LEARNING SEMINAR: THE TRIANGULATION CONJECTURE

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In this seminar we are trying to prove the following:

Theorem 1 ([3])**.** *Non-triangulable topological manifolds exist in all dimensions* ≥ 5.

This fits into the diagram in fig. 1. E_8 denotes the topological manifold with intersection form E_8. The idea is to make a smooth 4-manifold with boundary the Poincaré homology sphere P, then there is a nasty topological manifold which we can paste on to get a topological manifold which is not triangulable. Then [3] will tell us that this is strict in general for higher dimensions. $\Sigma\Sigma P$ denotes the double suspension of the Poincaré homology sphere. The double suspension theorem says that this can be given a triangulation.

The outline of this talk is as follows:

(1) Triangulations
(2) History
(3) Obstructions
(4) $\mathbb{Z}HS^3$
(5) Rokhlin invariant
(6) KS obstruction for triangulations

1. Triangulations

Definition 1. A simplicial complex is (V, S) where

- V is a set,
- $S \subset 2^V$,
- $\sigma' \subset \sigma$ and $\sigma \in S$ implies $\sigma' \in S$.

This is given a geometric realization as:

$$\left\{ x \in \mathbb{R}^V \,\middle|\, x \in \text{ convex hull of } \sigma \right\} . \tag{1}$$

We won't distinguish between this sort of combinatorial data and the associated topological space.

Definition 2. The star of a simplex is:

$$\mathrm{St}\,(\sigma) = \{\tau \mid \sigma \subset \tau\} \ , \tag{2}$$

the closure of some $S' \subset S$ is

$$\mathrm{Cl}\,(S') = \{\tau \mid \tau \subseteq \sigma \in S'\} \ , \tag{3}$$

and the link of a simplex is

$$\mathrm{Lk}\,(\sigma) = \{\tau \in \mathrm{Cl}\,(\mathrm{St}\,(\sigma)) \mid \tau \cap \sigma = \emptyset\} \ . \tag{4}$$

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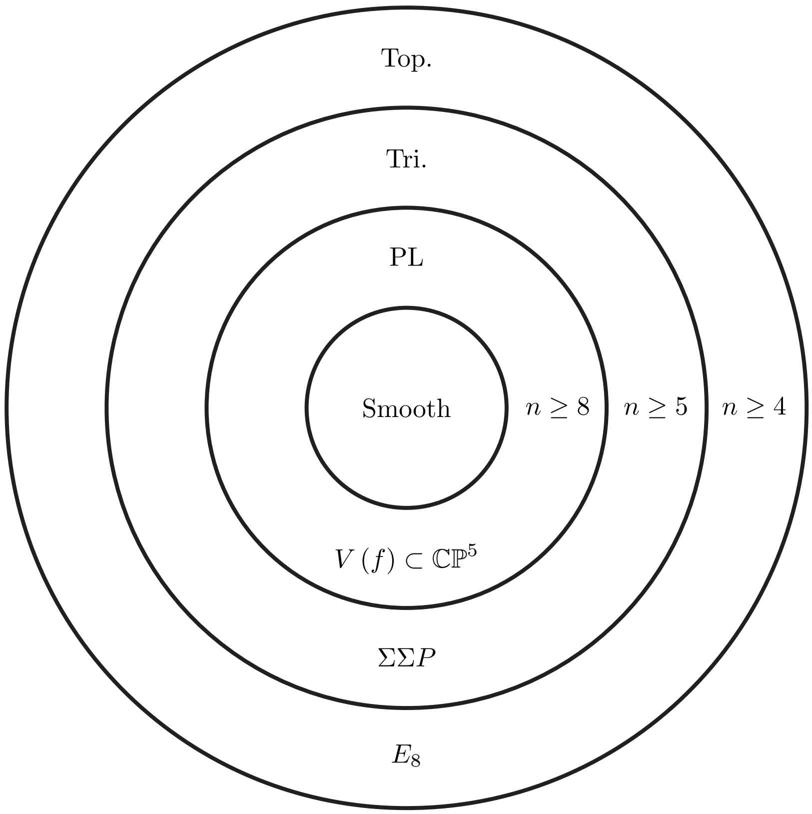

FIGURE 1. Venn diagram showing strict inclusions of types of topological manifolds.

Example 1. Consider the simplicial complexes in fig. 2. On the top, the star of the blue vertex (the middle vertex) is in red (and includes itself of course). On the bottom, the link of the blue vertex (the middle vertex) is in red.

Definition 3. Let K be a simplicial complex. A triangulation of M is a homeomorphism $K \to M$.

Definition 4. We say a triangulation K is combinatorial if $\mathrm{Lk}(\sigma) \simeq$ a sphere as PL manifolds for every σ.

This is a sufficient condition for it to be a piecewise linear manifold.

2. History

One question we might ask is if topological implies piece-wise linear. This was answered as follows:

