top}} (\mathbb{L}) \in H^2 (X; \mathbb{Z}) \qquad\qquad c_1^{\text{top}} (\mathbb{L}) = f_{\mathbb{L}}^* (x) \tag{3.57}$$

Alternatively, consider the following. Let $\mathcal{M}_{\mathbb{L}}$ be the moduli space (stack) of line bundles. For $X \in \mathbf{Sch}$ a scheme, we map

$$X \mapsto \{\text{isomorphism classes of line bundles on } X\} \ . \tag{3.58}$$

Then we have the following.

Theorem 3.12. *$\mathcal{M}_{\mathbb{L}}$ is an Artin stack, and the Chow ring is*

$$A^* (\mathcal{M}_{\mathbb{L}}) = \mathbb{Z} [x] \ . \tag{3.59}$$

Then this implies for $\mathbb{L}$ a line bundle on $X \in \mathbf{Sch}$ leads to $f\colon X \to \mathcal{M}_{\mathbb{L}}$, and we can define

$$c_1(\mathbb{L}) \coloneqq f_{\mathbb{L}}^* X \ . \tag{3.60}$$

If $X$ is smooth and proper over $\mathbb{C}$, then the algebraic and topological definitions of Chern classes are compatible, via the cycle class map, i.e.,

$$\begin{array}{ccc} A^*(X) & \longrightarrow & H^{2*}(X) \\ & & \\ c_1(\mathbb{L}) & \longmapsto & c_1^{\mathrm{top}}(\mathbb{L}) \end{array} \ . \tag{3.61}$$

**4.2. $\kappa$-classes.** In our setting, for $\mathbb{L}$ the relative cotangent bundle as in eq. (3.50):

$$\mathbb{L} \to \mathcal{M}_{g,1} \xrightarrow{\pi} \mathcal{M}_g \ , \tag{3.62}$$

we define

$$\psi = c_1(\mathbb{L}) \in H^2(\mathcal{M}_{g,1};\mathbb{Q}) \ . \tag{3.63}$$

Then we define Mumford's $\kappa$-classes as follows.

Definition 3.1 ($\kappa$-classes).

$$\kappa_i = \pi_*\left(\psi^{i+1}\right) \in H^{2i}(\mathcal{M}_g;\mathbb{Q}) \ . \tag{3.64}$$

Remark 3.2. The map $\pi_*$ is the Gysin push-forward

$$\pi_*\colon H^*(\mathcal{M}_{g+1};\mathbb{Q}) \to H^{*-1}(\mathcal{M}_g;\mathbb{Q}) \ . \tag{3.65}$$

See section 4.3 for further discussion of Gysin push-forward for oriented fiber bundles.

Definition 3.2 (tautological ring). The *tautological ring* $R^*(\mathcal{M}_g)$ is the subring of $H^*(\mathcal{M}_g;\mathbb{Q})$ generated by $\{\kappa_i\}$.

Remark 3.3. This subring is especially important for Gromov-Witten theory. Important formulas in Gromov-Witten theory such as the ELSV formula [**ELSV**] and the Pandharipande-Pixton relations [**PP**] are naturally expressed in terms of polynomials in the $\kappa$-classes.

Theorem 3.13 (Faber-Looijenga). *$R^i(\mathcal{M}_g) = 0$ for $i > g-2$ and*

$$R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q} \ . \tag{3.66}$$

Conjecture 2 (Faber). *The map*

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_j) \cong \mathbb{Q} \tag{3.67}$$

*is perfect.*

This has been computationally verified by Faber for $g < 24$; however, it is incompatible with Pixton's conjectural characterization of all relations among the $\kappa_i$.

**Corollary 3.14.**

$$\dim R^*(\mathcal{M}_g) < \sum_{n=1}^{g-2} p(n) \tag{3.68}$$

*where $p(n)$ is the partition number of $n$.*

THEOREM 3.15 (Hardy-Ramanujan).

$$p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \tag{3.69}$$

The original proof is a prototypical application of the circle method; there is also an elementary proof due to Erdős.

As a consequence of the Hardy-Ramnujan estimate for $p(n)$, we have

$$\dim R^*(\mathcal{M}_g) < g \cdot p(g) \tag{3.70}$$

$$< c^{\sqrt{g}} \ . \tag{3.71}$$

We will see that

$$\frac{\dim R^*(\mathcal{M}_g)}{\dim H^*(\mathcal{M}_g;\mathbb{Q})} \to 0 \tag{3.72}$$

as $g \to \infty$. So the tautological ring is, in this sense, a vanishingly small portion of the cohomology ring of $\mathcal{M}_g$.

Lecture 29; April 13, 2020

**4.3. Gysin map.** Recall definition 3.1, where the $\kappa$-classes were defined as the images of the powers of

$$\psi = c_1(\mathbb{L}) \in H^2(\mathcal{M}_{g,1};\mathbb{Q}) \tag{3.73}$$

under the map:

$$\pi_* : H^*(\mathcal{M}_{g+1};\mathbb{Q}) \to H^{*-2}(\mathcal{M}_g;\mathbb{Q}) \ . \tag{3.74}$$

This map is the *Gysin map* associated to the *orietented fiber bundle* $\pi$. We should think of $\pi_*$ as "integration along the fibers".

The construction is as follows. Choose compatible triangulations of $\mathcal{M}_g$ and $\mathcal{M}_{g,1}$. See [**H6**]. Then for each oriented $k$-face $\sigma \subseteq \mathcal{M}_g$, $\pi^{-1}(\sigma)$ is a $(k+2)$-chain on $\mathcal{M}_{g,1}$. Moreover, $\pi^{-1}(\partial\sigma) = \partial\left(\pi^{-1}(\sigma)\right)$, which implies that $\pi^{-1}$ is compatible with $\partial$, and

$$\pi^{-1}(k\text{-cycle}) = (k+2)\text{-cycle} \qquad \pi^{-1}(k\text{-boundary}) = (k+2)\text{-boundary} \ . \tag{3.75}$$

So we have a "wrong way" map on homology:

$$\pi^* \colon H_k(\mathcal{M}_g;\mathbb{Q}) \to H_{k+2}(\mathcal{M}_{g,1};\mathbb{Q}) \tag{3.76}$$

called the *Gysin pullback*. Now dualizing we get a pullback on the duals:

$$\pi^* \colon H^k(\mathcal{M}_g;\mathbb{Q})^* \to H^{k+2}(\mathcal{M}_{g,1};\mathbb{Q})^* \ . \tag{3.77}$$

Now we can define the Gysin pushforward as the adjoint (transpose) of the Gysin pullback:

$$\pi_* \colon H^{k+2}(\mathcal{M}_{g,1};\mathbb{Q}) \to H^k(\mathcal{M}_g;\mathbb{Q}) \ . \tag{3.78}$$

**4.4. Stable cohomology of $\mathcal{M}_g$.** The following was conjectured by Mumford and proved, decades later, by Madsen and Weiss.

