GEOMETRY AND ALGEBRA OF PSEUDO-HOLOMORPHIC CURVES

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Pseudo-holomorphic curves were introduced by Gromov in 1985, and have become perhaps the central tool in symplectic topology/geometry.

1. Introduction to symplectic geometry

Definition 1. A symplectic form on a manifold M^{2n} is a closed 2-form ω which, when viewed as a pairing $\omega : TM \otimes TM \to \mathbb{R}$, is non-degenerate (i.e. the induced map $TM \to TM^\vee$ is an isomorphism.) and skew-symmetric.

Example 1. $\mathbb{R}^{2n}$ with standard basis $\{x_1, \cdots, x_n, y_1, \cdots, y_n\}$ is a symplectic manifold with the symplectic form

$$
\tag{1} \omega_{\mathrm{std}} \coloneqq \sum_{i=1}^{n} ds_i \wedge dy_i
$$

Theorem 1 (Darboux)**.** *For any symplectic manifold M and point $p \in M$, there exists a function $f : U \to M$, (diffeomorphic onto its image), where $U \subseteq \mathbb{R}^{2n}$ is open, $p \in f(u)$, and $f^*(\omega_M) = \omega_{std}$.*

Define the symplectomorphisms $\mathrm{Diff}_0(M, \omega)$ to consist of the diffeomorphisms preserving the symplectic form. Of course this is contained inside $\mathrm{Diff}_0(M, \omega^n)$, which are volume preserving. Explicitly, the symplectomorphisms consist of the identity component of

$$
\tag{2} \{f : M \to M \mid f^*\omega = \omega\}_0 \subseteq \{f : M \to M \mid f^*(\omega^n) = \omega^n\}
$$

Theorem 2 (Gromov)**.** *With respect to the C^0-topology, the inclusion of symplectomorphisms inside the volume preserving diffeomorphisms is either closed or dense.*

If we have that this inclusion is closed, is somehow considered to be the rigidity regime, and similarly, this inclusion being dense is somehow considered to be the flexibility regime.

A *symplectic embedding* $f : (U, \omega_n) \to (V, \omega_V)$ is a map (diffeomorphism onto its image) such that $f^*\omega_V = \omega_U$. We introduce the following notation:

$$
\tag{3} \mathrm{B}^{2n}(r) < \left\{ (x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n} \mid \sum x_i^2 + y_i^2 \leq r^2 \right\}
$$

$$
\tag{4} \mathrm{B}^2(R) \times \mathbb{R}^{2n-2} < \left\{ (x_1, \cdots, x_n, y_1, \cdots, y_n) \mid x_1^2 + y_1^2 \leq R^2 \right\}
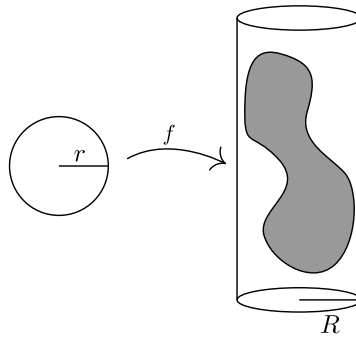$$

FIGURE 1. If we have a symplectic embedding of a sphere inside a cylinder, then the radius of the cylinder must be larger than the radius of the sphere by fig. 1.

Theorem 3. *If there exists a symplectic embedding:*

$$\text{(5)} \qquad\qquad \mathrm{B}^{2n}(r) \hookrightarrow \mathrm{B}^{2}(R) \times \mathbb{R}^{2n-2}$$

as in fig. 1, then $r \leq R$.

Eliashberg, Conley-Zehnder, and Gromov all independently showed this. Gromov's proof is the one that uses holomorphic curves, so we will focus more on this.

1.1. **Pseudo-holomorphic curves.** To prove theorem 3 we introduce the following notions:

Definition 2. An almost complex structure (acs) on M^{2n} is an endomorphism $J : TM \to TM$ such that $J^2 = -\,\mathrm{id}$.

Equivalently this is putting the structure of a $\mathbb{C}$ vector space on TM, and then J is just multiplication by i.

Warning 1. Such an (M, J) is *NOT* in general locally isomorphic to $\mathbb{C}^n$.

An acs J is said to be tamed by ω iff $\omega(v, Jv) > 0$, for all $v \in TM \setminus \{0\}$. J is said to be compatible with ω iff $(v, w) \mapsto \omega(v, Jw)$ is a Riemannian metric, that is, it is symmetric and positive definite.
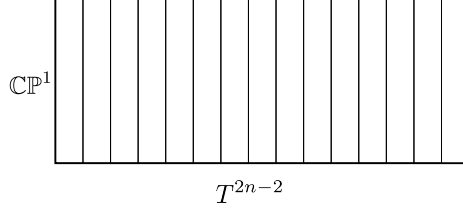
Lemma 1. *Compatible implies tame.*

Often times compatibility is more convenient, so people just work with this assumption instead.

Lemma 2. *On a symplectic vector space (V, ω), the spaces of compatible (resp. tame) J are contractible.*

A pseudo-holomorphic curve is a holomorphic map

$$\text{(6)} \qquad\qquad u : (C, j) \to (M, J)$$

where (C, j) is a Riemann surface and $du : TC \to u^*TM$ is $\mathbb{C}$-linear.

FIGURE 2. Foliation by holomorphic $\mathbb{CP}^1$.

Proof of theorem 3. WLOG, $\operatorname{im}(f)$ is contained in a compact subset of $\mathrm{B}^2(R) \times \mathbb{R}^{2n-2}$. To see this, just restrict f to $\mathrm{B}^{2n}(r-\epsilon)$, where $r - \epsilon \leq R$ for all $\epsilon > 0$, then $r \leq R$. Since $\operatorname{im}(f)$ is compact, it gives an embedding

$$(7) \qquad f : \mathrm{B}^{2n}(r) \hookrightarrow \mathrm{B}^2(R) \times \left(\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2} \right)$$

for large $N < \infty$. We now write $\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2} = T^{2n-2}$, and further embed:

$$(8) \qquad \mathrm{B}^2(R) \times \left(\mathbb{R}^{2n-2}/N\mathbb{Z}^{2n-2} \right) \hookrightarrow S^2 \times T^{2n-2}$$

where S^2 is the sphere of the same area as $\mathrm{B}^2(R)$. All together, we have a symplectic embedding:

$$(9) \qquad f : \mathrm{B}^{2n}(r) \hookrightarrow \left(S^2 \times T^{2n-2}, \omega_{S^2} + \omega_{T^{2n-2}} \right)$$

and want to show $r \leq R$.

Now equip $S^2 \times T^{2n-2}$ with a product acs $J = J_{S^2} \oplus J_{T^{2n-2}}$, so it is biholomorphic to $\mathbb{CP}^2 \times \left(\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2} \right)$ as a complex manifold. Now consider the space

$$(10) \qquad \mathcal{M}_{0,1}\left(S^2 \times T^{2n-2}, J \right) := \left\{ u : \mathbb{CP}^1 \to S^2 \times T^{2n-2} \mid du \text{ is } \mathbb{C} \text{ linear } \right\} / \sim$$

where we have defined the relation:

$$(11) \qquad u \sim u \circ \varphi \mid \varphi \iff \mathbb{CP}^1 \to \mathbb{CP}^1, \varphi(0) = 0$$

So we are effectively quotienting out by the $\mathrm{PGL}_2\,\mathbb{C}$ action on such maps, as long as they fix a point.

So if we have a map

$$(12) \qquad u : \mathbb{CP}^1 \to \mathbb{CP} \times \left(\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2} \right)$$

this splits as:

$$(13) \qquad u_1 : \mathbb{CP}^1 \to \mathbb{CP}^1 \qquad\qquad u_2 : \mathbb{CP}^1 \to \mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}$$

We always get a lift as follows:

$$(14)$$
$$\begin{array}{ccc} \mathbb{CP}^1 & \longrightarrow & \mathbb{C}^{n-1}/\mathbb{Z}^{2n-2} \\ & \overset{\text{lift}}{\dashrightarrow} & \uparrow \\ & & \mathbb{C}^{n-1} \end{array}$$

since $\pi_1\left(\mathbb{CP}^1 \right) = 0$. Therefore by Liouville's theorem, u_2 is constant.

We have a foliation by holomorphic $\mathbb{CP}^1$ as in fig. 2.

