Spherical varieties

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But first. . . ☺

Our building will officially be renamed "PMA"! ☺

See here for the full announcement.

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Notation

- k is a field.
- G is an algebraic group over k (an algebraic variety which is also a group, i.e. group scheme of finite type over a field).
- X is an algebraic variety over k equipped with an action of G.

We are entering the subsection:

What are they?

Spherical varieties are algebraic varieties equipped with an action of a certain type of algebraic group G subject to a finiteness condition.

- The type of G will be called *reductive*.
- First we motivate and define this term.
- Then we make precise what we mean by "finiteness condition" on the action of G on the variety.

We are entering the subsection:

Why define such a thing?

One name of the game is generalizing features of toric varieties and flag varieties.

- Another name of the game is **classification**.
- Classifying all algebraic varieties (up to birational equivalence) would be the dream. This is just too hard to do.
- Instead, one imposes some structure, to make the problem tractable.
- For us, this structure comes in the form of a group action.

Spherical varieties are sometimes taken to be normal. We won't assume this, but this is also natural from this point of view: if we can't classify singular things, it is reasonable to insist on better behavior.

Example

If the group is a torus, $G = (\mathbb{G}_m)^n$, then we get the notion of a toric variety. This structure restricted the class of varieties enough to get the classification via fans that we have seen.

Ok, fine. But why do I care about classification?

- Classifications often come in a combinatorial package (ADE classification, classification of toric varieties by fans, etc.).
- If one has an honest classification then one can use this combinatorial data instead of the variety itself to perform constructions.

Example

Intrinsic 2d mirror symmetry works somewhat in this way. In particular, the Gross-Siebert program[a] starts with a variety, converts it into combinatorial data, and uses this data to define the mirror variety.

[a] My brevity is not intended to simplify this and other programs of 2d mirror symmetry. It is much more complicated and deep than my description might imply.

Where do they show up?

- Toric and flag varieties are both examples of spherical varieties (as expected).
- Spherical varieties are intimately related to the Langlands program (both geometric and arithmetic). Executive summary: classically one studies automorphic forms on the upper half-plane by calculating period integrals. Now one would like to generalize this, and there is a sense in which key features of this example are encoded in the notion of a spherical variety.
- See David Ben-Zvi's talks at MSRI about the relative Langlands program for more on this.

We are entering the subsection:

Map of the zoo

$$
\begin{array}{ccccc}
\text{abelian} & \cap & \text{simple} & & = \emptyset \\
\Downarrow & & \Downarrow & & \\
\text{nilpotent} & \cap & \text{semisimple} & & = \emptyset \\
\Downarrow & & \Downarrow & & \\
\text{solvable} & \cap & \text{reductive} & & = \text{abelian}
\end{array}
$$

Definition (solvable)

An algebraic group G is *solvable* if and only if it admits a subnormal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_k = \{1\} \tag{1}$$

such that each G_i / G_{i+1} is abelian. In other words it is built out of abelian groups by extensions.

Simple, semisimple, and reductive

The *radical of G*, written $R(G)$, is the maximal normal subgroup which is connected, and solvable.

(Such a subgroup exists because extensions and quotients of solvable algebraic groups are solvable.)

Definition 1 (simple)

G is *simple* if and only if it does not contain any (proper, nontrivial, and connected) normal subgroups.

Definition 2 (semisimple)

G is *semisimple* if and only if $R(G) = \{1\}$.

Definition 3 (reductive)

G is *reductive* if and only if $R(G) \cong (\mathbb{G}_m)^n$ for some n.

Tl;dr:

$$\textbf{Simple} = \text{no } \underline{\text{normal}} \text{ subgroups;} \tag{2}$$
$$\textbf{semisimple} = \text{no } \underline{\text{solvable}}\ \underline{\text{normal}} \text{ subgroups;} \tag{3}$$
$$\textbf{reductive} = \text{solvable normal subgroups are } \underline{\text{abelian}}. \tag{4}$$

Important subgroups

Definition 4

Write $T \subset G$ for a maximal torus.