CONJECTURE 3 (Mumford). *$\deg \kappa_j = 2j$ and*

$$H^i(\mathcal{M}_g;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\ldots]_i \tag{3.79}$$

*for $g \gg i$.*

This implies that

$$\dim H^i(\mathcal{M}_g;\mathbb{Q}) = \begin{cases} 0 & i \text{ odd} \\ p(i/2) & i \text{ even} \end{cases} \tag{3.80}$$

for $g \gg i$ where $p$ is the permutation number.

An early major step toward confirming Mumford's conjecture was the following stabilization theorem of Harer.

THEOREM 3.16 (Harer [**H3**]). *$H^i(\mathcal{M}_g;\mathbb{Q}) \cong H^i(\mathcal{M}_{g+1};\mathbb{Z})$ for $g \gg i$.*

This was followed by a striking result of Tillmann, on the structure of the stable cohomology.

THEOREM 3.17 (Tillmann [**T**]). *The stable cohomology of $\mathcal{M}_g$ is the cohomology of an infinite loop space.*

The next steps are to understand which infinite loop space is appearing in this context and to compute its cohomology.

DEFINITION 3.3. The Grassmannian of affine $k$-planes in $\mathbb{R}^n$ is

$$\mathrm{AG}(k,n) := \{\text{affine } k\text{-planes in } \mathbb{R}^n\} \; . \tag{3.81}$$

We write $\mathrm{AG}^+(k,n)$ for the 1-point compactification of $\mathrm{AG}(k,n)$.

Note that

$$\mathrm{AG}^+(k,n) \subseteq \mathrm{AG}^+(k,n+1) \subseteq \dots \tag{3.82}$$

so we can define

$$\mathrm{AG}^+(k,\infty) = \varinjlim_n \mathrm{AG}^+(k,n) \tag{3.83}$$

The following theorem was conjectured by Madsen-Tillmann [**MT**].

THEOREM 3.18 (Madsen-Weiss [**MW**]). *The stable cohomology of $\mathcal{M}_g$ is the cohomology of the infinite loop space*

$$\Omega^\infty\left(\mathrm{AG}^+(2,\infty)\right) = \varinjlim_n \Omega^n\left(\mathrm{AG}^+(2,n)\right) \; . \tag{3.84}$$

The cohomology ring of $\Omega^\infty\left(\mathrm{AG}^+(2,\infty)\right)$ is naturally identified with the polynomial ring in the $\kappa$-classes, so this theorem of Madsen and Weiss confirms Mumford's conjecture.

We will give a rough idea of the proof following [**H5**]. Write $S_g$ for the compact oriented surface of genus $g$.

DEFINITION 3.4. Define the configuration space $\mathcal{C}(S_g,\mathbb{R}^n)$ to be the collection of oriented subsurfaces of $\mathbb{R}^n$ which are diffeomorphic to $S_g$.

Write $\mathcal{E}(S_g,\mathbb{R}^n)$ for the collection of smooth embeddings of $S_g$ in $\mathbb{R}^n$.

REMARK 3.4. Roughly speaking, the configuration space is something like a Hilbert scheme.

Note that

$$\mathcal{C}(S_g,\mathbb{R}^n) = \mathcal{E}(S_g,\mathbb{R}^n)\,/\,\mathrm{Diff}^+(S_g) \tag{3.85}$$

where $\mathrm{Diff}^+(S_g)$ consists of the oriented diffeomorphisms.

Then we have

$$\mathcal{E}\left(S_g, \mathbb{R}^\infty\right) = \varinjlim_n \mathcal{E}\left(S_g, \mathbb{R}^n\right) \tag{3.86}$$

and

$$\mathcal{C}\left(S_g, \mathbb{R}^\infty\right) = \varinjlim_n \mathcal{C}\left(S_g, \mathbb{R}^n\right) = \mathcal{E}\left(S_g, \mathbb{R}^\infty\right) / \operatorname{Diff}^+\left(S_g\right) \ . \tag{3.87}$$

**Proposition 3.19.** *$\mathcal{E}\left(S_g, \mathbb{R}^\infty\right)$ is contractible.*

**Proposition 3.20.** *For $g \gg 2$, each connected component of $\operatorname{Diff}^+\left(S_g\right)$ is contractible.*

As a consequence, we have that

$$\mathcal{C}\left(S_g, \mathbb{R}^\infty\right) = B \operatorname{Mod}\left(S_g\right) \tag{3.88}$$

where Mod denotes the mapping class group. This implies that

$$H^*\left(\mathcal{C}\left(S_g, \mathbb{R}^\infty\right)\right) \cong H^*\left(\operatorname{Mod}\left(S_g\right)\right) \cong H^*\left(\mathcal{M}_g, \mathbb{Q}\right) \tag{3.89}$$

where the second isomorphism comes from Teichmüller theory.

Indeed, we have a properly discontinuous action of $\operatorname{Mod}\left(S_g\right)$ on Teichmüller space, a space of hyperbolic metrics on $S_g$. Teichmüller space is a contractible manifold of dimension $3g-3$, and the quotient is $\mathcal{M}_g$. Lecture 30; April 15, 2020

**Proposition 3.21.** *(1) $\mathcal{E}\left(S_g, \mathbb{R}^\infty\right)$ is contractible, and*
*(2) each connected component of $\operatorname{Diff}^+\left(S_g\right)$ is contractible for $g \geq 2$.*

Then this means

$$E \operatorname{Mod}\left(S_g\right) \coloneqq \mathcal{E}\left(S_g, \mathbb{R}^\infty\right) / \operatorname{Diff}^+_0\left(S_g\right) \ , \tag{3.90}$$

where $\operatorname{Diff}^+_0$ denotes the connected component of the identity of $\operatorname{Diff}^+$, is contractible. It also has a free action of the mapping class group:

$$\operatorname{Mod}\left(S_g\right) = \operatorname{Diff}^+\left(S_g\right) / \operatorname{Diff}^+ 0\left(S_g\right) \ . \tag{3.91}$$

I.e. $E \operatorname{Mod}\left(S_g\right)$ is contractible with free $\operatorname{Mod}\left(S_g\right)$ action, and

$$B \operatorname{Mod}\left(S_g\right) \coloneqq \mathcal{C}\left(S_g, \mathbb{R}^\infty\right) \tag{3.92}$$

is the quotient.