Now we have that

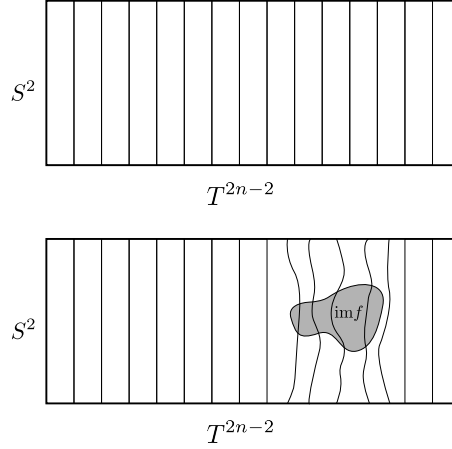$$(15) \qquad \mathcal{M}_{0,1}^d\left(S^2 \times T^{2n-2} \right) \subseteq \mathcal{M}_{0,1}\left(S^2 \times T^{2n-2} \right)$$

FIGURE 3. Holomorphic curves with respect to J_0 and J_1.

is the set of curves u with $u_* \left[\mathbb{CP}^1 \right] = d \left[S^2 \right] \times \text{pt}$. In particular, we focus on $\mathcal{M}^1_{0,1} \left(S^2 \times T^{2n-2} \right)$, which is just the total space $S^2 \times T^{2n-2}$. Moreover, this map is simply the evaluation map sending a curve with a marked point to the image of the marked point.

In fig. 3, we can see the holomorphic curves with respect to

$$(16) \qquad J_0 = J_{\mathbb{CP}^1} \oplus J_{\mathbb{C}^{n-1}/\mathbb{Z}^{2n-2}}$$

along with the holomorphic curves with respect to J_1, which is some compatible acs on $S^2 \times T^{2n-2}$ which agrees with $f_* J_{\text{std}}$ over $\text{im}\,(f)$.

It is a classical result that the space of ω-compatible almost-complex structures is nonempty and contractible. Therefore we can choose a path of complex structures $\{J_t\}_{t \in [0,1]}$ between J_0 and J_1. Now the collection of $t \in [0,1]$, such that 1-pointed genus 0, curve in $S^2 \times T^{2n-2}$ with respect to J_t in class $\left[S^2 \right] \times \text{pt}$ projects as

$$(17) \qquad \begin{array}{c} \mathcal{M}^1_{0,1} \left(S^2 \times T^2, \{J_t\} \right) \\ \downarrow^{\pi} \\ [0,1] \end{array}$$

Now we know $\pi^{-1}(0) = S^2 \times T^{2n-2}$, but we don't know much more besides this. To continue, we need to appeal to some big non-trivial results about moduli spaces of holomorphic curves:

(1) Gromov compactness: π is proper (i.e. π^{-1} of compact is compact)
(2) For generic choice of J_1 and $\{J_t\}_t$, the total space $\mathcal{M}^1_{0,1} \left(S^2 \times T^{2n-2}, \{J_t\} \right)$ is a manifold with boundary $\pi^{-1}(-) \cup \pi^{-1}(1)$. Schematically, we have something like in fig. 4. To show this, we need transversality and non-linear Fredholm theory.

Now consider the evaluation map

$$(18) \qquad \text{ev} : \mathcal{M}^1_{0,1} \left(S^2 \times T^{2n-2}, \{J_t\} \right) \to S^2 \times T^{2n-2}$$
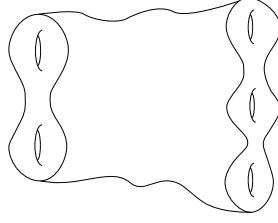
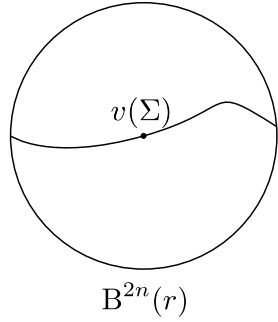FIGURE 4. Schematic of the space which is projected onto the interval $[0, 1]$.

FIGURE 5. The curve $v(\Sigma)$ on $B^{2n}(r)$.

We know $\mathrm{ev}_* \left[\pi^{-1}(0) \right] = \left[S^2 \times T^{2n-2} \right]$, but we have this cobordism, so actually $\mathrm{ev}_* \left[\pi^{-1}(1) \right] = \mathrm{ev}_* \left[\pi^{-1}(0) \right]$. In particular, this implies that

$$
\text{(19)} \qquad \mathrm{ev} : \mathcal{M}_{0,1}^1 \left(S^2 \times T^{2n-2}, J_1 \right) \to D^2 \times T^{2n-2}
$$

is surjective.

Consider a holomorphic curve $u : \mathbb{CP}^1 \to S^2 \times T^{2n-2}$ which passes through $f(0)$ where f is our embedding. We have therefore actually produced a curve $v : \Sigma \to B^{2n}(r)$ passing through the origin, where Σ is as in the following diagram:

$$
\text{(20)} \qquad
\begin{array}{ccc}
\Sigma := u^{-1} \left(f \left(B^{2n}(r) \right) \right) & \longhookrightarrow & \mathbb{CP}^1 \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\
B^{2n}(r) & \overset{f}{\longhookrightarrow} & S^2 \times T^{2n-2}
\end{array}
$$

where v is proper, since it is the pullback of the proper map u. The picture here is as in fig. 5. Now we have the following inequalities:

$$
\text{(21)} \qquad \mathrm{Area}\left(v(\Sigma)\right) = \int_\Sigma v^* \omega
$$

$$
\text{(22)} \qquad \leq \int_{\mathbb{CP}^1} u^* \omega
$$

$$
\text{(23)} \qquad = \left\langle u_* \left[\mathbb{CP}^1 \right], [\omega] \right\rangle
$$

$$
\text{(24)} \qquad = \mathrm{Area}\left(S^2 \right) = \mathrm{Area}\left(B^2(R) \right) = \pi R^2
$$

Now the last ingredient is the following:

Proposition 1 (Monotonocity). *If* $v : \Sigma \to \mathrm{B}^{2n}(r) \subseteq \mathbb{C}^n$ *is a proper, non-constant, holomorphic (with respect to J_{std}) map covering the origin, then*

$$\text{(25)} \qquad \int_\Sigma v^* \omega \geq \pi r^2$$

Sketch proof 1 of proposition 1. Observe a holomorphic curve in $\mathbb{C}^n$ is a minimal surface, (actually this is true only under the assumption that this is a minimal embedding rather than holomorphic.) □

Proof 2 of proposition 1. We have our standard form:

$$\text{(26)} \qquad \omega_{\mathrm{std}} := \sum_i dx_i \wedge dy_i = d\lambda_{\mathrm{std}}$$

which has a standard primitive:

$$\text{(27)} \qquad \lambda_{\mathrm{std}} = \sum_i \frac{1}{2} \left(x_i \, dy_i - y_i \, dx_i \right)$$

now define

$$\text{(28)} \qquad F(t) := \frac{1}{t^2} \int_{V^{-1}(\mathrm{B}^{2n}(t))} v^* \omega = \frac{1}{t^2} \left[\mathrm{Area}\left(v(\Sigma) \cap \mathrm{B}^{2n}(t) \right) \right]$$

Now it is enough to just prove F is increasing, since

$$\text{(29)} \qquad \lim_{t \to 0^+} F(t) = \pi \, (\text{ order of vanishing of } v \text{ at } 0)$$

To see this we can calculate:

$$\text{(30)} \qquad F(t) = \frac{1}{t^2} \int v^* (d\lambda) = \frac{1}{t^2} \int_{v^{-1}(\partial \mathrm{B}^{2n}(t))} v^* \lambda_{\mathrm{std}}$$

where we have used Stokes theorem. Now we use coordinates

$$\text{(31)} \qquad \begin{aligned} \mathbb{R}_s \times S_m^{2n-1} &\longrightarrow \mathbb{R}^{2n} \setminus 0 \\ (s, m) &\longmapsto e^{s/2} m \end{aligned}$$

In these coordinates, $\lambda_{\mathrm{std}} = e^s \alpha_{\mathrm{std}}$ where

$$\text{(32)} \qquad \alpha_{\mathrm{std}} := \lambda_{\mathrm{std}}|_{\mathrm{unit}\ S^{2n-1}}$$

Then we can pick up this calculation from above:

$$\text{(33)} \qquad F(t) = \frac{1}{t^2} \int v^* \left(t^2 \alpha_{\mathrm{std}} \right) = \int_{V^{-1}(\partial \mathrm{B}^{2n}(t))} v^* (\alpha_{\mathrm{std}})$$

Now we can use Stokes theorem again to get:

$$\text{(34)} \qquad F(t_1) - F(t_2) = \int_{v^{-1}(\mathrm{B}^{2n}(t_1) \setminus \mathrm{B}^{2n}(t_2))} v^* (d\alpha_{\mathrm{std}})$$

Finally we can calculate that $d\alpha_{\mathrm{std}}$ is ≥ 0 on complex lines, so the RHS is nonnegative since v is holomorphic. □

The result clearly follows from proposition 1. □

2. Non-linear Fredholm theory for moduli of pseudo-holomorphic curves

Let (X, J) be an almost complex manifold, and let (C, j) be a closed Riemann surface. E.g. $\mathbb{CP}^1$ or T^2. More generally, we will allow for C to have nodes (locally $\{xy = 0\}$) but to keep the discussion simple, we won't talk much about this. Then we have the associated moduli space:

$$(35) \qquad \mathcal{M}(C, X) := \{u : C \to C \mid du\, \mathbb{C}\text{-linear}, u \in \mathcal{C}^\infty\}$$

We can always write:

$$(36) \qquad du = \frac{1}{2}(du + J \circ du \circ j) + \frac{1}{2}(du - J \circ du \circ j) = \underbrace{(du)^{0,1}}_{\mathbb{C}\text{-antilinear}} + \underbrace{(du)^{1,0}}_{\mathbb{C}\text{-linear}}$$

Our equation $(du)^{0,1} = 0$ is elliptic. Elliptic regularity just means $u \in C^2$ implies $u \in C^\infty$.

