L-SPACES, FOLIATIONS, AND LEFT-ORDERABILITY

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The broad question being driven at here, is what sophisticated symplectic notions can tell us about more familiar notions on 3-manifolds. Are there any relationships between Heegard Floer (or monopole Floer) homology and more classical invariants of low-dimensional topology?

1. L-space conjecture

An L-space is, roughly-speaking, is a 3-manifold with minimal Heegard Floer homology. Recall that Heegard Floer homology and monopole Floer homology are isomorphic.

The so-called L-space conjecture is as follows:

Conjecture 1. *Suppose Y is an irreducible 3-manifold which is closed and oriented. Also suppose Y is a rational homology sphere. Then TFAE:*

(1) Y is not an L-space
(2) Y has a coorientable taut foliation
(3) $\pi_1(Y)$ is left orderable

We will see what these things actually mean in turn. If we have a closed orientable 3-manifold Y, then we pick a Heegard diagram $\mathcal{H}$, which is a genus g surface Σ paired with $\vec{\alpha} = \alpha_1, \cdots, \alpha_g$ and $\alpha\beta = \beta_1, \cdots, \beta_g$. Now from this we can define the chain complex $\widehat{\mathrm{CF}}(\mathcal{H})$, and then taking homology, we get $\widehat{\mathrm{HF}}(Y)$. Recall that $\widehat{\mathrm{CF}}$ is generated by g-tuples of intersection points $\vec{x} = (x_1, \cdots, x_g)$ where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation σ.

$\widehat{\mathrm{CF}}$ carries a $\mathbb{Z}/2\mathbb{Z}$ grading. We only consider this as a relative grading so we don't have to worry about signs. We orient each α_i and β_i, and now each of our intersection points has a sign, and the grading of each intersection point $\vec{x}$ is given by

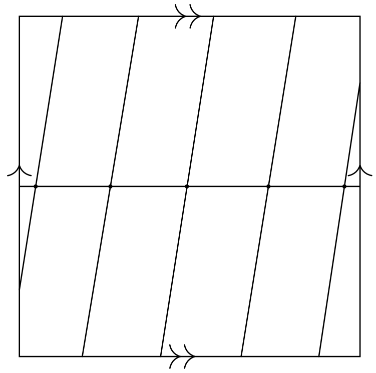
$$\mathrm{gr}(\vec{x}) = \prod_{i=1}^{g} \mathrm{sign}(x_i)\,\mathrm{sign}(\sigma) \tag{1}$$

Exercise 1. Let M be the matrix whose ij entry is the intersection number of α_i and β_j counted with sign. Show:

a. $|\det M| = \left|\chi\left(\widehat{\mathrm{CF}}\right)\right| = \left|\chi\left(\widehat{\mathrm{HF}}(Y)\right)\right|$

b. $|\det M| = \begin{cases} |H_1(Y,\mathbb{Z})| & Y \text{ is } \mathbb{Q}HS^3 \\ 0 & \text{o/w} \end{cases}$

Date: June 19, 2018.
All errors introduced are my own.



Corollary 1. $\operatorname{rank}\widehat{\mathrm{HF}}(Y) \geq \chi\left(\widehat{\mathrm{HF}}(Y)\right) = |H_1(Y,\mathbb{Z})|$ *if* Y *is a* $\mathbb{Q}$ *homology sphere.*

Definition 1. An *L-space* is a 3-manifold such that

$$\operatorname{rank}\widehat{\mathrm{HF}}(Y) = |H_1(Y,\mathbb{Z})| \tag{2}$$

Recall that we have the splitting

$$\widehat{\mathrm{HF}} = \bigoplus_{\mathfrak{s}\in\mathrm{spin}^c(Y)} \widehat{\mathrm{HF}}(Y,\mathfrak{s}) \tag{3}$$

$\mathrm{spin}^c(Y)$ is affine copy of $H^2(Y) - H_1(Y)$.

Proposition 1. *For* Y *a* $\mathbb{Q}HS^3$, *we have* $\operatorname{rank}\widehat{\mathrm{HF}}(Y,\mathfrak{s}) \neq 0$ *for any* $\mathfrak{s}$.

This leads us to another equivalent definition:

Theorem 1. Y *is an L-space iff* $\operatorname{rank}\widehat{\mathrm{HF}}(Y,\mathfrak{s}) = 1$ *for all* $\mathfrak{s} \in$ *spin*$^c(Y)$.

Example 1. Consider the lens space $Y = L(p,q)$. This is obtained by gluing two solid tori together, and then you can take this gluing to be the Heegard decomposition. Note there are no holomorphic disks between any generators. In fact, there are no disks at all. For any pair, there's an obstruction in $H_1(\Sigma)/\langle\alpha,\beta\rangle \cong H_1(Y)$ which can be identified with $\mathrm{spin}^c(Y)$. This tells us that every generator lives in a different spin^c structure, so Y is an L-space.

Example 2. Any Y^3 with spherical geometry. This includes lens spaces of course. The branched double covers of nonsplit alternating links is always an L space. Surgeries on certain knots also yield L-spaces. For example, $1/1$ surgery on the trefoil yields an L-space, and then for any $p/q \geq 1$ yields an L-space as well.

