**RTG Geometry & Topology Summer School, 2018**
**Bordered algebras and a bigraded knot invariant**
**Problem set 1:**

**Exercise 1** (Black & White graphs)

(a) Show that for any knot projection the number of black spanning trees is equal to the number of white spanning trees.

(b) Show that the number of Kauffman states is equal to the number of spanning trees of the black graph.

(c) Suppose we have a knot projection of a knot $K$ with the property that the Black graph has a unique spanning tree. Show that $K$ is the unknot.

**Exercise 2** (Kauffman states and the Alexander polynomial)

Let $(D, e)$ be an oriented knot projection $D$ with a marked edge $e$, and let $A(D, e)$ denote the Kauffman state sum (discussed in class). A pointed Reidemeister move is a Reidemeister move on the diagram that is supported in the complement of the marked edge.

(a) Show that if $(D_1, e)$ and $(D_2, e)$ are related by a pointed Reidemeister move, then $A(D_1, e) = A(D_2, e)$.

(b) Show that if $e_1$ and $e_2$ are two edges on $D$, then $A(D, e_1) = A(D, e_2)$.

(c) Let $D_+$ be a diagram with a distinguished positive crossing, and $e$ a fixed marking (away from the crossing). Changing the crossing gives $D_-$, and taking the oriented resolution gives $D_0$. Find a relationship between the state sums $A(D_+, e)$, $A(D_-, e)$ and $A(D_0, e)$.

(d) Use the above results to show the following. If $L$ is an oriented link and $(D_1, e_1)$ and $(D_2, e_2)$ are two marked diagrams representing $L$ then $A(D_1, e_1) = A(D_2, e_2)$. (From now on we can use the notation $A_L$.)

(e) Show the $A_L$ agrees with the Alexander polynomial of $L$.

**Exercise 3** (Pretzel knots)

(a) Show that the Alexander polynomial of the pretzel knot $P(-3, 5, 7)$ is equal to 1.

(b) Compute the Alexander polynomial of the pretzel knots

$$P(2a + 1, 2b + 1, 2c + 1).$$

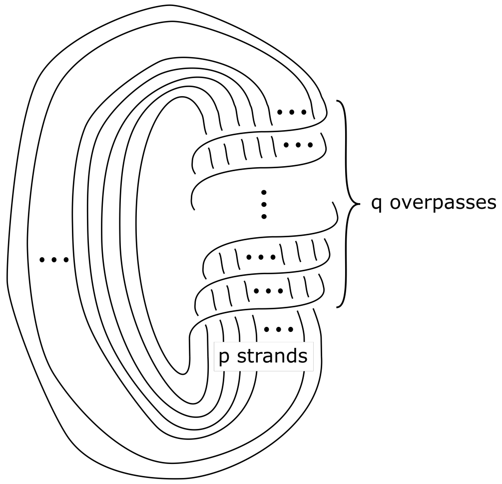


Figure 1: The p,q-torus knot

**Exercise 4** (Alternating knots)

Let $K$ be an alternating knot and $D$ an alternating projection of $K$.

(a) Show that the absolute value of $A_K(-1)$ is equal to the number of black trees of the diagram $D$.

(b) Let $d_{max}$ denote the maximal Alexander grading among all Kauffman states of $D$. Find a Seifert surface of $K$ whose genus agrees with $d_{max}$. (Hint: use the Seifert algorithm for the diagram $D$)

(c) Show that for an alternating knot $K$ the Seifert genus of $K$ agrees with the degree of its (symmetrized) Alexander polynomial.

**Exercise 5** (Fibered knots)

Show that both the trefoil knot and the figure-eight knot are fibered.

**Exercise 6** Let $T_{p,q}$ denote the $p,q$-torus knot, as shown in Figure 1. For a knot $K$, denote by $u(K)$ the *unknotting number* of $K$, which is the minimal number of crossings that need to be reversed in order to transform $K$ into the unknot (minimal over all possible projections).

(a) Show that $T_{p,q} = T_{q,p}$

(b) Show that $u\left(T_{p,q}\right) \leq \frac{(p-1)(q-1)}{2}$