Now consider the k, p maps:

$$(37) \qquad W^{k,p} = \{f \text{ distribution on } \mathbb{R}^n \mid \forall |\alpha| \leq k, D^\alpha f \in L^p\}$$

We can think of this as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to

$$(38) \qquad \|f\|_{k,p} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_p$$

How do we generalize this to $W^{k,p}(C, X)$? We can think of this two ways:

(1) Well we can embed $X \subseteq \mathbb{R}^N$, and then

$$(39) \qquad W^{k,p}(C, X) \subseteq W^{k,p}(C, \mathbb{R}^N)$$

is the set of maps f which almost everywhere land in C.

(2) (This second definition only makes sense when $W^{k,p} \subseteq C^0$.) We can view $W^{k,p}(C, X) \subseteq C^0(C, X)$ as the set of functions $f : C \to X$ which are locally of class $W^{k,p}$ with respect to the smooth atlases on C and X.

$$(40) \qquad \begin{array}{ccc} C & \xrightarrow{\ f\ } & X \\ \varphi \uparrow & & \psi \uparrow \\ \mathbb{R}^2 \lhook\joinrel\longrightarrow U & \xrightarrow{\psi^{-1}f\varphi} V & \longleftarrow\joinrel\rhook \mathbb{R}^{2n} \end{array}$$

Then $\psi^{-1}f\varphi \in W^{k,p}$.

We now assume $W^{k,p}(C, X)$ is a smooth Banach manifold. The charts are, for $u \in C^\infty(C, X)$,

$$(41) \qquad \begin{array}{c} W^{k,p}(C, u^*TX) \longrightarrow W^{k,p}(C, X) \\ \\ \xi \longmapsto \exp_u \xi \end{array}$$

Note that $W^{k,p}(C, u^*TX)$ is well defined since the pullback u^* is a smooth bundle. This is really true for any map $\exp : TX \to X$ whose derivative at the zero section

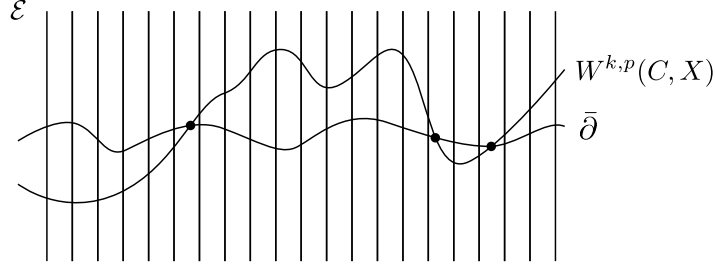FIGURE 6. The three intersection points pictured comprise the moduli space: $\mathcal{M}(C, X) = \left\{ u : C \to X \mid (du)^{0,1} = 0 \right\}$

is id. If we wanted to check if the transition maps between two of these are smooth, we would just write down the following:

$$W^{k,p}(C, u^*TX) \xrightarrow{\overset{\exp_v^{-1} \circ \exp_u : u^*TX \to v^*TX}{\longleftarrow}} W^{k,p}(C, X) \leftarrow W^{k,p}(C, v^*TX)$$

(42)
$$\xi \longmapsto \exp_u \xi$$

$$\exp_v \eta \longleftarrow\!\shortmid \eta$$

Similarly, there is a bundle $\mathcal{E} \to W^{k,p}(C, X)$ where

$$(43) \qquad \mathcal{E} = \bigcup_{u \in W^{k,p}(C,X)} W^{k-1,p}\left(C, \overline{\mathrm{Hom}}_{\mathbb{C}}(TC, u^*TX)\right)$$

Note that $\overline{\mathrm{Hom}}_{\mathbb{C}}(TC, u^*TX)$ denotes the $\mathbb{C}$ anti-linear maps $TC \to u^*TX$. The sections of $\mathcal{E} \to W^{k,p}(C, X)$ map $u \mapsto (du)^{0,1} = \overline{\partial}(u)$. The section measures how much these maps fail to be holomorphic. The picture to have in mind is as in fig. 6.

The conclusion of this discussion, is that if $D\overline{\partial}$ is surjective at some point $p \in \mathcal{M}(C, X)$, that is, some holomorphic map $u : C \to X$, then $\mathcal{M}(C, X)$ is a smooth (Banach) manifold near p.

At $u : C \to X$,

$$(44) \qquad D_u\overline{\partial} : W^{k,p}(C, u^*TX) \to W^{k-1,p}\left(C, \overline{\mathrm{Hom}}_{\mathbb{C}}(TC, u^*TX)\right)$$

measures the failure of $\exp_u \xi$ to be $\mathbb{C}$-linear for small ξ.

Definition 3. Suppose we have two Banach spaces X and Y. Then an operator $A : X \to Y$ is called *Fredholm* iff $\ker A$ is finite dimensional, $\mathrm{im}\, A$ is closed and $\mathrm{coker}\, A$ is finite dimensional.

Theorem 4. *Any elliptic operator of order d on a compact manifold M is Fredholm from $W^{k,p}$ to $W^{k-d,p}$, in particular $D_u\overline{\partial}$.*

In other words we can strike out the word Banach above, and write that if $D\overline{\partial}$ is surjective at some point $p \in \mathcal{M}(C, X)$, then $\mathcal{M}(C, X)$ is a smooth manifold near p.

We define the index of the operator to be

$$\text{(45)} \qquad \text{ind}\,(A) := \dim\,(\ker A) - \dim\,(\text{coker}\, A)$$

There ar emany ways to calculate, such as the Atiyah-Singer index theorem, but on Riemann surfaces we don't need such a poweful statemt. We simply offer the result:

$$\text{(46)} \qquad \text{ind}\,\left(D_i\bar\partial\right) = (1-g)\dim X + \langle 2c_1\,(TX)\,, u_*\,[C]\rangle$$

So this gives the dimension of the moduli space of curves.

2.1. Generic transversality.

We now introduce a new space $\mathcal{J}$ of almost complex structures on X. For example, we might take this to consist of the almost complex structures on X of class $W^{k,p}$, so this would be some Banach manifold.

We can just cross everything from before with $\mathcal{J}$ to get:

$$\text{(47)} \qquad \begin{array}{c} \mathcal{E} \times \mathcal{J} \\ \downarrow\upharpoonleft \\ W^{k,p}\,(C,X) \times \mathcal{J} \end{array}$$

where the sections just map $(u,J) \mapsto \left(\bar\partial_J u, J\right)$.

Before, the zero set was just $\mathcal{M}\,(C,X)$, and now the zero set is

$$\text{(48)} \qquad \mathcal{M}\,(C,X,\mathcal{J}) := \{J \in \mathcal{J}, u : C \to X \,|\, u \text{ is } J \text{ holomorphic }\}$$

Now we simply have $\pi : \mathcal{M}\,(C,X,\mathcal{J}) \to \mathcal{J}$, where $\pi^{-1}\,(\{J\}) = \mathcal{M}\,(C,X,J)$. Now it's much easier for the derivative to be surjective, since we have enlarged the base. Now $\mathcal{M}\,(C,X,\mathcal{J})$ is a Banach manifold at (u,J) iff the following is surjective:

$$\text{(49)} \qquad W^{k,p}\,(C,u^*TX) \oplus T_J\mathcal{J} \to W^{k-1,p}\,\left(C,\overline{\text{Hom}}_{\mathbb{C}}\,(TC,u^*TX)\right)$$

Explicit calculation can show that this is surjective if u is *somewhere injective* on every component of C.

Theorem 5 (Sard-Smale). *The regular values of π are of second Baire category. That is, a countable intersection of open dense subsets. In particular, it is dense.*

Of course the whole total space being smooth doesn't mean every fiber is smooth, but the complement of the nonregular points is at least dense.

3. Topologies on moduli spaces and compactness

We can impose many regularity conditions on the maps in $\mathcal{M}\,(C,X)$, and similarly, we can put many different topologies on this. A very strong topology is the $\mathcal{C}^\infty$ topology where all derivatives would have to converge, on the other hand, the $\mathcal{C}^0$ topology is very weak. However, regularity tells us that these topologies are the same.