### 4.5. Review of classifying spaces.
Let $G$ be a discrete group

Remark 3.5. There is a version for non-discrete groups as well, e.g. $\mathrm{GL}_n$ has the infinite Grassmanian of $n$-planes in $\mathbb{R}^\infty$ as its classifying space. But we will focus on discrete groups.

**Proposition 3.22.** *(1) There exists a contractible CW-complex $EG$ with free $G$-action.*
*(2) The quotient $BG = EG/G$ is well-defined up to homotopy.*
*(3) The singular cohomology of $BG$ with coefficients in $A$ is the same as the group cohomology of $G$: $H^*\left(BG; A\right) \cong H^*\left(G; A\right)$.*
*(4) $BG$ is a $K\left(\pi, 1\right)$-space for $G$, i.e.*

$$\pi_i\left(BG\right) \cong \begin{cases} G & i = 1 \\ 1 & i > 1 \end{cases} \ . \tag{3.93}$$

PROOF. (1) Consider the $\Delta$-complex $EG$ with $n$-faces given by $(n+1)$-tuples $(g_0, \ldots, g_n)$ of elements of $G$, not necessarily distinct. Note that $G$ acts freely on $EG$ by

$$g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n) \tag{3.94}$$

for $g \in G$. Observe that at the vertex $g$, there is a loop edge $(g, g)$. Now we show $EG$ is contractible. To see this, consider the linear flow toward $e$ in $(e, g_0, \ldots, g_n)$. This is a homotopy from the inclusion $(g_0, \ldots, g_n) \hookrightarrow EG$ to the constant map $(g_0, \ldots, g_n) \to (e)$. These glue together to make a homotopy from $1_{EG}$ to the constant map $EG \to (e)$. This is not a deformation retraction because $e$ is not fixed, it flows along the loop edge $(e, e)$.

(2) Let $(EG)'$ be any other contractible space on which $G$ acts freely. Consider the quotients $X = EG/G$, $X' = (EG)'/G$. Then $EG$ is the universal cover of $X$, and $(EG)'$ is the universal cover of $X'$. This implies

$$\pi_1(X, x) \cong G \cong \pi_1(X', x') \ , \tag{3.95}$$

for any choice of base points $x$ and $x'$.

The homotopy lifting theorem, plus the isomorphism on fundamental groups above, gives rise to a map from the 1-skeleton $f\colon (X')^{(1)} \to X$, taking $x'$ to $x$. We then proceed by cellular approximation. Consider a 2-cell $e^2 \subseteq X'$. Then $\partial e^2$ is contractible in $X'$, which implies $f\left(\partial e^2\right)$ is contractible in $X$. Hence $f$ extends to $(X')^{(1)} \cup e^2$. Similar arguments show that $f$ extends to all of $X'$, cell-by-cell.

(3) Since we know $BG$ is independent (up to homotopy) of the choice of $EG$, we take the particular model from (1). In this model, each $n$-face of BG is the image of a unique $n$-face of $EG$ of the form $(e, g_1, \ldots, g_n)$, and hence corresponds to a unique $n$-tuple of elements of $G$. In this way, the cellular chain complex of $BG$ computes the group homology of $G$.

□

This is relevant for us because, as we have discussed, $\mathcal{M}_g$ is a $K(\pi, 1)$ for $\mathrm{Mod}(S_g)$, whose universal cover is Teichmüller space.

This implies

$$H^*(\mathcal{M}_g; \mathbb{Q}) \cong H^*(\mathcal{C}(S_g, \mathbb{R}^\infty); \mathbb{Q}) \ . \tag{3.96}$$

Hence we can understand and study the cohomology of $\mathcal{M}_g$ from multiple perspectives. Viewing $\mathcal{M}_g$ as an algebraic variety (or Deligne-Mumford stack) endows this ring with a mixed Hodge structure and leads to many of the ideas we have discussed in this course. Studying the same ring as the cohomology of $\mathcal{C}(S_g, \mathbb{R}^\infty)$ brings techniques from stable homotopy theory into play, and is especially fruitful if one considers $g$

Understanding the cohomology of $\mathcal{M}_g$ directly is much easier using tools from algebraic geometry such as mixed hodge theory. But from this point of view, the cohomology agrees with that of $\mathcal{C}(S_g, \mathbb{R}^\infty)$, which can be understood better using stable homotopy theory.

**4.6. Geometric relationship between $\mathcal{C}(S_g, \mathbb{R}^\infty)$ and $\Omega^\infty \mathrm{AG}^+(2, \infty)$.** These are related by the *scanning map*. Fix $S_g \subseteq \mathbb{R}^n$. Choose a small $\epsilon > 0$. For each point $x \in \mathbb{R}^n$, intersect $S_g$ with the ball $B_\epsilon(x)$. For $\epsilon$ sufficiently small, each $B_\epsilon(x) \cap S_g$ is an "almost flat" disk in $\mathbb{R}^n$. Fix a homeomorphism $B_\epsilon(x) \cong \mathbb{R}^n$. So this gives us a map

$$\mathbb{R}^n \to \mathrm{AG}^+(2, \mathbb{R}^n) \tag{3.97}$$

which sends

$$x \mapsto \begin{cases} \text{pt at } \infty & B_\epsilon(x) \cap S_g = \emptyset \\ T_y S_g & \text{o/w} \end{cases} \tag{3.98}$$

where $y$ is the barycenter of $B_\epsilon(x) \cap S_g$. The idea is that as we pull the ball away from the surface, this barycenter moves towards infinity. Note that because $S_g$ is compact, if we are "far away", we do map to the point at infinity. So this maps extends to

$$S^n \to \mathrm{AG}^+(2, \mathbb{R}^n) \tag{3.99}$$

where the point at infinity goes to the point at infinity.

