Sheaves on stacks

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November 2, 2020

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Preliminaries, reminders, and notation

- $\mathbf{S}$ is a **site**, i.e. a category equipped with a Grothendieck topology.
- E.g. $\mathbf{S} =$ schemes over a fixed scheme B.
- Let $\mathbf{X}$ be a category **fibered** over $\mathbf{S}$.
- For $U \in \mathbf{S}$, write $\mathbf{X}(U)$ for the fiber.
- If $\mathbf{X}(U)$ is a **groupoid** for all $U \in \mathbf{S}$ we say this is fibered in groupoids.
- In this case we have a functor:

$$\begin{aligned} \mathbf{S}^{\mathsf{op}} &\longrightarrow \mathbf{Grpd} \\ U &\mapsto X(U) \subset \mathbf{X} \,. \end{aligned} \quad . \tag{1}$$

- In order for a prestack to be a **stack** we require that, for all $U \in S$ with a covering $\{U_i \to U\}_{i \in I}$, the functor

$$\mathbf{X}(U) \to \mathbf{X}\left(\{U_i \to U\}_{i \in I}\right)$$

is an equivalence of categories.

Quasi-coherent sheaves on a prestack

Let $\mathbf{X} \xrightarrow{p} \mathbf{Sch}/B$ be a category fibered in groupoids.

Definition

A **quasi-coherent sheaf** $\mathcal{F}$ on $\mathbf{X}$ is

1. a quasi-coherent sheaf $\mathcal{F}_x$ on $p(x)$ for all $x \in \mathbf{X}$,
2. and for all $f\colon x \to y$ in $\mathbf{X}$, an isomorphism

$$\alpha_f\colon \mathcal{F}_y \xrightarrow{\simeq} (pf)^* \mathcal{F}_x \ .$$

which satisfy the cocycle condition: for any two morphism $f\colon x \to y$ and $g\colon y \to z$, the following diagram commutes:

$$\begin{array}{ccc} pf^* \circ pg^* \mathcal{F}_z & = & (pg \circ pf)^* \mathcal{F}_z \\ \downarrow{\scriptstyle pf^*\alpha_g} & & \downarrow{\scriptstyle \alpha_{g\circ f}} \\ pf^* \mathcal{F}_y & \xrightarrow{\alpha_f} & \mathcal{F}_x \ . \end{array}$$

Artin stacks

Definition (Artin stack)

An **Artin stack** is a stack in groupoids $\mathbf{X}$ over the fppf site[a] such that

1. the diagonal map of $\mathbf{X}$ is **representable**, and
2. there is some scheme and smooth surjection to $\mathbf{X}$.

[a] The underlying category is $\mathbf{Aff}/S$.

- Let $\mathbf{A}, \mathbf{B} \to \mathbf{S}$ be categories fibered in groupoids. A morphism $f\colon \mathbf{A} \to \mathbf{B}$ is **representable** if for every scheme U and morphism of stacks $\mathbf{U} \to \mathbf{B}$, the fiber product $\mathbf{A} \times_{\mathbf{B}} \mathbf{U}$ is representable.
- Intuition: the intersection of any two algebraic spaces in an algebraic stack is an algebraic space.
- More formally, the diagonal morphism $\Delta\colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is representable if and only if for any pair of morphisms of algebraic spaces $\mathbf{A}, \mathbf{B} \to \mathbf{X}$, the fiber product $\mathbf{A} \times_{\mathbf{X}} \mathbf{B}$ is representable.

Hierarchy

$$\begin{array}{c}\{\text{affine } B\text{-schemes}\} \\ \downarrow \\ \{B\text{-schemes}\} \\ \downarrow \\ \{B\text{-algebraic spaces}\} \\ \downarrow \\ \{\text{DM stacks over } S\} \\ \downarrow \\ \{\text{Artin stacks over } S\} \\ \downarrow \\ \{\text{stacks over } S\} \\ \downarrow \\ \{\text{categories fibered in groupoids}\}\end{array} \; .$$

Quotient stacks

- Let G be an affine smooth group scheme over a scheme B.
- Let X be an S-scheme with an action of G.
- Define $[X/G]$ to be the following category fibered over $\mathbf{Sch}/B$.
  - An object over U is a principal G-bundle $E \to U$ together with an equivariant map $E \to X$.
  - A morphism from $E \to U$ to $E' \to U$ is a bundle map that is compatible with the corresponding equivariant maps.
- This forms an **Artin stack**.
- If the stabilizers of the geometric points are finite and reduced, then this is a Deligne-Mumford stack.

Quotient stack vs. quotient space

Proposition (Stacks Project, Lemma 64.34.1)

Let X be an algebraic space over a scheme. If the stabilizers are ***trivial*** *(i.e. the action is* ***free****) the quotient $\overline{X/G}$ exists as an* ***algebraic space****.*

- There is a (destructive) functor $\mathbf{QC}\left([X/G]\right) \to \mathbf{QC}\left(\overline{X/G}\right)$.
- The quotient space is generally more **coarse** than the quotient stack.
- We can detect this at the level of $\mathbf{QC}$:

$$\mathbf{QC}\left([X/G]\right) = \left\{ M \in \mathcal{O}_X\text{-}\mathbf{Mod} \,\middle|\, M \text{ is } G \text{ equivariant} \right\}$$
$$\mathbf{QC}\left(X/G\right) = \mathcal{O}_X^G\text{-}\mathbf{Mod} \; .$$

- E.g. if $X = \bullet$ then $\overline{\bullet/G} = \bullet$ and

$$\mathbf{QC}\left([\bullet/G]\right) = \mathbf{Rep}\left(G\right)$$
$$\mathbf{QC}\left(\bullet\right) = \mathbf{Vect} \; .$$