Sketch proof. Suppose u_i converges to u in $\mathcal{C}^0$. The derivatives might however exhibit bad behaviour. Luckily, elliptic estimates imply that $\|u_i\|_{\mathcal{C}^k}$ is bounded. Given this boundedness, Arzelá-Ascoli implies that there is a subsequence u_{i_k} which is $\mathcal{C}^\infty$ convergent to something, but it's already $\mathcal{C}^0$ convergent to u, so it must be $\mathcal{C}^\infty$ convergent to u, which means the whole sequence has to converge to u in $\mathcal{C}^\infty$. $\square$
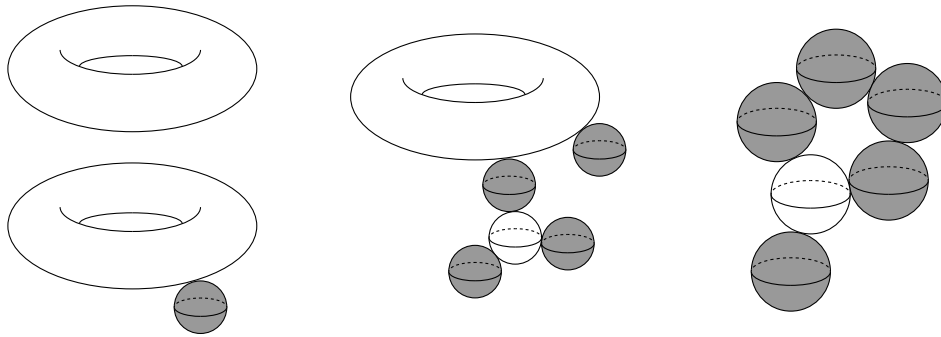
FIGURE 7. Several examples of nodal Riemann surfaces. This is a classical Riemann surface, only now we allow for nodes. Note that the spheres colored gray are the regions where we insist on map u be non-constant for reasons related to stability.

Let's talk about a moduli space without a fixed domain. Define $\overline{\mathcal{M}}_g(X)$ to consist of the pairs (C, u) such that C is a nodal Riemann surface of genus g, and $u : C \to C$ holomorphic such that $\{\varphi : C \to C \mid u \circ \varphi = u\}$ is finite. When this is the case, u is said to be *stable*..

Example 2. Nodal Riemann surfaces of genus 1 of course consist of the smooth Riemann surfaces of genus 1, but also when we attach additional copies of Riemann spheres as in fig. 7.

Consider some map from a surface with many nodes. By taking a resolution where we remove a node, and then glue a cylinder to the circular boundaries this creates, we can therefore get an induced map on a surface with fewer nodes. This motivates the following. A neighborhood of $u : C \to X$ in $\overline{\mathcal{M}}_g(X)$ consists of maps $u' : C' \to X$ where C' is a ϵ-small resolution of C, with ϵ-close acs, and u' is ϵ-close to u in $\mathcal{C}^0$.

Why is this stability condition from above required for this moduli space to make sense? To see this, we consider the example in fig. 7. Consider some curve u defined on the top left surface. Now consider the bottom left surface. If we don't insist on stability, and allow for the map to be constant on the attached Riemann sphere, then if we take the curve u, and just extend it to be a constant on the sphere, we get a curve u' defined on the bottom left surface such that u and u' are actually regarded as different curves. But now, for any such unstable map, we can always just take a resolution to get the stabilization, so our moduli space isn't Hausdorff. However, as long as we insist on stability we have the following:
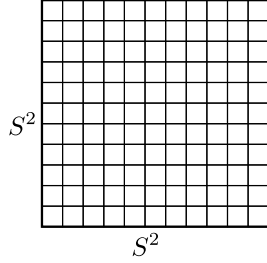
Lemma 3. $\overline{M}_g(X)$ *is Hausdorff*

This topology is called the Gromov topology. In fact, we have:

Theorem 6 (Gromov). $\overline{M}_g(X)$ *is compact.*

4. SYMPLECTOMORPHISM GROUP OF $S^2 \times S^2$, REEB ORBITS ON S^3

We now consider

$$(50) \qquad\qquad \mathrm{Symp}\left(S^2 \times S^2, \omega_0 \oplus \omega_0\right)$$

That is, the symplectomorphisms of this space with two factors having the same area. We will use holomorphic curves to determine homotopy type of this group. We will first work to understand holomorphic curves with respect to the acs $J_{\text{split}} = J_0 \oplus J_0$. So we want maps in classes $(1,0)$ and $(0,1)$ in $H_2\left(S^2 \times S^2\right) = \mathbb{Z} \oplus \mathbb{Z}$, with domain $\mathbb{CP}^1$, that is, with genus 0. The only such maps are $S^2 \times \text{pt}$ and $\text{pt} \times S^2$ in $S^2 \times S^2$. This gives us the foliation in section 4.

Let's think about transversality for a holomorphic immersion $u : C \to X$. So we have our linearized operator $D : W^{k,p}\left(C, u^*TX\right) \to W^{k-1,p}\left(C, \overline{\text{Hom}}_{\mathbb{C}}\left(TC, u^*TX\right)\right)$ which always fits into the following sequence with exact columns:

(51)
$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
W^{k,p}\left(C, TC\right) & \xrightarrow{D^T} & W^{k-1,p}\left(C, \overline{\text{Hom}}_{\mathbb{C}}\left(TC, TC\right)\right) \\
\downarrow & & \downarrow \\
W^{k,p}\left(C, u^*TX\right) & \xrightarrow{D} & W^{k-1,p}\left(C, \overline{\text{Hom}}_{\mathbb{C}}\left(TC, u^*TX\right)\right) \\
\downarrow & & \downarrow \\
W^{k,p}\left(C, u^*TX/TC\right) & \xrightarrow{D^N} & W^{k-1,p}\left(C, \overline{\text{Hom}}_{\mathbb{C}}\left(TC, u^*TX/TC\right)\right) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

where the rows are short exact sequences, and we are just taking the quotient. Then we get the following exact sequence:

$$0 \to \ker D^T \to \ker D \to \ker D^N \to \text{coker}\, D^T \to \text{coker}\, D \to \text{coker}\, D^N \to 0$$

Note that $\ker D$ is the tangent space to $\mathcal{M}\left(C, X\right)$, and $\ker D^N$ is the tangent space of holomorphic maps to X in the same topological type of C.

We should think of D^T as controlling deformations of holomorphic structure on C

(52)
$$\ker D^T = H^0\left(C, TC\right)$$

since this consists of the infinitesimal automorphisms of C and

(53)
$$\text{coker}\, D^T = H^1\left(C, TC\right)$$

are the infinitesimal deformations of acs on C.

Let $u : C \to X$ be one of these curves $\text{pt} \times S^2$ or $D^2 \times \text{pt}$. Then

(54)
$$D^N : W^{k,p}\left(C, u^*TX/TC\right) \to W^{k-1,p}\left(C, \overline{\text{Hom}}_{\mathbb{C}}\left(TC, u^*TX/TC\right)\right)$$

is the Cauchy-Riemann operator on this bundle. But $u^*TX/TC = \mathcal{O}_{\mathbb{CP}^1}$ is just the trivial bundle for $C = \mathbb{CP}^1$, so

$$(55) \qquad \ker D^N = H^0\left(\mathbb{CP}^1, \mathcal{O}\right) = \mathbb{C} \qquad \operatorname{coker} D^N = H^1\left(\mathbb{CP}^1, \mathcal{O}\right) = 0$$

The coker vanishing means the moduli spaces are transverse.

We now consider an arbitrary compatible J on $S^2 \times S^2$. Suppose we have $u : \mathbb{CP}^2 \to \left(S^2 \times S^2, J\right)$, in the class $(1,0)$ or $(0,1)$. Since we're on $S^2 \times S^2$, the map $\pi_2\left(S^2, S^2\right) \xrightarrow{\sim} H_2\left(S^2 \times S^2\right)$ is an isomorphism, Therefore

$$(56) \qquad\qquad u^*\left(T\left(S^2 \times S^2\right)\right) = TS^2 \oplus \mathbb{C}$$

as complex vector bundles.

If u is a self transverse immersion, $u : \mathbb{CP}^1 \looparrowright S^2 \times S^2$, then the number of double points satisfies

$$(57) \qquad\qquad c_1\left(N_u\right) + 2\#\left(\text{double points}\right) = \left(u_*\left[\mathbb{CP}^1\right]\right) \cap \left(u_*\left[\mathbb{CP}^1\right]\right)$$

Since u is an immersion, $u^*\left(T\left(S^2 \times S^2\right)\right) = TS^2 \oplus N_u$. Now we have:

$$(58) \qquad c_1\left(TS^2 \oplus N_u\right) = c_1\left(TS^2\right) + c_1\left(N_u\right) = c_1\left(u^*\left(T\left(S^2 \times S^2\right)\right)\right)$$

$$(59) \qquad\qquad\qquad\qquad\qquad = c_1\left(TS^2\right) + c_1\left(\mathbb{C}\right)$$

so $c_1\left(N_u\right) = c_1\left(\mathbb{C}\right)$, and since complex line bundles over S^2 are classified by the first Chern class, we get $N_u \cong \mathbb{C}$. This all means that u is an embedding.