In other words a point in $\mathcal{C}(S_g, \mathbb{R}^n)$ gives us a map $S^n \to \mathrm{AG}^+(2, \mathbb{R}^n)$, which is a point in $\Omega^n \mathrm{AG}^+(2, \mathbb{R}^n)$. This gives us a map

$$\mathcal{C}(S_g, \mathbb{R}^n) \to \Omega^n \left(\mathrm{AG}^+(2, \mathbb{R}^n)\right) \ . \tag{3.100}$$

Now passing to the limit, we get a map

$$\mathcal{C}(S_g, \mathbb{R}^\infty) \to \Omega^\infty \left(\mathrm{AG}^+(2, \mathbb{R}^\infty)\right) \ , \tag{3.101}$$

called *the scanning map.* This is well-defined up to homotopy. Then the theorem of Madsen-Weiss is that the scanning map induces isomorphisms on $H^i$ for $i < 2g/3$.

The rough idea is to let $g \to \infty$. Then show that

$$\mathcal{C}(S_\infty, \mathbb{R}^\infty) \to \Omega^\infty \left(\mathrm{AG}^+(2, \mathbb{R}^\infty)\right) \tag{3.102}$$

is a weak equivalence.

CHAPTER 4

# Kontsevich's graph complex

First we describe this graph complex $K^\bullet$ as a $\mathbb{Q}$ vector space. We will then add a differential and a compatible Lie bracket, so that it becomes a chain complex and, moreover, a differential graded Lie algebra (dgla). In fact, the differential will be given as the bracket with one particular element of the graph complex. Lecture 31; April 20, 2020

The generators of $K^\bullet$ in degree $n$ are isomorphism classes of pairs $(G, \omega)$ where $G$ is a graph with $n$ edges, no loops, and no multiple edges; and $\omega$ is a total ordering of $E(G)$. The relations among these are given by:

$$(G, \sigma(\omega)) = \operatorname{sign}(\sigma)(G, \omega) \tag{4.1}$$

for $\sigma \in S_n$. Notationally we will write $G = (G, \omega)$.

The operation on $K^\bullet$ we will focus on, is inserting a graph $G_1$ at some vertex $v$ of some other graph $G_2$. There are many ways of doing this. To specify such an insertion, we give a map

$$\varphi\colon \operatorname{Star}_{G_2}(v) \to V(G_1)\ . \tag{4.2}$$

Here, $\operatorname{Star}_{G_2}(v)$ is the set of edges containing $v$. Then the resulting graph is explicitly given by the graph $G_\varphi$ with vertices:

$$V(G_\varphi) \coloneqq V(G_1) \amalg V(G_2) \setminus \{v\} \tag{4.3}$$

and edges in bijection with the disjoint union

$$E(G_\varphi) \cong E(G_1) \amalg E(G_2)\,; \tag{4.4}$$

under this bijection, each edge $(v, v') \in \operatorname{Star}_{G_2}(v)$ corresponds with an edge $(\varphi(v), v')$ in $\operatorname{Star}_{G_\varphi}(\varphi(v))$.

Now we introduce a non-commutative binary operation on $K^\bullet$ by setting

$$G_1 \circ G_2 = \sum_{v \in V(G_2)} \sum_{\varphi\colon\ \operatorname{Star}(v) \to V(G_1)} G_\varphi \tag{4.5}$$

where $G_\varphi$ has edge ordering $\omega_1 \wedge \omega_2$, and extending linearly

**Proposition 4.1.** *The operation*

$$[G_1, G_2] = G_1 \circ G_2 - (-1)^{|E(G_1)| \cdot |E(G_2)|}\, G_2 \circ G_1 \tag{4.6}$$

*is a graded Lie bracket:*

$$[\ ,\ ] : K^i \wedge K^j \to K^{i+j}\ . \tag{4.7}$$

Proof. We need to check:

(1) $[\ ,\ ]$ is well-defined.

(2) $[\ ,\ ]$ is graded skew-symmetric: if $a \in K^i$ and $B \in K^j$, then

$$[a, b] = -(-1)^{i \cdot j}\,[b, a]\ . \tag{4.8}$$

(3) $[\ ,\ ]$ satisfies the graded Jacobi identity: if $a \in K^i$, $b \in K^j$, $c \in K^k$,

$$(-1)^{ik}[a,[b,c]] + (-1)^{ij}[b,[c,a]] + (-1)^{jk}[c,[a,b]] = 0 \ . \tag{4.9}$$

(1) We need to show that

$$[(G_1,\omega),G_2] = \operatorname{sign}(\sigma)[(G,\sigma(\omega)),G_2] \ . \tag{4.10}$$

This is true by inspection.

(2) Now we check skew-symmetry. For $a \in K^i$ and $b \in K^j$,

$$[G_1,G_2] = G_1 \circ G_2 - (-1)^{\deg G_1 \deg G_2} G_2 \circ G_1 \tag{4.11}$$

and

$$[G_2,G_1] = G_2 \circ G_1 - (-1)^{\deg G_2 \deg G_1} G_1 \circ G_2 \tag{4.12}$$

so then

$$-(-1)^{\deg G_1 \deg G_2}[G_2,G_1] = [G_1,G_2] \tag{4.13}$$

tautologically.

(3) To check the graded Jacobi identity, the key observation is that every term appears twice with opposite signs.

Exercise 0.1. Complete the verification of the graded Jacobi identity.

□

Now we have the graded $\mathbb{Q}$ vector space $K^\bullet$, endowed with a graded Lie bracket $[\ ,\ ]$. We use the Lie bracket to define a differential

$$d\colon K^i \to K^{i+1} \tag{4.14}$$

by

$$d := [\bullet\!\!-\!\!\!-\!\!\bullet, -] \ . \tag{4.15}$$

**Proposition 4.2.** *The graph complex $K^\bullet$ with this differential $d$ is a differential graded Lie algebra (dgla).*

Remark 4.1. A nice way of thinking of dglas is that they are Lie algebra objects in the category of chain complexes (in this case, of $\mathbb{Q}$ vector spaces).

Proof. We need to show:

(1) $d^2 = 0$, and
(2) for $a \in K^i$, $d[a,b] = [da,b] + (-1)^i[a,db]$.

The key observation is that every term appears twice with opposite sign. □

Exercise 0.2. Complete the verification of the required properties, by checking that the signs do cancel.