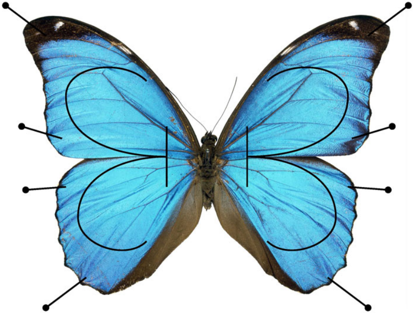
Definition 5

A maximal connected solvable subgroup $B \subset G$ is called a *Borel subgroup*.

Theorem

All maximal tori (resp. Borel subgroups) are conjugate.

The butterfly garden



Grothendieck's vision of a pinned reductive group: "the body is a maximal torus T, the wings are the opposite Borel subgroups B, and the pins rigidify the situation."

Picture and description from here.

First observations

- Simple $\implies$ semisimple $\implies$ reductive.
- $G/R(G)$ is always semisimple.
- If G is abelian then G is reductive.
- If G is solvable and nonabelian then G is not reductive.

Examples of reductive algebraic groups

Example

$(\mathbb{G}_m)^n$ is abelian, so reductive.

Example

The following algebraic groups are simple, so reductive:

$$\mathrm{SL}_n \, (\text{for } n \geq 2), \qquad \mathrm{Sp}_{2n} \text{ and}, \qquad \mathrm{SO}_n \, (\text{for } k = \overline{k}) \, . \tag{5}$$

Example

A maximal torus of SL_n consists of diagonal matrices, and a Borel subgroup of SL_n consists of upper triangular matrices.

Example

GL_n and O_n are reductive.

Example of quotient by radical

Example 1

If $G = \mathrm{GL}_n$ the radical is just scalar matrices aI_n for $a \neq 0$, i.e. $\mathbb{G}_m$. The quotient is SL_n. This is semisimple (in fact simple).

The following is a more interesting example from [Mil]. Consider the algebraic group GL_{m+n}. This is given by block matrices

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \tag{6}$$

where A is $m \times m$ and C is $n \times n$.
The radical consists of matrices of the form:

$$\begin{bmatrix} aI_m & B \\ 0 & cI_n \end{bmatrix} \tag{7}$$

and the semisimple quotient is:

$$G/R(G) = \mathrm{PGL}_m \times \mathrm{PGL}_n \ . \tag{8}$$

Unipotent characterization

Recall an operator T is called *unipotent* if and only if there is some $N \in \mathbb{Z}_+$ such that

$$(T - 1)^N = 0 \ . \tag{9}$$

An algebraic group is called *unipotent* if it acts by unipotent operators in any rational representation.

Lemma

G is reductive if and only if it does not contain any normal subgroups which are (proper, connected, and) unipotent.

- If G is unipotent then G is not reductive.
- Just as $G/R(G)$ was semisimple, the quotient of any algebraic group by its maximal normal subgroup which is (connected and) unipotent is reductive.

Examples which are not reductive

Example

The additive group $\mathbb{G}_a$ (and any $(\mathbb{G}_a)^n$) are not reductive. This is because we can view $a \in \mathbb{G}_a$ as

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} . \tag{10}$$

So this is actually a unipotent group, and therefore cannot be reductive.

Example

The Borel subgroup B of GL_n is not reductive. This consists of upper triangular matrices, and has nontrivial unipotent normal subgroup consisting of upper-triangular matrices with 1 on the diagonal. In fact, B is solvable.

Example of unipotent quotient

Again consider the Borel subgroup of GL_n consisting of upper triangular matrices. This has maximal normal **unipotent** subgroup given by upper triangular matrices with 1 on the diagonal.

Just as the quotient of G by the radical was semisimple, the quotient by this is reductive. Indeed the quotient is the torus:

$$\mathrm{GL}_n \Big/ \left\langle \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\rangle \cong (\mathbb{G}_m)^n \quad . \tag{11}$$

We are entering the subsection:

Warning

Reductive groups admit a very rich representation theory. This is just the tip of the iceberg.

Rep. theoretic characterization of reductive groups

Definition

Let V be a (finite-dimensional) k vector space. A representation of G is a map $G \to \mathrm{GL}(V)$.