For general J-holomorphic $u : \mathbb{CP}^1 \to S^2 \times S^2$ in class $(1,0)$ or $(0,1)$, u is an embedding (and immersed). Hence, $\ker D^N = \mathbb{C}$ and $\operatorname{coker} D^N = 0$, as previously. In particular, u is transverse.

Recall $\mathcal{M}_{0,0}^{1,0}$ consists of the genus 0 J-holomorphic curves in $S^2 \times S^2$ in class $(1,0) \in H_2\left(S^2 \times S^2\right)$. Since u is transverse we have that $\mathcal{M}_{0,0}^{1,0}\left(S^2 \times S^2, J\right)$ and $\mathcal{M}_{0,0}^{0,1}\left(S^2 \times S^2, J\right)$ are cut-out transversely for every compatible acs J. This is a miracle that happens only in dimension 4.

For any J, there exists a path J_t from $J = J_0$ to $J_1 = J_{\text{split}}$. Then we look at the parameterized moduli space

$$(60) \qquad\qquad \begin{array}{c} \mathcal{M}_{0,0}^{1,0}\left(S^2 \times S^2, \{J_t\}_{t \in [0,1]}\right) \\ \downarrow \pi \\ [0,1] \end{array}$$

where π is proper by Gromov compactness.

Define the energy to be the value given by

$$(61) \qquad\qquad\qquad \int_\omega : H_2\left(S^2 \times S^2\right) \to \mathbb{R}$$

Since the two factors have the same area, $(1,0)$ and $(0,1)$ have the minimal positive value of $\int \omega$. Note that if $u : C \to C$ is J-holomorphic and non-constant, then $\in_C u^*\omega > 0$, which is precisely the definition of J being tamed by ω.

In addition to being proper, π is also a submersion. This is just the statement that all fibers cut out transversely. But we know the fiber $\pi^{-1}\left(J_{\text{split}}\right) = S^2$, which means $\pi^{-1}\left(J\right) = S^2$. Thus for any compatible J, the J-holomorphic curves in classes $(1,0)$ and $(0,1)$ from a pair of ⋔ foliations. Even with respect to a strange acs, such as in fig. 8, we get a foliation by these spheres. The spheres are disjoint because of positivity of the intersection.
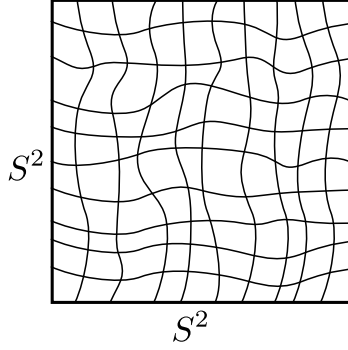
FIGURE 8. For arbitrary acs J, we get this same well-behaved foliation by spheres, only now it might exhibit nonstandard behaviour.

Finally, we consider the actual group of symplectomorphisms of $S^2 \times S^2$. Consider the classifying space:

$$(62) \qquad B \operatorname{Symp} \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right) \left\{ (V, \omega) \mid \exists (v, \omega) \cong \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right) \right\}$$

Think of this as a set of submanifolds of $\mathbb{R}^\infty$. So given a symplectic manifold, we can always consider the collection of compatible almost complex structures, which is contractible. So the forgetful map:

$$(63)$$
$$\left\{ (V, \omega, J) \mid \exists (v, \omega) \cong \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right), J \text{ compatible } \right\} \to B \operatorname{Symp} \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right)$$

has contractible fibers, so this is a homotopy equivalence.

Remark 1. These classes $(1,0)$ and $(0,1)$ are precisely the classes in H_2 of minimal positive energy and zero self intersection.

Given (V, ω, J) in the above set, we get two transverse foliations by J-holomorphic copies of S^2 in the classes $(1,0)$ and $(0,1)$.

Now consider the set

$$(64) \qquad \left\{ (V, \omega, J, \omega') \mid (v, \omega) \cong \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right), J \text{ compatible}, \omega' \text{ split wrt} J \right\}$$

This condition on ω' is equivalent to ω' being a sum of an area form of total area 1, on leaf space of foliation i. Now we can again map this to the set of triples (V, ω, J), and the fibers will again be contractible, so this is again a homotopy equivalence. Now consider the following facts:

Fact 1.
Note J is compatible with ω', because at each point we can decompose the tangent space with respect to he foliations, and both J and ω' are split with respect to this decomposition.
ω and ω' are cohomologous.

Together, using Moser's theorem, these imply that (V, ω) and (V, ω') are in fact symplectomorphic.

Therefore this set is the same as:

$$(65)$$
$$\left\{ (V, \omega, J, \omega') \mid (V', \omega') \cong \left(S^2 \times S^2, \omega_0 \oplus \omega_0 \right) J \text{ compatible with } \omega, \omega' \text{ split wrt } J \right\}$$
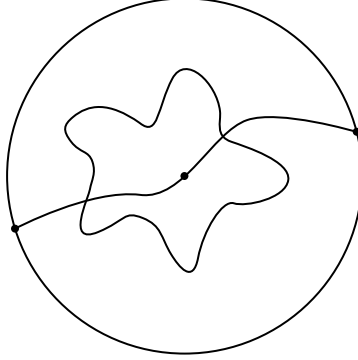
FIGURE 9. Some star-shaped region in $\mathbb{C}^2$ which corresponds to a contact form on S^3.

now we can map this to the set

(66)
$$\left\{ (V, \omega', J) \mid (V, \omega') \cong \left(S^2 \times s^2, \omega_0 \oplus \omega_0 \right), J \text{compatible with } \omega', \omega' \text{ split wrt } J \right\}$$

which is an equivalence. Finally, we can deformation retract this to

(67) $\{(V, \omega', J) \mid \text{ same conditions, except require leaves have same acs.}\}$

and this is also an equivalence. But this is the same as

(68) $$\left\{ (V, \omega', J) \mid (V, \omega', J) \cong \left(S^2 \times S^2, \omega_0 \oplus \omega_0, j_0 \oplus j_0 \right) \right\}$$

so we have:

(69) $B \operatorname{Aut} \left(S^2 \times S^2, \omega_0 \oplus \omega_0, j_0 \oplus j_0 \right) = (\operatorname{SO}(3) \times \operatorname{SO}(3)) \rtimes \mathbb{Z}/2$

By similar methods, we can show:

Theorem 7. Symp $\left(\mathbb{CP}^2, \omega_{std} \right)$ *is homotopy equivalent to* $\operatorname{PU}(3) \cong \operatorname{PGL}_3 \mathbb{C}$.

Proposition 2. *For any compatible J on $\mathbb{CP}^2$ and $p, q \in \mathbb{CP}^2$, there exists a unique pseudo-holomorphic $\mathbb{CP}^1 \to \mathbb{CP}^2$ of degree 1 through p, q*

If we take some $S^3 \subseteq \mathbb{C}^2$, this carries what is called a contact structure $\xi \subseteq TS^3$, where $\xi := TS^3 \cap J\left(TS^3 \right)$. For a 1-form α on S^3 with $\ker \alpha = \xi$, we can define a vector field R_α on S^3 by $\alpha\left(R_\alpha \right) = 1$ and $d_\alpha\left(R_\alpha, \cdot \right) = 0$.

Corollary 1. *Any Reeb vector field R_α for the standard contact structure ξ on S^3 has a closed orbit.*

Proof. Consider $\mathbb{CP}^2$ with $\mathbb{C}^2$ inside it. Contact forms on S^3 correspond to star-shaped regions in $\mathbb{C}^2$ as in fig. 9. Then we perform a neck-stretch degeneration of J along this hypersurface corresponding to α and holomorphic curves in the neck region converge to cylinders over Reeb orbits. □

5. EXAMPLES OF BUBBLING

Consider some maps u_n from a Riemann surface to X. Recall that, to this, we can associate the energy

(70) $$E\left(u_n \right) = \int u_n^* \omega = \int |du_n|^2$$
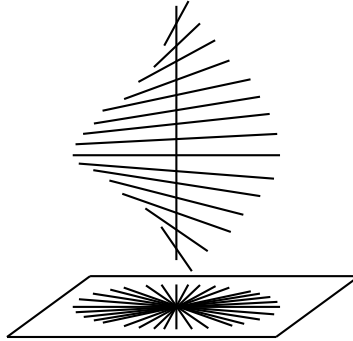
FIGURE 10. The blowup of $\mathbb{CP}^2$.

The point is, the RHS is useful for analysis, but the middle is more easily computable. In particular, if these maps are in the same homology class, then these have the same energy. Bubbling occurs when energy concentrates around some point. This is intuitively clear, because energy is really just area, so a large amount of area is trapped in this sort of localized space. Gromov compactness tells us that when u_n does not admit a convergent subsequence, then we have to have this sort of bubbling phenomenon. We now consider some concrete example to illustrate how this isn't just some sort of analytic annoyance, but rather a geometric phenomenon.