**Corollary 4.3.** $[\ ,\ ]$ *induces a bracket on* $H^*(K^\bullet)$.

Lecture 32; April 22, 2020

## 1. Genus

Observation 1. *The genus of the graphs $G_1$ and $G_2$ are given by*

$$g(G_1) = \#E(G_1) - \#V(G_1) + 1, \text{ and} \tag{4.16}$$

$$g(G_2) = \#E(G_2) - \#V(G_2) + 1. \tag{4.17}$$

*Then we have:*

$$g(G_\varphi) = \#E(G_1) + \#E(G_2) - \#V(G_1) - \#V(G_2) + 1 + 1 \tag{4.18}$$

$$= g(G_1) + g(G_2) \ . \tag{4.19}$$

In other words, the bracket is additive on genus:

$$[\ ,\ ] : K^\bullet_{g_1} \times K^\bullet_{(g_2)} \to K^\bullet_{(g_1+g_2)} \ . \tag{4.20}$$

Since the differential is bracket with a graph of genus 0, it preserves genus. So the chain complex $K^\bullet$ splits as a direct sum

$$K^\bullet = \bigoplus_g K^\bullet_{(g)} \tag{4.21}$$

where $K^\bullet_{(g)}$ is generated by graphs of genus $g$.

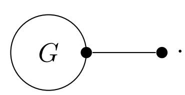
## 2. Stability

Claim 4.1. (1) If $G$ is stable then $dG$ is a sum of stable graphs.
(2) If $G$ is unstable then $dG$ is a sum of unstable graphs.

Proof. (1) If $G$ is stable, then consider

$$dG = [\bullet\!\!-\!\!\bullet, G] \ . \tag{4.22}$$

This contained unstable terms in the naive expansion. But such terms will appear with the opposite orientation. E.g. the two ways we can insert $\bullet\!\!-\!\!\bullet$ to get

(4.23) 

We can also get vertices of valence 2. For every edge of $G$, we can get a vertex of valence 2 by inserting a fixed vertex of $\bullet\!\!-\!\!\bullet$ at either of the vertices adjacent to that edge. As it turns out, the signs are set up exactly so that these cancel.

(2) If $G$ is unstable, then it has a vertex $w$ of weight 0 and valence less than 3. Consider

$$dG = [\bullet\!\!-\!\!\bullet, G] \ . \tag{4.24}$$

By inspection, every graph that appears is unstable.

□

**Corollary 4.4.** *The complex $K^\bullet$ splits as a direct sum*

$$K^\bullet \cong \bigoplus_g \left( K^{\bullet st}_{(g)} \oplus K^{\bullet unst}_{(g)} \right), \tag{4.25}$$

*where $K^{\bullet st}_{(g)}$ and $K^{\bullet unst}_{(g)}$ are generated by stable and unstable graphs, respectively, of genus $g$.*

## 3. Relationship to $\Delta_g$

Recall $C_\bullet(\Delta_g)$ splits as:

$$C_\bullet\left(\Delta_g^{\mathrm{lw}}\right) \oplus C_\bullet\left(\Delta_g, \Delta_g^{\mathrm{lw}}\right) \tag{4.26}$$

where the second summand is generated by stable graphs with no loops or weights. Now taking duals of everything, we get a corresponding splitting of cochain complexes:

$$C^\bullet(\Delta_g) \cong C^\bullet\left(\Delta_g^{\mathrm{lw}}\right) \oplus C^\bullet\left(\Delta_g, \Delta_g^{\mathrm{lw}}\right) \ . \tag{4.27}$$

Observation 2. *There is a canonical isomorphism* $K_{(g)}^{\bullet st} \cong C^\bullet\left(\Delta_g, \Delta_g^{lw}\right)$.

Also recall that $\Delta_g^{\mathrm{lw}}$ is contractible, so $C_\bullet\left(\Delta_g^{\mathrm{lw}}\right)$ is acyclic. A similar argument shows that $K_{(g)}^{\bullet\mathrm{unst}}$ is acyclic for $g \geq 2$.

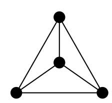
This means

$$H^*(\Delta_g; \mathbb{Q}) \cong H^* K_{(g)}^\bullet \tag{4.28}$$

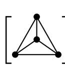
for $g \geq 2$.

Recall we have some interesting classes in $H_\bullet(\Delta_g; \mathbb{Q})$.

Example 4.1. Let $g = 3$. Then the three wheel:

(4.29) $K_4 =$  .

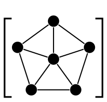
is in the kernel of the differential on $C_\bullet(\Delta_g)$ because every edge is contained in a triangle. This lives in top degree $C_5(\Delta_3)$, where the incoming differential is:

(4.30) $d\colon 0 \to C_5(\Delta_3) \ni \Big[$  $\Big]$ .

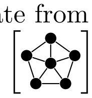
In particular, $[K_4]$ is not in the image of $d$.

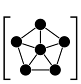
Alternatively, a cycle is nontrivial in homology if and only if there exists a cocycle which pairs nontrivially with it. In this example. the cochain $[K_4]$ is itself a cocycle, which gives another way of seeing that the cycle $[K_4]$ is nonzero in homology.

Claim 4.2. The class

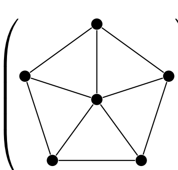
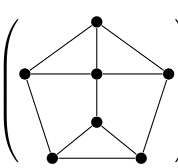
(4.31) $\Big[$  $\Big] \in H_*(\Delta_5; \mathbb{Q})$

is nonzero.

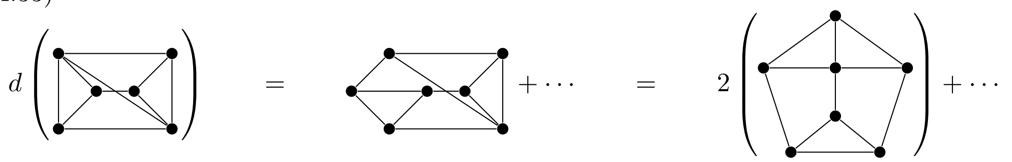
In this case, the claim is not immediate from degree considerations. We seek a cocycle in $C^*(\Delta_5; \mathbb{Q})$ that pairs nontrivially with $\Big[$  $\Big]$.

However, $\Big[$  $\Big]$ is not a cocycle itself, as was the case for $[K_4]$.

As it turns out, Lecture 33; April 27, 2020

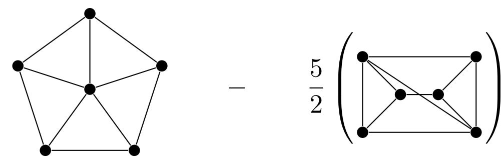
(4.32) $d\Bigg($  $\Bigg) = 5\Bigg($  $\Bigg)$ .