Definition

A *semisimple (or completely reducible) representation* is a direct sum of simple (or irreducible) representations.

Theorem

Assume char $k = 0$. *G is reductive iff every (finite-dimensional) representation is semisimple.*

The direction($\Longleftarrow$) is easy to show. Normal unipotent subgroups of G act trivially on semisimple representations of G. So if G admits a faithful semisimple representation then G is reductive.

Summary so far:

- Algebraic varieties were very hard to classify, so instead we just look at ones which have some kind of group action.
- In particular, we ask for a **reductive group action** satisfying some other finiteness constraint which we will meet shortly.
- This is a good type of group to ask for, because:
  - it generalizes abelian groups and
  - it generalizes simple groups so in particular
  - it has good representation theory, i.e. it acts on things in an understandable way.

We are entering the subsection:

Usual definition of spherical varieties

Definition (spherical variety)

X is a *spherical variety* if and only if it contains an open dense B orbit.

Now we will identify some equivalent characterizations of spherical varieties. The punchline will be that this is fundamentally some kind of finiteness condition.

Complexity

Definition (complexity)

The complexity of X, written $c(X)$, is the minimal codimension of a B orbit.

Theorem 1

X is spherical if and only if $c(X) = 0$.

Proof.

Open dense orbits are the proper codimension 0 orbits. □

Finitely many B orbits

Theorem

X is spherical if and only if it has finitely many B orbits.

Lemma 1 (Theorem 4.5.5 [Per])

If $Y \subset X$ is a closed B-stable subvariety then $c(Y) \leq c(X)$.

Proof.

($\implies$): Let $Y \subseteq X$ be some minimal subvariety containing infinitely many orbits. Lemma 1 implies $c(Y) = 0$. The complement of this orbit is a closed G-stable subvariety which must have infinitely many B-orbits, contradicting minimality of Y.
($\impliedby$): Nonzero complexity implies infinitely many G orbits (and hence B orbits) since any maximal orbit has nonzero codimension, and orbits are disjoint. □

Borel invariant rational functions

Theorem

X is spherical if and only if the only B invariant rational functions are constant: $k(X)^B = 0$.

This follows from Rosenlicht's theorem [Ros63] Theorem 2.3.

Theorem (Rosenlicht)

The transcendence degree of $k(X)^B$ over k is $c(X)$.

The idea is that

$$c(X) = \dim(\text{“}X/B\text{”}) \tag{12}$$
$$= \text{transcendence degree}\left(k(\text{“}X/B\text{”})\right) \tag{13}$$
$$= \text{transcendence degree}\left(k(X)^B\right). \tag{14}$$

So $c(X) = 0$ if and only if $k(X)^B = k$.

Summary so far

- We are considering G-varieties. We want a good type of G, and good condition on the action.
- Reductive G is good because it has nice representation theory.
- The condition we put on the action is a finiteness condition given by any of the equivalent conditions in the following theorem.

Theorem

The following are equivalent:

- *X is spherical (i.e. X contains an open dense B orbit),*
- *$c(X) = 0$ (i.e. the maximal B orbit is codimension 0),*
- *X has finitely many B orbits,*
- *$k(X)^B = k$.*

We are entering the subsection:

Good features of spherical varieties

This slide could alternatively be titled: other talks people can give about spherical varieties.

- Luna-Vust theory.
  - Birational models of a given spherical variety are classified by colored fans.
  - "Explicitly classify those spherical SL_2 spaces with an open G/N orbit." - Tom Gannon
- Projective spherical varieties are Mori dream spaces.
- The Chow groups of a spherical variety are equal to the G invariant Chow groups and are finitely generated. If the variety is smooth, these are the homology groups.
- If $\operatorname{char}(k) = 0$ then all singularities are rational.
- Flesh out connections with automorphic forms.

References

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- Nicolas Perrin, *Introduction to spherical varieties*, Available at: hcm.uni-bonn.de/fileadmin/perrin/spherical.pdf.
- Maxwell Rosenlicht, *A remark on quotient spaces*, An. Acad. Brasil. Ci. **35** (1963), 487–489. MR 171782