Recall the blowup construction. In general, we can take any complex manifold X of complex dimension 2 and $x \in X$, and define the blowup:

$$(71) \qquad \mathrm{Bl}_x X = \tilde{X} = X \setminus \{x\} \cup \mathbb{P}\left(T_x X\right)$$

This is easy to write as a set, but we have to work a bit harder for the topology. Consider $\mathbb{C}^2$ with the origin regarded as our point x. Then at the origin we put a copy of $\mathbb{P}^1$ in here, so all of the lines which cross at the origin now cross this line at different points. The picture to image here is as in fig. 10. Explicitly:

$$(72) \qquad \left\{((x,y),[X:Y]) \in \mathbb{C}^2 \times \mathbb{P}^1 \,|\, xY - yX = 0\right\} = \{(p,l) \,|\, p \in l\}$$

then for a generic manifold we can just glue in this construction locally.

Now let X be the blowup of $\mathbb{CP}^2$ at $(0,0,1)$. We know this projects to $\mathbb{CP}^2$. Consider some line which comes close to the origin, but does not meet it. This is shown in red in fig. 11. If the red line were to actually touch the boundary, it would be exactly coinciding with one line through the origin, and only intersect all of the other lines at a single point. Therefore we see that this intersection, as viewed in the blowup, gets an extremely large "length" as viewed in the picture, or in proper dimensions, a large area, and therefore energy. Explicitly, the domain gets closer and closer to being two spheres joined at a point.

As another example, consider the curve given by $y^2 = x^2\left(x+1\right)$ as in fig. 12. We intersect the curve with the red line, and as we approach the node, when we lift to the blowup with the limiting behaviour on the right. So again we see this bubbling phenomenon, since if the line were to actually pass through the node, it would not have this energy accumulation.

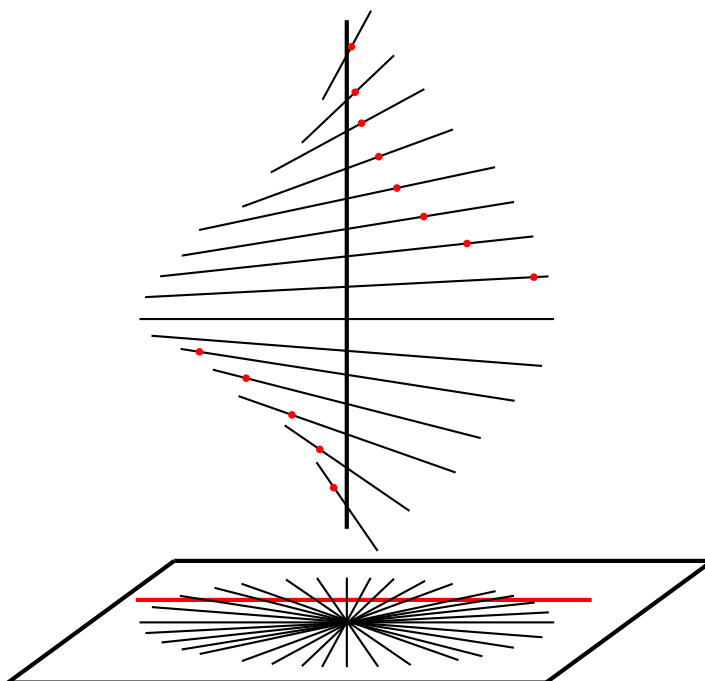The domain of this example approaches becoming three spheres joined as in fig. 13.

FIGURE 11. The red line comes close to the origin, but does not touch it. The intersections between the red line, and the lines going through the origin in $\mathbb{CP}^2$, are plotted in red in the blowup.
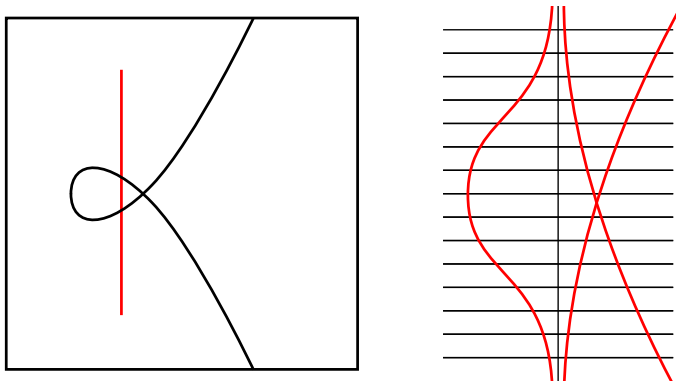

FIGURE 12. The elliptic curve $y^2 = x^2 (x + 1)$ is plotted, and then we intersect the curve with the red line near, but not intersecting, the node. We then lift this to the blowup on the right.
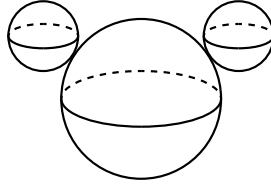
FIGURE 13. The domain of the elliptic curve example approaches
having something like this as its domain.

6. FUKAYA CATEGORIES

So far we have regarded holomorphic-curves as geometric objects. Now we're
going to forget some of these things and concern ourselves more over enumerative
questions.

6.1. **Construction of classical Fukaya category.** Fix a symplectic manifold
(X, ω), and assume it is exact. So $\omega = d\lambda$ for some 1-form λ. This forces X to
be noncompact.[1] Now consider exact Lagrangians L. This means in addition to
$\omega|_L = 0$, we also have $\lambda|_L = df$ for some f.

Remark 2. One reason for ω to be exact, is to get control over the moduli spaces.
Suppose the manifold contains a bunch of holomorphic spheres. If this was the case,
then we could just glue in some extra sphere to any disk, and we would get a disk
of higher energy. Then we could do this for an n-fold cover of the sphere, and we
would get disks of arbitrary energy. But when it's exact, we have no holomorphic
spheres.

Exactness also helps achieve transversality. It rules out bubbling holomorphic
spheres which are hard to make transverse.

Definition 4. The Fukaya category $\mathcal{F}(X, \lambda)$ has objects exact Lagrangians, and
morphism spaces are

$$(73) \qquad \mathrm{CF}^*(L, K) := \bigoplus_{p \in L \cap K} \mathbb{Z}/2\mathbb{Z}$$

for any two exact Lagrangians L_0 and L_1.

Recall that the Floer complex CF^* is equipped with a differential

$$(74) \qquad d : \mathrm{CF}^*(L, K) \to \mathrm{CF}^*(L, K)$$

where p maps to the count

$$(75) \qquad p \mapsto \sum_q \# \mathcal{M}(D) q$$

where the disks D are as in the first disk in fig. 14. Note that $\mathcal{M}$ has dimension
$\mathrm{ind}(q) - \mathrm{ind}(p) - 1$, and $\#\mathcal{M}$ is the count of elements in case $\dim M = 0$. Also
note that $\mathcal{M}$ is the moduli space of disks up to reparameterization.

Now we want to define a sort of composition

$$(76) \qquad \mu^2 : \mathrm{CF}^*(L_0, L_1) \otimes \mathrm{CF}^*(L_1, L_2) \to \mathrm{CF}^*(L_0, L_2)$$

---

[1] This is because ω^n is a volume form, and $\omega = d\lambda$, so $\omega^n = 0 \in H^{2n}(X)$.
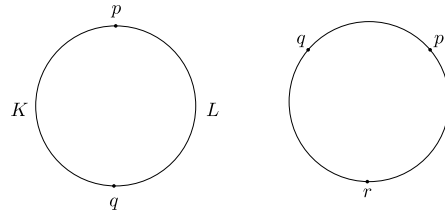
FIGURE 14. (Left) Disk with two punctures which are summed over in the differential on the Floer complex. (Right) Disk with three punctures which are summed over in the "composition" map on the Floer complex.
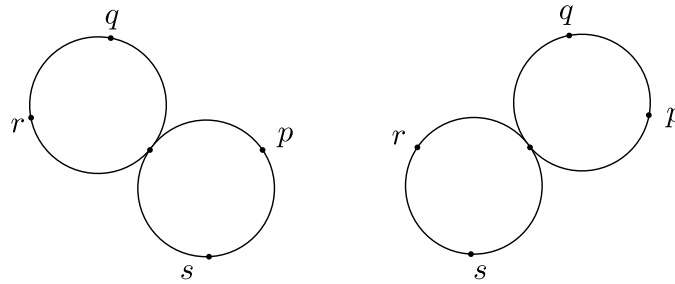

FIGURE 15. The two disks which we might count in our attempt at showing μ^2 is associative.

where we explicitly map:

$$(77) \qquad p \otimes q \mapsto \sum_r \#\mathcal{M}(D)\, r$$

where D denotes disk as in the right of fig. 14. Similarly, d is sometimes denoted μ^1.