Next, one computes that:
(4.33)



Moreover, each of the elided terms, which come from other ways of splitting the vertex of valence 4 (which is unique mod automorphisms), is a graph with an automorphism that acts by an odd permutation on the edge set. Hence all of the remaining terms are zero in $K^\bullet$. As a consequence, we have:

(4.34)



is a cocycle. Since $[W_5]$ appears with nonzero coefficient in this cocycle, we see that the cycle $[W_5]$ is nonzero in homology.

THEOREM 4.5. *Write $W_g$ for the $g$-wheel. Then $[W_g] \neq 0$ in $\widetilde{H}_{2g-1}(\Delta_g; \mathbb{Q})$ for $g$ odd.*

Note that this implies there is a graph cocycle $\sigma \in K^{2g}_{(g)}$ such that $\sigma(W_g) \neq 0$. Finding one with rational coefficients is an open problem. However, by the universal coefficients theorem, to prove existence of such a cocycle it is enough to find such a cocycle in $K^{2g}_{(g)} \otimes \mathbb{R}$ or $\otimes \mathbb{C}$.

PROOF SKETCH.

IDEA 1 (Drinfeld, Kontsevich, Willwacher). Associate an integral on a configuration space to each graph with $g+1$ vertices and $2g$ edges.

Consider

$$C = \left\{ z_0, \ldots, z_g \in \mathbb{C}^{g+1} : \operatorname{Im}(z_i) > 0, z_i \neq z_j \right\} / \mathbb{R}_{>0}, \mathbb{R} \tag{4.35}$$

where $\mathbb{R}_{>0}$ acts is by scaling, and $\mathbb{R}$ acts by translation. This is a noncompact manifold of dimension $2g$.

Now we define

$$\sigma(G, \omega) = \int_C \bigwedge_{(v_j, v_k) \in E(G)} \frac{1}{i} \frac{d \log(z_j - z_k)}{\overline{z}_j - \overline{z}_k} \tag{4.36}$$

$$= \int_C \bigwedge_{(v_j, v_k) \in E(G)} \frac{1}{i} \frac{dz_j - dz_k}{z_j - z_k \left(\overline{z_j - z_k}\right)} \tag{4.37}$$

$$= \int_C \bigwedge_{(v_j, v_k) \in E(G)} \frac{1}{i} \frac{dz_j - dz_k}{|z_j - z_k|} . \tag{4.38}$$

Note that we wedge these 1-forms in the order given by the ordering $\omega$. Then we have three facts about this.

(1) This is obviously a graph cochain because of the properties of $\wedge$.
(2) We also have $d\sigma = 0$. This follows from an application of Stokes theorem, and the vanishing lemma of Kontsevich [**K2**, Lemma 6.4].

(3) Finally, tricks from analytic number theory allow one to compute the integral and conclude that:

$$\sigma\left(W_g\right) \in \pi^{\mathbb{Z}} \cdot \mathbb{Q}^{\times} \cdot \zeta\left(g\right) \tag{4.39}$$

where $\zeta$ is the Riemann $\zeta$ function. Since $\zeta$ is nonzero at positive integers, $\sigma\left(W_g\right)$ is nonzero, as required.

□

As a consequence, we have

$$\widetilde{H}_{2g-1}(\Delta_g; \mathbb{Q}) \neq 0 \tag{4.40}$$

for $g$ odd. So

$$H^{4g-6}\left(\mathcal{M}_g; \mathbb{Q}\right) \neq 0 \tag{4.41}$$

for $g$ odd and $g \geq 3$. This is was not expected.

Conjecture 4 (Kontsevich [**K1**], Church-Farb-Putman [**CFP2**]).

$$\bigoplus_g H^{4g-4-k}\left(\mathcal{M}_g; \mathbb{Q}\right) \tag{4.42}$$

*is finite-dimensional for $g \gg k$.*

Equivalently, the conjecture says that

$$H^{4g-4-k}\left(\mathcal{M}_g; \mathbb{Q}\right) = 0 \tag{4.43}$$

for $g \gg k$, which we have shown to be false for all odd $g$.

We still want to discuss that stronger result that

$$H^{4g-6}\left(\mathcal{M}_g; \mathbb{Q}\right) \neq 0 \tag{4.44}$$

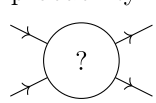
for $g \neq 2, 4, 6$; and that the dimension of $H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$ grows exponentially with $g$.

Now we provide some motivation for the construction of Lecture 34; April 29, 2020

$$\sigma \colon K^{2g}_{(g)} \to \mathbb{R} \tag{4.45}$$

from (4.36),(4.37), and (4.38). Integrals associated to finite graphs have been studied for decades under the guise of Feynman integrals. Roughly speaking, the idea from physics is that we have unobserved particles which collide interact in all possible ways. Each interaction is encoded in a (labeled) graph. The edges represent paths of particles through space time, and the vertices are collisions or interactions. If we send two particles into a chamber, then the theory tells us that a detector at the other side will register each possible observation with a certain predictable probability:

(4.46)




.

So now we compute the outcome by considering every possible way for particles to combine in question zone, and hope it coincides with the observed phenomenon.

Each graph contributes a probability to each possible observed phenomenon. Feynman expressed this contribution as an integral. In the early 2000s, people realized that these Feynman integrals were related to algebraic geometry and number theory.

A basic object which comes up in the related algebraic geometry is the graph hypersurface $X_G$, defined to be the following vanishing locus:

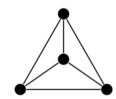
$$X_G = \mathbb{V}\left(\psi_G\right) \tag{4.47}$$

where $\psi_G$ is the polynomial given by:

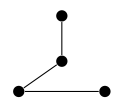
$$\psi_G = \sum_{\text{spanning trees } T} \left( \prod_{e \in T} x_e \right) \quad . \tag{4.48}$$

Digression 1 (Spanning trees). Recall that a tree is a connected graph of genus 0. A spanning tree of a graph $G$ is a tree that contains all vertices of $G$.

For instance, for

(4.49) $G \quad =$ 

a spanning tree is given by:

(4.50) $T \quad =$  .

The number of edges in any spanning tree $T$ is:

$$\#E\,(T) = \#V\,(G) - 1 \ . \tag{4.51}$$

How many spanning trees of are there of $K_n$? For $n = 3$, we clearly have 3. For $n = 4$, we have $\binom{6}{3} = 20$ subgraphs with 3 edges, but they are not all spanning trees. We need to subtract the loops, of which there are 4. So we get 16.

