AN INTRODUCTION TO SYMPLECTIC GEOMETRY.

LECTURES BY HENRY HORTON
NOTES BY JACKSON VAN DYKE

1. Linear symplectic geometry

Definition 1. A *linear symplectic form* on a finite dimensional[1] vector space V is a bilinear form $\omega : V \times V \to \mathbb{R}$ such that

    (1) ω is skew-symmetric:

$$(1) \qquad\qquad \omega(u, v) = -\omega(v, u)$$

    (2) ω is non-degenerate

$$(2) \qquad\qquad \forall v \in V, \omega(u, v) = 0 \implies u = 0$$

A vector space (V, ω) is called a symplectic vector space.

Remark 1. The non-degenerate condition is equivalent to the map:

$$(3) \qquad\qquad \begin{array}{c} V \xrightarrow{\ \tilde{\omega}\ } V^* \\[6pt] w \longmapsto (\tilde{w} : v \mapsto \omega(u, v)) \end{array}$$

being an isomorphism.

Example 1. The space $\mathbb{R}^{2n}$ admits a standard symplectic form

$$(4) \qquad\qquad \omega_{\text{std}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

In the standard basis $\{e_1, \cdots, e_{2n}\}$. This can be written as a 2-form:

$$(5) \qquad\qquad \omega_{\text{std}} = \sum_{k=1}^{n} de_k \wedge de_{k+n}$$

Example 2. Two vector spaces (V_1, ω_1) and (V_2, ω_2) are symplectomorphic if there exists a linear isomorphism $T : V_1 \to V_2$ such that $T^* \omega_2 = \omega_1$. Here

$$(6) \qquad\qquad (T^* \omega_2)(u, v) = \omega_2(Tu, Tv)$$

Theorem 1. *Any symplectic vector space* (V, ω) *admits a symplectic basis*

$$(7) \qquad\qquad \{x_1, \cdots, x_n, y_1, \cdots, y_n\}$$

such that

$$(8) \qquad \omega(x_j, y_k) = \delta_{jk} \qquad\qquad\qquad \omega(x_i, x_j) = 0$$

All error introduced are my own.

[1]We focus on finite dimensional symplectic geometry. One can of course do infinite dimensional symplectic geometry, only this can look a bit different.

Proof. This is somehow like the Gram-Schmidt process. Pick $x_1 \in V$ nonzero. Since this is non-degenerate, there must be some $y_1 \in V$ such that $\omega(x_1, y_1) \neq 0$. Now rescale y_1 to get $\omega(x_1, y_1) = 1$. Now let $V_1 = \text{Span}(x_1, y_1)$. Then take the symplectic complement:

$$(9) \qquad V_1^\omega = \{u \in V \mid \forall v \in V_1, \omega(u, v) = 0\}$$

Then the claim is the following:

Claim 1.1. $V = V_1 \oplus V_1^\omega$

Warning 1. This is not true in general.

Let $u \in V$ and write $a = \omega(u, x_1)$ and $b = \omega(u, y_1)$. Then

$$(10) \qquad u = \underbrace{(bx_1 - ay_1)}_{\in V_1} + \underbrace{(u - bx_1 + ay_1)}_{\in V_1^\omega}$$

Now suppose $u \in V_1 \cap V_1^\omega$ which implies $u = ax_1 + by_1$ which implies $0 = \omega(u, x_1) = -b$ and $0 = \omega(u, y_1) = a$ which means $u = 0$. Together this implies the claimed decomposition.

Now we can see that ω restricts to V_1^ω as a symplectic form, so repeat this process for V_1^ω. Since V is finite dimensional, this will eventually stop. $\qquad \square$

Corollary 1. *If (V, ω) is a symplectic vector space, then there exists n such that $\dim(V) = 2n$. Furthermore, (V, ω) is symplectomorphic to $(\mathbb{R}^{2n}, \omega_{std})$ via*

$$(11) \qquad x_k \mapsto e_k \qquad\qquad y_k \mapsto e_{n+k}$$

2. Symplectic manifolds

Definition 2. Let M be a smooth manifold. Then a *symplectic form* on M is a 2-form $\omega \in \Omega^2(M)$ such that

   (1) ω is non-degenerate, that is, $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is non-degenerate for all $p \in M$.