But this isn't quite associative. If we compose three morphisms:

$$(78) \qquad \mathrm{CF}^*(L_0, L_1) \otimes \mathrm{CF}^*(L_1, L_2) \otimes \mathrm{CF}^*(L_2, L_3) \to \mathrm{CF}^*(L_0, L_3)$$

we can send

$$(79) \qquad p \otimes q \otimes r \mapsto \sum_s \#\mathcal{M}(D)\, s$$

but there is now some ambiguity in what disk we are counting. We might choose either one of the disks in fig. 15.

In order to relate these, we define another relation:

$$(80) \qquad \mu^3 : \mathrm{CF}^*(L_0, L_1) \otimes \mathrm{CF}^*(L_1, L_2) \otimes \mathrm{CF}^*(L_2, L_3) \to \mathrm{CF}^*(L_0, L_3)$$

which sends:

$$(81) \qquad p \otimes q \otimes r \mapsto \sum_s \#\mathcal{M}(D)\, s$$

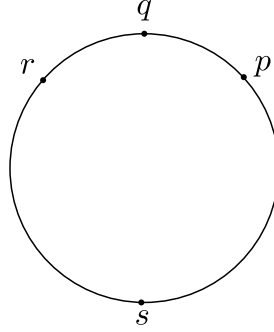where the disks D are as in fig. 16.

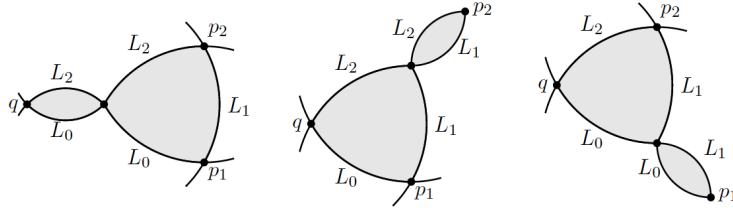FIGURE 16. μ_3 sends $p \otimes q \otimes r$ to the count of disks of this sort.

FIGURE 17. The three potential degenerations of the disk with three punctures, which give us the Leibniz rule for μ^1 and μ^2.

Fact 2. μ^3 *is a chain homotopy between* $\mu^2\left(\mu^2\left(\cdot,\cdot\right),\cdot\right)$ *and* $\mu^2\left(\cdot,\mu^2\left(\cdot,\cdot\right)\right)$

6.2. **Degeneration.** These disks with marked points (or punctures) we have been discussing have so-called degenerations. These motivate some structure relations on CF. When we have two points on the boundary, we have that the disk can degenerate to two disks intersecting at a point, with one marked point on each. This gives us $d^2 = 0$.

Now with three points, we get degenerations as in fig. 17.

This gives us

$$(82) \qquad \mu^2\left(\mu^1\left(\cdot\right),\cdot\right) + \mu^2\left(\cdot,\mu^1\left(\cdot\right)\right) + \mu^1\left(\mu^2\left(\cdot,\cdot\right)\right) = 0$$

which is exactly the statement hat μ^2 is a chain map.

Exercise 1. Repeat this for the μ^3 case.

This isn't a category in the usual sense as we've seen it so far. In fact, it's an $\mathcal{A}^\infty$ category, so we also have higher homotopies, which tells us there's only one way to compose morphisms. Generalizing the above, the Fukaya category has the higher operations

$$(83) \qquad \mu^k : \mathrm{CF}\left(L_0, L_1\right) \otimes \cdots \otimes \mathrm{CF}\left(L_{k-1}, L_k\right) \to \mathrm{CF}\left(L_0, L_k\right)\left[2 - k\right]$$

of degree $k \geq 1$ satisfying the $\mathcal{A}_\infty$ relations, counting disks as in fig. 18.

The conditions that these μ^k must satisfy can be written explicitly as:

$$(84) \qquad \sum_{l=1}^{j} \sum_{j=0}^{k-l} (-1)^* \mu^{j+1-l}\left(p_k, \cdots, p_{j+l+1}, \mu^l\left(p_{j+l}, \cdots, p_{j+1}\right), p_j, \cdots, p_1\right) = 0$$
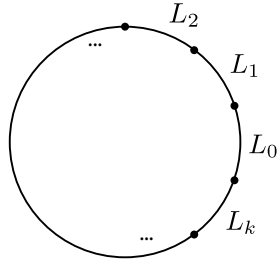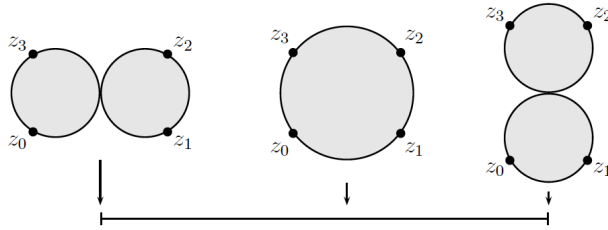
FIGURE 18. The type of disk counted by μ^k.

FIGURE 19. The associahedron for $k = 3$.

where $* = j + \deg(p_1) + \cdots + \deg(p_j)$, though this if of course not of interest to us since we are working over $\mathbb{Z}/2\mathbb{Z}$. These can be understood as coming from summing over contributions from trees which can also be viewed as degenerations of disks. We consider the *associahedron* $\overline{R}_{k,1}$ to a given $k \in \mathbb{Z}$. This is the compactification of the space of potential conformal structures with $k + 1$ marked points. The top-dimensional facets correspond to nodal degenerations of a standard disk D to a union of disks, where each component has at least 2 of the k-marked points, and the higher codimension faces correspond to nodal degenerations with more components. See fig. 19 for the example $k = 3$.

Remark 3. This is all sort of forced upon us. If we consider how to count disks of some sort, then the conditions we will look for are exactly the conditions for an $\mathcal{A}_\infty$ category.

6.3. **Wrapped Fukaya category.** If our exact symplectic manifold (X, λ) is cylindrical at ∞, we can define the so-called wrapped Fukaya category $\mathcal{W}(X)$. Cylindrical just means that near infinity, X is isomorphic to $(\mathbb{R}_{s \geq 0} \times Y, e^s \alpha)$ for some contact form α on Y. Take the same objects as $\mathcal{F}(X)$, only now the morphisms are define by flowing the first argument by the Reeb flow of α, which happens to also be the Hamiltonian vector field of e^s. In other words,

$$(85) \qquad \mathrm{CW}^*(L, K) = \mathrm{Hom}_{\mathcal{W}(X)}(L, K) = \mathrm{CF}^*(\Phi L, K)$$

where Φ is the flow of this vector field.

Example 3. Consider the cylinder $X = \mathbb{R} \times S^1 = T^* S^1$ as in the left of fig. 20.

$$(86) \qquad \mathrm{Hom}_{\mathcal{W}(T^* S^1)}(K, K) = \mathrm{HF}^*(L, K) = \mathbb{F}_2\left[t, t^{-1}\right]$$
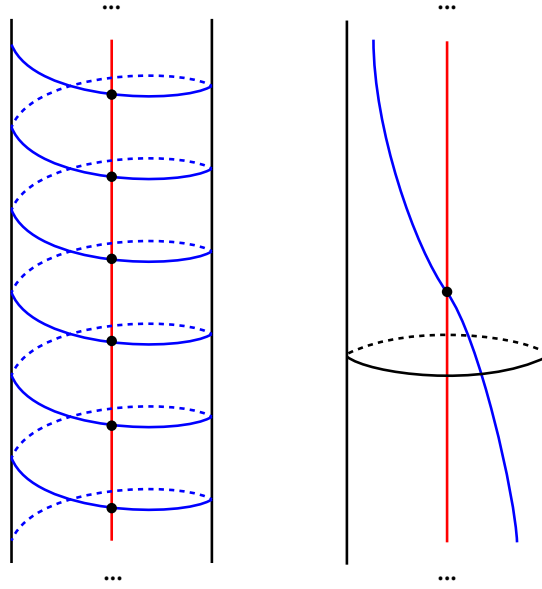
FIGURE 20. (Left) The cylinder with with two exact Lagrangians K and L, one obtained by twisting the other at infinity. (Right) The cylinder with an exact Lagrangian twisted in a finite sense.
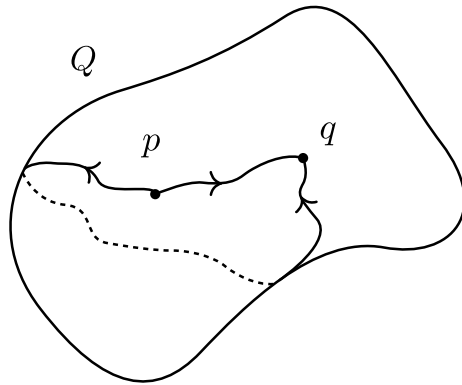

FIGURE 21. Two geodesics both from p to q on Q.