Exercise 3.1. Show that $K_n$ has $n^{n-2}$ spanning trees.

Fact 4. *Choosing a base vertex $v \in V\,(G)$ induces an abelian group structure on the set of spanning trees. The isomorphism class of the group is independent of the choice of base vertex, and this group is called the Jacobian of $G$, written* $\mathrm{Jac}\,(G)$*.*

In fact, one can improve on the result stated in the exercise above as follows. The Jacobian group of the complete graph $K_n$ is:

$$\mathrm{Jac}\,(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2} \ . \tag{4.52}$$

This relates to algebraic geometry as follows. Write $X$ for a semistable family of nodal curves. Let $G$ denote the dual graph of the central fiber. The general fiber $X_\eta$ is a smooth compact curve of genus $g$. This has Jacobian $\mathrm{Jac}\,(X_\eta)$ which is an abelian variety of dimension $g$. The Néron model given us a flat family of group schemes characterized by a universal property. Then the Jacobian of the graph $G$ is the component group of the special fiber of the Néron model.

Now

$$X_G = \mathbb{V}\,(\psi_G) \subseteq \mathbb{P}^{\#E(G)-1} \tag{4.53}$$

and the degree is given by the genus

$$\deg\,(X_G) = g\,(G) \ . \tag{4.54}$$

Now we observe that the Feynman integral (amplitude) of $G$ is a "period" on a space related to $X_G$.

Digression 2 (Periods). Let $X$ be a variety defined over $\mathbb{Z}$ (or $\mathbb{Q}$). Now consider $X_{\mathbb{C}}$. Now we can take integrals of closed algebraic differential forms with coefficients in $\mathbb{Q}$ over singular cycle classes in $H_*\,(X, \mathbb{Z})$.

EXAMPLE 4.2. Let $X$ be an elliptic curve, uniformized by the complex plane:

$$X = \mathbb{C}/\langle 1, \tau\rangle \ . \tag{4.55}$$

Then $\tau$ is a period, since we could take the algebraic differential form $dz$, and integrate over the loop in $X$ that is the image of the interval $[0, \tau]$.

CONJECTURE 5 (Kontsevich). *$\#X_G(\mathbb{F}_q)$ is a polynomial in $q$.*

Note that one can recover

$$\dim_{\mathbb{Q}} H^*(X_{\mathbb{C}}, \mathbb{Q}) \tag{4.56}$$

from $\#X_G(\mathbb{F}_q)$ for $q = p^n$, by the Weil conjectures (proved by Grothendieck and Deligne).

Having $\#X(\mathbb{F}_q)$ be polynomial in $q$ essentially means that the motive of $X$ is of Tate type, i.e., the space $X$ is built inductively out of affine spaces. For instance $\mathbb{P}^n = \mathbb{A}^n \sqcup \cdots \sqcup \mathbb{A}^0$ is of Tate type, as is every toric variety.

This conjecture was verified for graphs with up to 12 edges by Stanley and Stembridge [**S1**, **S2**]. But it was shown to be false in general by Belkale-Brosnan [**BB**].

There is a well-known phenomenon that on varieties of Tate type, periods are often multiple zeta values. And the integrals used to build a graph cocycle $\sigma$ witnessing the nontriviality of the cycle $[W_g]$ are periods. In this context, it is somewhat less surprising that $\sigma(W_g)$ involves $\zeta(g)$.

This $\zeta$-value (and multipole $\zeta$ values) appears as a period on $X$ when $H^*(X, \mathbb{Q})$ includes nontrivial extensions of Tate motives, i.e., multiple pieces of the weight filtration involve shifts of the Hodge structure of affine space, and the weight filtration is not split, i.e., the mixed Hodge structure does not decompose as a direct sum of Hodge structures of different weights. We should therefore expect that there is some nontrivial extension of Tate motives related to top weight cohomology of $\mathcal{M}_g$. However, already for $g = 3$, we see that this extension does not live in the cohomology of $\mathcal{M}_g$, since $H^6(\mathcal{M}_3; \mathbb{Q}) \cong \mathbb{Q}$. Lecture 35; May 1, 2020

THEOREM 4.6 (Willwacher [**W**]). *The Lie subalgebra*

$$\prod_g H^{2g}\left(K^\bullet_{(g)}\right) \subseteq \prod_g H^*\left(K^\bullet_{(g)}\right) \tag{4.57}$$

*is isomorphic to the Grothendieck Teichmüller Lie algebra* GRT.

GRT is the Lie algebra associated to the pro-unipotent completion of

$$\pi_1\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right) = \pi_1(\mathcal{M}_{0,4}) \ , \tag{4.58}$$

and is conjecturally closely related to the pro-unipotent completion of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$.

This group acts on $\mathcal{M}_{0,4}$, so it acts on

$$\pi_1^{\text{ét}}(\mathcal{M}_{0,4}) \tag{4.59}$$

so it acts on the group ring

$$\mathbb{C}\left[\pi_1^{\text{ét}}(\mathcal{M}_{0,4})\right] \tag{4.60}$$

and this is expected to be faithful.

THEOREM 4.7 (Hain [**H1**]). *Let $X$ be a variety. $\mathbb{Q}\left[\pi_1^{ét}(X)\right]$ carries a canonical/functorial mixed Hodge structure.*

Grothendieck proposed that there should be a category of motives, i.e., a geometrically defined abelian category generated by classes of algebraic varieties that satisfies a universal property with respect to Weil cohomology theories and explains the intricate airthmetic and geometric relations among various algebraically defined cohomology theories. Then this category of motives should come with a map to Galois representations and a map to Hodge structures. The generalized Hodge conjecture says that the functor from motives to Hodge structures should be fully faithful. Similarly, there should be a category of mixed motives intimately related to mixed Hodge structures, and one might look for mixed motives in the mixed Hodge structure on $\mathbb{Q}\left[\pi_1^{\text{ét}}(X)\right]$, for an algebraic variety $X$.

One of Grothendieck's programs is to study the motives which appear in

$$\mathbb{Q}\left[\pi_1^{\text{ét}}(\mathcal{M}_{g,n})\right] \ , \tag{4.61}$$

THEOREM 4.8 (Brown [**B**]). *The mixed Hodge structure on* $\pi_1(\mathcal{M}_{0,4})$ *generates the category of mixed Tate motives over* $\mathbb{Z}$.