   (2) ω is closed ($d\omega = 0$)

Remark 2.   (1) Nondegeneracy gives a bundle isomorphism

$$(12) \qquad \tilde{\omega} : TM \to T^*M$$

   The point is that even though this is always an isomorphism (for example by choosing a Riemannian metric) with a symplectic manifold, $\tilde{\omega}$ gives a differently flavored identification.

   (2) Each tangent space is a symplectic vector space, so since these have to be even dimensional, so does the manifold.

   (3) Nondegeneracy is also equivalent to $\omega^{\wedge n}$ being a volume form. Therefore a symplectic manifold always comes with an orientation.

   (4) Because ω is closed it defines a de Rham cohomology class $\omega \in H^2(M; \mathbb{R})$. Because we're orientable and nondegenerate, we get that if M is closed (compact, boundaryless) then $[\omega] \neq 0$. This means there is no exact symplectic form for closed M. However if M is noncompact, ω can be exact, and this is a nice thing to study. In fact we can explicitly write:

$$(13) \qquad \omega_{\text{std}} = -d\left(\sum_{k=1}^{n} e_{k+n} de_k\right)$$

Example 3. $\mathbb{R}^{2n}$ is of course still a symplectic manifold.

Example 4. Consider the two-sphere:

$$(14) \qquad S^2 = \left\{ p \in \mathbb{R}^3 \,\|p\| = 1 \right\}$$

Now we can write the tangent space explicitly as:

$$(15) \qquad TS^2 = \left\{ p_1 v \in \mathbb{R}^3 \mid \|p\| = 1, p \cdot v = 0 \right\}$$

and define

$$(16) \qquad \omega_p (u, v) = p \cdot (u \times v)$$

where $\times$ is just the cross-product in $\mathbb{R}^3$. Then this is in fact a symplectic form. To see this is nondegenerate, we can notice that $\omega_p (u, u \times p) = 1$. ω is closed because S^2 is two-dimensional and $d\omega \in \Omega^3 \left(S^2 \right)$.

Note that the 2-sphere is the only symplectic sphere. That is, S^{2n} admits no symplectic form for $n > 1$. This is because the second cohomology is zero for all such S^{2n}.

Example 5. If Σ_g is a compact orientable surface of genus g, and ω is any area form, it is certainly non-degenerate, and for the same reason as for the sphere, it is closed.

Example 6. If Q is a smooth n-dimensional manifold, consider local coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ for T^*Q. We have a 1-form

$$(17) \qquad \lambda = \sum_{k=1}^{n} p_k dq_k$$

and therefore a 2-form $\omega = -d\lambda$. If we think of $\alpha \in \Omega^1 (Q)$ as a map $\alpha : Q \to T^*Q$, then λ is characterized by $\alpha^* \lambda = \alpha$. This is often called the tautological 1-form. (T^*Q, ω) is symplectic.

Example 7. We can always take products of symplectic manifolds. If (M, ω_M) and (N, ω_N) are symplectic, then $(M \times N, -\omega_M \times \omega_N)$ is symplectic. Note that $\omega_M \times \omega_N$ is also a symplectic form, but having the sign is better because of the relationship with symplectomorphisms.

Definition 3. Two symplectic manifolds (M, ω_M) and (N, ω_N) are symplectomorphic if there exists a diffeomorphism $\varphi : M \to N$ such that $\varphi^* \omega_N = \omega_M$.

Theorem 2 (Darboux). *Every point $p \in M$ in a symplectic manifold (M, ω) has a neighborhood U such that $(I, \omega|_U)$ is symplectomorphic to $\left(\mathbb{R}^{2n}, \omega_{std} \right)$.*

Remark 3. In Riemannian geometry, the analogous result certainly does not hold. Riemannian geometry has local invariants like curvature (and maybe some global invariants too) but the point is that symplectic manifolds have only global invariants.

3. HAMILTONIAN VECTOR FIELDS

Recall gradient vector fields in Riemannian geometry. Let $H : M \to \mathbb{R}$ be a function. Then a metric g induces an isomorphism $\tilde{g} : TM \to T^*M$ and $\tilde{g} : \Gamma (TM) \to \Gamma (T^*M)$ and then $\operatorname{grad} (H) = \tilde{g}^{-1} (dH)$. We then have an analogous construction defined by

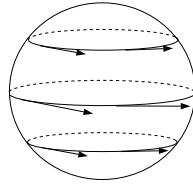$$(18) \qquad X_H = \tilde{\omega}^{-1} (dH)$$

FIGURE 1. Hamiltonian flow on S^2 where H is the height function.