Recall on the chain complex level

$$0 \to \mathrm{CF}^- \xrightarrow{i} \mathrm{CF}^\infty \xrightarrow{\pi} \mathrm{CF}^+ \to 0 \tag{4}$$

and on the homology level we get a long exact sequence:

$$\cdots \xrightarrow{\delta_-} \mathrm{HF}^-(Y,\mathfrak{s}) \xrightarrow{i_*} \mathrm{HF}^\infty(Y,\mathfrak{s}) \xrightarrow{\pi_*} \mathrm{HF}^+(Y,\mathfrak{s}) \xrightarrow{\delta_+} \cdots \tag{5}$$

Now we define the reduced Floer homology:

$$\mathrm{HF}^+_{\mathrm{red}}(Y,s) = \operatorname{coker}\pi_* \cong \ker i_* = \mathrm{HF}^-_{\mathrm{red}}(Y,s) \tag{6}$$

This gives us a third equivalent definition of al L-space:

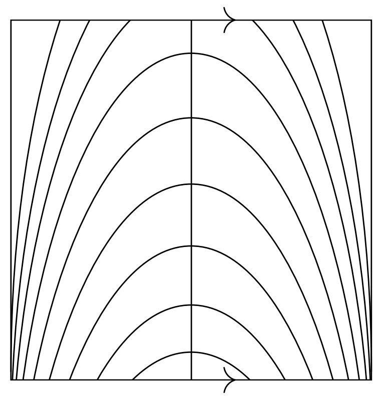


FIGURE 1. The Reeb foliation of $D^2 \times S^1$.

Theorem 2. *A $\mathbb{Q}HS^3$ is an L-space iff* $\mathrm{HF}_{red}(Y) = 0$.

2. Taut foliations

Suppose Y is a 3-manifold, then a cooriented foliation $\mathcal{F}$ of Y is a decomposition into oriented surfaces, called leaves, which locally looks like

$$\mathbb{R}^3 = \coprod_{z \in R} \mathbb{R}^2 \times \{z\} \tag{7}$$

Theorem 3 (Lickorish,Novikov-Zieschang)**.** *Every 3-manifold has a foliation.*

Example 3. Consider $Y = F \times S^1$. Then the foliation can just be

$$\mathcal{F} = \coprod_{\theta \in S^1} F \times \{\theta\} \tag{8}$$

Example 4. S^3, or any lens space, has a foliation with only one closed leaf $S^1 \times S^1$. Each solid torus is foliated as follows: If this is part of a bigger foliation, it is called a Reeb component.

We now consider foliations with certain properties.

Definition 2. A foliation is *taut* if for every leaf L, there is a closed curve that is transverse to the foliation, so it intersects every leaf transversely, intersecting L.

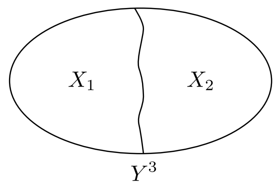
Theorem 4. *Let $\mathcal{F}$ be a foliation. Then TFAE*

(1) A foliation is taut.
(2) There is a single transverse closed curve intersecting every leaf.
(3) There exists a Riemannian metric such that all of the leaves are minimal surface.

Proposition 2. *If $\mathcal{F}$ has a Reeb component, then it is not taut.*

Theorem 5 (Novikov)**.** *If we have a Reebless foliation, then $\pi_1(Y)$ is infinite and Y is irreducible.*

Theorem 6 (Gabai)**.** *If Y is irreducible, and not a $\mathbb{Q}HS^3$, then Y has a taut foliation, which is in particular Reebless.*



3. Connection to Floer homology

Theorem 7 (Ozswáth, Zabó)**.** *If Y has coorientable taut foliation (CTF) then Y is not an L-space.*

Sketch of proof. Eliashberg-Thurston showed that a CTF can be perturbed in a sufficiently nice way to a weakly symplectically, semi-fillable contact structure. This roughly means we can cap it off as the boundary of a 4-manifold with a symplectic structure. Eliashberg also showed that a symplectic filling of this type can in fact be embedded in a closed symplectic 4-manifold, so we can sort of cap it off on the other side too, as in section 3. We know that the SW invariant for a 4-manifold is nontrivial, but we also know about these relative SW invariants of X_1 and X_2, but we know that these live in the reduced monopole Floer homology $\mathrm{HF}_{\mathrm{red}}(Y)$, which is nonzero, so this is not an L-space. □

4. Left-orderability

Definition 3. A nontrivial group G is left orderable (LO) if there is a strict total order $<$ such that $g < h$ implies $fg < fh$ for all $f \in G$.

Warning 1. We have made a point to say nontrivial, and by usual convention, the trivial group is not considered to be LO. This is because it makes a lot of other statements cleaner.

Example 5. Any finite group is not LO. More generally, if G has any torsion at all, then G is not left-orderable.

There is maybe no reason to see connections between these things and Floer theory, however we do have that

Proposition 3. *If $\pi_1(Y)$ is finite, then $\pi_1(Y)$ is not LO. Also, this being finite implies that that Y has spherical geometry which, as we saw earlier, are L-spaces.*

Proposition 4. *The branch double cover (BDC) of a nonsplit alternating link implies that $\pi(Y)$ is not LO.*

5. Recent developments

We know the conjecture holds for Seifert fibered spaces. This is due to Lisca-Stipsicz for the case of things which are Seifert fibered over orientable base-orbifolds. Bayer-Gordon-Watson checked the nonorientable case.

This is now know to be true for graph manifolds. This is based heavily on work of Boyer-Clay. This has also been shown for cyclic branch covers of knots and links. We also know exactly when surgeries on knots are L-spaces, so this is more well understood for this case. Nathan Dunfield has made extensive computational checks in the process of searching for a counterexample.