Recall we defined graph cocycles $\sigma_g \in K^{2g}_{(g)}$ via Feynman integrals. These cocycles satisfy:

$$\langle \sigma_g, [W_g] \rangle \neq 0 \tag{4.62}$$

for $g$ odd. Along with Theorem 4.8, the work of Brown and Willwacher shows that

$$\{\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots\} \tag{4.63}$$

generate a free Lie subalgebra of GRT.

This is one half of the following conjecture.

CONJECTURE 6 (Deligne-Drinfeld-Ihara).

$$\mathbf{Lie}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots) \cong \mathrm{GRT} \ . \tag{4.64}$$

The consequence for us is that we have many new cohomology classes in

$$H^{2g-1}(\Delta_g; \mathbb{Q}) \cong H^{4g-6}(\mathcal{M}_g; \mathbb{Q})^\vee \ , \tag{4.65}$$

coming from nontrivial Lie words in the generators $\sigma_3, \sigma_5, \ldots$ of this free Lie algebra.

EXAMPLE 4.3. For instance,

$$[\sigma_3, \sigma_5] \implies H^{4g-6}(\mathcal{M}_g; \mathbb{Q}) \neq 0 \tag{4.66}$$

and

$$[\sigma_3, \sigma_9], [\sigma_5, \sigma_7] \implies \dim H^{4g-6}(\mathcal{M}_{12}, \mathbb{Q}) \geq 2 \ , \tag{4.67}$$

As a consequence of the freeness of the Lie algebra generated by $\{\sigma_3, \sigma_5, \ldots\}$, we immediately get nonvanishing of $H^{4g-6}(\mathcal{M}_g; \mathbb{Q})$ for even $g \geq 8$. Lecture 36; May 4, 2020

**Corollary 4.9.** $H^{4g-6}(\mathcal{M}_g, \mathbb{Q}) \neq 0$ *for* $g \neq 2, 4, 6$.

PROOF. For $g$ odd we have $[\sigma_g] \neq 0$. For $g$ even, $g \geq 9$,

$$[\sigma_2, \sigma_{g-3}] \neq 0 \in H^{2g-1}(\Delta_g; \mathbb{Q}) \ . \tag{4.68}$$

$\square$

Now we show that

$$\dim H^{4g-6}\left(\mathcal{M}_g, \mathbb{Q}\right) \tag{4.69}$$

grows exponentially with $g$. It is enough to show that

$$\dim_{\mathbb{Q}} \mathrm{Lie}\left\langle \sigma_3, \sigma_5, \ldots \right\rangle_g \tag{4.70}$$

grows exponentially with $g$.

For $V$ any graded vector space, we have a Poincaré series

$$f_V(t) = \sum_n \dim V_n t^n \ . \tag{4.71}$$

For

$$V = \langle \sigma_3, \sigma_5, \ldots \rangle \tag{4.72}$$

we get

$$f_V(t) = \frac{t^3}{1-t^2} \tag{4.73}$$

$$= t^3 + t^5 + t^7 + \ldots \ . \tag{4.74}$$

We are interested in

$$f_{\mathrm{Lie}(V)} = \sum A_n t^n \ , \tag{4.75}$$

so we will estimate the integers $A_n$.

Digression 3 (Universal enveloping algebra). For any Lie algebra $L$, the universal enveloping algebra is the universal unital associative algebra $U(L)$ such that there is a linear map

$$L \xrightarrow{\varphi} U(L) \tag{4.76}$$

which satisfies:

$$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) \ . \tag{4.77}$$

It is universal in the sense that for any unital associative algebra $A'$ and

$$\varphi' \colon L \to A' \ , \tag{4.78}$$

such that $\varphi$ also satisfies (4.77), there is a unique map of unital associative algebras

$$U(L) \xrightarrow{\psi} A' \tag{4.79}$$

such that

$$\varphi' = \psi \circ \varphi \ . \tag{4.80}$$

We will estimate the integers $A_n$ by playing with two different descriptions of the universal enveloping algebra of the free Lie algebra $U(\mathrm{Lie}(V))$.

Observation 3. *(1) The universal property implies that*

$$U(\mathrm{Lie}(V)) = \bigoplus_{n\geq 0} V^{\otimes n} \ . \tag{4.81}$$

*(2) Poincaré-Birkhoff-Witt theorem: If $\{x_1, x_2, \ldots\}$ is an ordered basis for $L$, then*

$$\left\{ x_1^{b_1} x_2^{b_2} \ldots : n \geq 0, b_i \geq 0 \right\} \tag{4.82}$$

*is a basis for $U(L)$.*

This means that as a graded vector space,

$$U(L) \cong \mathrm{Sym}^\bullet L \ . \tag{4.83}$$

In our situation, this means

$$U(\mathrm{Lie}(V)) = \mathrm{Sym}^\bullet(\mathrm{Lie}(V)) \ . \tag{4.84}$$

First we have

$$f_V(t) = \frac{t^3}{1-t^2} \ . \tag{4.85}$$

If we write $A_n = \dim \mathrm{Lie}(V)_n$, we can write:

$$f_{U(\mathrm{Lie}(V))} = \frac{1}{1-f_V(t)} \tag{4.86}$$

$$= \prod_{n\geq 0} \frac{1}{(1-t^n)^{A_n}} \ . \tag{4.87}$$

Now apply

$$t\frac{d}{dt}\log \tag{4.88}$$

to both sides to get

$$\frac{t^3\left(3-t^2\right)}{(1-t^2)(1-t^2-t^3)} = \sum_{d\geq 0} dA_d \frac{t^d}{1-t^d} \ . \tag{4.89}$$

Now the order of growth of the coefficients is controlled by the norm of the smallest pole. The LHS has a unique smallest pole at:

$$\alpha \sim 0.75488\ldots \ . \tag{4.90}$$

This means for

$$\beta_0 = \frac{1}{\alpha} = 1.32\ldots \tag{4.91}$$

we can write

$$p(t) = \sum_n a_n t^n \tag{4.92}$$

where $a_n \to \beta_0^n$. Now we can perform Möbius inversion on

$$a_n = \sum_{d|n} d\cdot A_d \tag{4.93}$$

to get

$$A_n = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)a_d \ . \tag{4.94}$$

Now $|\mu(N)| \leq N$, and it follows that the $d=n$ term dominates in the expression above, and $A_n$ grows faster than $\beta^n$ for any $\beta < \beta_0$.

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