Just as $\operatorname{grad} H$ is orthogonal to level surfaces of H, X_H is always tangent to level surfaces. Just like the gradient flow we have a Hamiltonian flow. The one-parameter semigroup of diffeomorphisms generated by X_H, φ_t^H is called the *Hamiltonian flow* of H.

Example 8. Let's consider $\left(\mathbb{R}^2, dx \wedge dy\right)$ and the Hamiltonian $H(x, y) = x^2 + y^2$. Then this yields $X_H = (2y, -2x)$ and then φ_t^H rotates counter-clockwise about the origin by t radians.

Example 9. On $\left(S^2, d\theta \wedge dz\right)$ define $H(\theta, z) = z$ to be the height. Then $X_h = \partial/\partial\theta$, so we have $\varphi_t^H(\theta, z) = (\theta + t, z)$ as in fig. 1

4. Almost-complex structures

Definition 4. Let M be a smooth manifold, then an almost-complex structure on M is a bundle endomorphism $J : TM \to TM$ such that $J^2 = -\operatorname{id}$. The pair (M, J) is called an almost-complex manifold.

Remark 4. A proper complex manifold has holomorphic coordinate charts.

Let $\mathcal{J}(M)$ denote the space of all almost complex structures on M. If (M, ω) is symplectic, we say that $J \in \mathcal{J}(M)$ is *tamed* by ω if $\omega(v, Jv) > 0$ for all $p \in M$ and $v \in T_pM$. This is like saying complex lines are positively oriented with respect to the symplectic form.

$J \in \mathcal{J}(M)$ is called compatible with ω if it is tamed by ω and J gives a symplectomorphism on each tangent space, that is, for all $p \in M$ and all $u, v \in T_pM$, we have $\omega_p(Ju, Jv) = \omega_p(u, v)$. We write $\mathcal{J}(M, \omega)$ for the space of ω-compatible almost complex structures.

Theorem 3. $\mathcal{J}(M, \omega)$ *is nonempty and contractible for any symplectic manifold* (M, ω).

We won't go through it here, but the proof establishes a one-to-one correspondence between Riemannian metrics M and compatible almost complex structure. That is for every Riemannian metric, we get a unique $J \in \mathcal{J}(M, \omega)$, then the space of Riemannian metrics is convex and contractible, so this is done.

Recall that a complex manifold M of complex dimension n is a smooth manifold equipped with an atlas such that the local coordinate neighborhoods are identified with $\mathbb{C}^n$. So it must be even dimensional, and the transition functions for overlaps are holomorphic functions.

Claim 3.1. If M is a complex manifold, then in charts, we get

$$(19) \qquad\qquad J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

for coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, so these local almost complex structures paste together to make a global almost complex structure.

The converse is not true, and if $J \in \mathcal{J}(M)$ induces a complex structure then it's called integrable.

A Kähler manifold is a triple (M, J, ω) where (M, ω) is symplectic, and J is a compatible almost complex structure. ω is called a Kähler form, and the Kähler metric is

$$(20) \qquad\qquad g(u, v) = \omega(u, Jv)$$

So Kähler manifolds always have a Riemannian structure.

Example 10. Our standard example $\left(\mathbb{R}^{2n}, J_0, \omega_{\mathrm{std}}\right)$ is a Kähler manifold.

Example 11. Complex projective spaces $\mathbb{CP}^n$ are Kähler. The Kähler form ω_{FS} is called the Fubini-Study form. If $\varphi : \mathbb{C}^{n+1} \setminus \{0\} \to S^{2n+1}$ which maps $z \mapsto z/|z|$, then w_{FS} is the unique 2-form on $\mathbb{CP}^n$ whose pullback under the projection

$$(21) \qquad\qquad \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$

is $\varphi^*\left(\omega_{\mathrm{std}}|_{S^{2n+1}}\right)$.

Proposition 1. *A complex submanifold of a Kähler manifold is Kähler. The Kähler form of this submanifold is just the restriction.*

Corollary 2. *Smooth complex projective varieties[2] are Kähler.*

Example 12. Any $K3$ surface is Kähler since it is a smooth complex projective variety.

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[2] These are smooth manifolds arising as a zero-set of some homogeneous polynomials.