Here we are effectively twisting for infinite time. If we were twisting a finite amount it would look like the right in fig. 20.

Consider $\mathcal{M}(T^*Q)$ where T^*Q is exact, and cylindrical at ∞. If we choose a Riemannian metric on Q, we get a contact form on S^*Q, whose Reeb flow is the geodesic flow. This means that if we consider the wrapped Floer cochains, $\mathrm{CW}^*\left(T_p^*Q, T_q^*Q\right)$, as vector spaces this is the same as $\mathbb{F}_2\left[\text{geodesics } p \rightsquigarrow q\right]$ as in fig. 21.
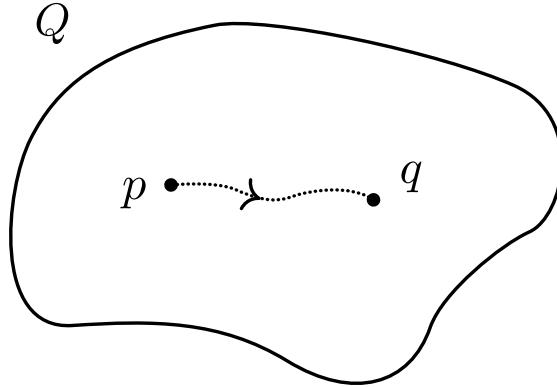
FIGURE 22. We can discretize this geodesic, so all of the adjacent points are "close enough" to find a canonical path between them.

Theorem 8 (Abbondandolo-Schwarz).

$$\text{(87)} \qquad \text{HW}^* \left(T_p^* Q, T_q^* Q \right) = H_{-*} \left(\Omega_{p,q} Q \right)$$

$$\text{(88)} \qquad\qquad = H_{-*} \left(\{ \gamma : [0,1] \to Q \, | \, \gamma(0) = p, \gamma(1) = q \} \right)$$

Remark 4. Another description of the map $\text{Hom}_{-*} \left(\Omega_{p,q} Q \right) \to \text{HW}^* \left(T_p^* Q, T_q^* Q \right)$ is as follows. Suppose we have Q and two points p and q. If these points are sufficiently close, there is a canonical shortest path, which corresponds to a cycle since the differential can only decrease the length. Now decide to only care about these short ones.

Now take any path from p to q, and discretize as in fig. 22. Each discrete point and its adjacent neighbor has a relevant path, but the composition at large may not be one we care about, so this isn't really a category.

Now consider the category **C** whose objects are the fibers $T_p^* Q$ and whose morphisms are "freely generated" by the unique short geodesics between pairs of close by points.

There is a tautological functor from $\mathcal{C}$ to $\mathcal{W}(T^*Q)$.

Claim 8.1. $\text{Hom}_{\mathcal{C}} \left(T_p^* Q, T_q^* Q \right) = C_{-*} \left(\Omega_{p,q} Q \right)$

Theorem 9 (Abouzaid). *The objects $T_q^* Q \in \mathcal{W}(T^*Q)$ generate $\mathcal{W}(T^*Q)$*

6.4. Arnold's conjecture.

Conjecture 1 (Arnold). *For any exact compact Lagrangian $L \subseteq T^*Q$, L is isotopic to the zero section.*

The picture here is as in and then the projection of L down to Q is a simple homotopy equivalence This is trivial for $Q = S^1$, and this is known for $Q = S^2$ and $Q = \mathbb{RP}^2$.

Theorem 10 (Fukaya-Seidel-Smith-Nadler-Zaslow-Abouzaid). *$L \subseteq T^*Q$ compact exact, then $L \to Q$ is a simple homotopy equivalence.*
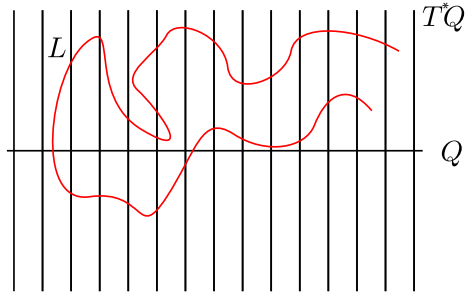
We will prove a weaker version of this. First we need:

FIGURE 23. We have some Lagrangian L in T^*Q, and we want to see what happens when we project this down to Q.
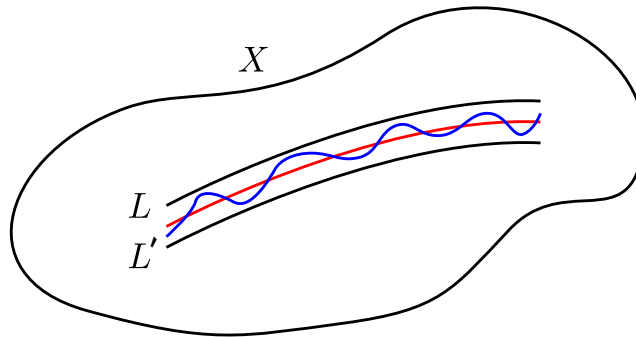

FIGURE 24. We take an exact Lagrangian L in X, and a tubular neighborhood surrounding it. Then we perturb L to get another exact Lagrangians L' in X.

Theorem 11 (Weinstein). *A neighborhood of L is symplectomorphic to a neighborhood of the zero section in T^*L.*

Proposition 3 (Floer). *If $L \subseteq X$ is exact, then $\mathrm{HF}^*(L, L) = H^*(L)$.*

Proof. We have the Weinstein neighborhood theorem 11 which gives us a canonical tubular neighborhood of such an L as in fig. 24. Now to compute this, we need to perturb L slightly. Consider the Hamiltonian perturbation L' of L given inside T^*L by the graph of df for some Morse function $f : L \to \mathbb{R}$, as in fig. 24.

Now the intersections of L' and L are the zeros of df, so they are also critical points of f, so

$$(89) \qquad \mathrm{CF}^*(L', L) = \mathrm{CM}^*(L, f)$$

as abelian groups, where CM^* denotes the Morse complex. Moreover, for appropriate choice of acs J, the differentials are actually the same.

Exactness is necessary, because these disks of small energy are identified with the Morse trajectories. We might also have big disks which goes out through the entire manifold, and the exactness rules them out. This is because for a disk u,

$$(90) \qquad E(u) = \int_{D^2} u^*\omega = \int_{D^2} u^*(d\lambda) = \int_{\partial D^2} u^*\lambda = 0$$
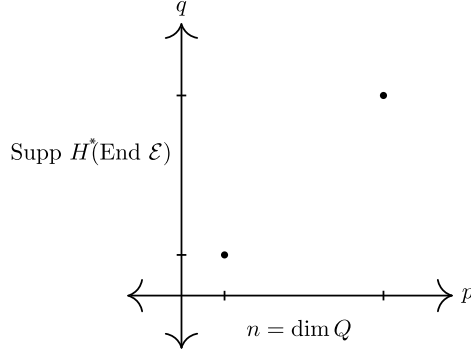
FIGURE 25. We have a range of n on the p axis, and $\mathrm{Supp}\, H^*\left(\mathrm{End}\,\mathcal{E}\right)$ on the q axis.

and the boundary is contained in L, where λ is just df, so this is 0. $\qquad\square$

Now we return to the conjecture. We have the following hypotheses: Q is simply connected, and Whitney class $w_2 = 0$. $L \subseteq T^*Q$ is compact, exact, and has Maslov class 0. Given L, we can associate a local system $\mathcal{E}_L$ on Q whose fiber over $q \in Q$ is $\mathrm{CW}^*\left(T_q^*Q, L\right)$, which we notice is finite dimensional, since L is compact.

The fact that the cotangent fibers T_q^*Q generate the Fukaya category means there is an isomorphism

$$(91) \qquad \mathrm{Hom}_{\mathbf{LocSys}/Q}\left(\mathcal{E}_L, \mathcal{E}_L\right) \xrightarrow{\sim} \mathrm{HF}^*\left(L, L\right) = H^*\left(L\right)$$

so we can tell what the endomorphism algebra of L is. This leads to a spectral sequence

$$(92) \qquad H^p\left(Q, H^q\left(\mathrm{End}\left(\mathcal{E}_L\right)\right)\right) \to H^*\left(L\right)$$

Since the thing the spectral sequence is converging to, $H^*\left(L\right)$, is defined in degree $0, \cdots, n$ as in fig. 25, we conclude that $H^*\left(\mathrm{End}\,\mathcal{E}_L\right)$ is concentrated in a single degree, which implies that $H^*\mathcal{E}_L$ is also concentrated in a single degree. Moreover it is one-dimensional. So $H^*\left(L\right) = H^*Q \otimes \mathrm{End}\, H^*\mathcal{E}_L$. So if we just count ranks, we see that $H^*\mathcal{E}_L$ has to have rank 1

In general, one really has to use the geometry of the holomorphic curves in proving the Arnold conjecture, which is fruitful in 4-dimensions, but not so much in higher dimensions.