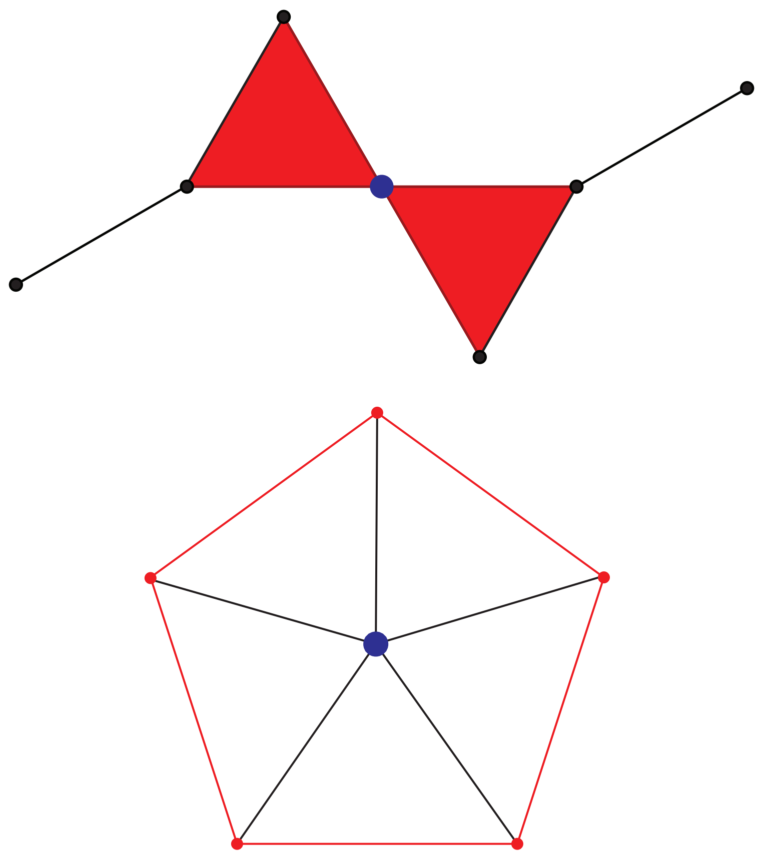

Figure 2. The geometric realizations of two simplicial complexes.

- $n \leq 3$: Yes, [4]
- $n = 4$: No, [1] (E_8 manifold)
- $n \geq 5$: No, [2]

We also might wonder when topological implies triangulable:

- $n < 3$: same as above.
- $n = 4$: No, [5] (E_8 manifold)
- $n \geq 5$: No, [3]

3. Obstructions

Recall a manifold is orientable exactly when there is no obstruction to performing the following lift:

$$\begin{tikzcd} & \mathrm{BSO}\left(n\right) \arrow[d] & \mathbb{Z}/2 = \mathrm{O}\left(2\right)/\,\mathrm{SO}\left(2\right) \arrow[l, hook'] \\ M \arrow[r] \arrow[ur, dashed] & \mathrm{BO}\left(n\right) & \end{tikzcd} \quad . \tag{5}$$

In particular when the Stiefel-Whitney class $w_1 \in H^1\left(M, \mathbb{Z}/2\right)$ is zero.

We could also ask when a manifold has a spin structure, i.e. when we have the following lift:

$$\begin{tikzcd} & \mathrm{BSpin}(n) \arrow[d] & B(\mathbb{Z}/2) \arrow[l, hook'] \\ M \arrow[r] \arrow[ur, dashed] & \mathrm{BSO}(n) & \end{tikzcd} \tag{6}$$

Similarly, we could ask for a lift

$$\begin{tikzcd} & \mathrm{BPL} \arrow[d] & K(3,\mathbb{Z}/2) = \mathrm{TOP}/\mathrm{PL} \arrow[l, hook'] \\ M \arrow[r] \arrow[ur, dashed] & \mathrm{BTOP} & \end{tikzcd} \tag{7}$$

where

$$\mathrm{TOP} = \varinjlim_n \mathrm{TOP}(n) \qquad \mathrm{TOP}(n) = \left\{ (\mathbb{R}^n, 0) \xrightarrow{\text{homeo}} (\mathbb{R}^n, 0) \right\} \tag{8}$$

and the analogous definitions for PL. The obstruction is exactly the Kirby-Siebenmann class $\Delta(M) \in H^4(M, \mathbb{Z}/2)$.

When we have a triangulation $K \to M$ we can explicitly write this class as the Rolkin invariant of the sum of some links in our complex. We define the Rolkin invariant and state this precisely in section 5 and definition 7.

4. $\mathbb{Z}HS^3$

Example 2. If we have a knot we can do $1/n$ surgery. The knot complement has $H_1 = \mathbb{Z}$ generated by the meridian. So if we glue the torus back in with these twists we get a homology three sphere.

Example 3. $\{x^p + y^q + z^r = 0\} \cap S^5 \subset \mathbb{C}^3$

Now we will consider the Homology cobordism group:

$$\Theta_3^H = \left\{ Y \text{ oriented } \mathbb{Z}HS^3 \right\} / \sim \tag{9}$$

where $Y \sim Y'$ if there exists W with $\partial W = Y \cup -Y$ such that

$$H_*(W, Y) = 0\ . \tag{10}$$

This forms an abelian group under connect sum #.

We know:

- this is infinitely generated,
- $\mathbb{Z}$ summand,

and we don't know if

- there is any torsion,
- $\mathbb{Z}^\infty$ summand.

5. Rokhlin invariant

This is a homomorphism

$$\mu : \Theta^3_H \to \mathbb{Z}/2\ . \tag{11}$$

Theorem 2 (Rokhlin)**.** *Every closed smooth* Spin 4*-manifold has signature* σ *divisible by* 16.

Definition 5. Let Y be a Spin 3-manifold

$$\mu\left(Y\right)=\frac{\sigma\left(X\right)}{8}\pmod{2} \tag{12}$$

for X such that $\partial X=Y$.

Lemma 3. *Suppose* $\partial X=\partial X'=Y$ *then*

$$\sigma\left(X\right)-\sigma\left(X'\right)=\sigma\left(X\cup_{Y}X'\right)=0\pmod{16}\ . \tag{13}$$

Claim 1. If Y is $\mathbb{Z}HS^{3}$ then $\mu\left(Y\right)\in\mathbb{Z}/2\mathbb{Z}$.

Fact 1. *Unimodular even symmetric bilinear forms have signature* $0 \pmod{8}$.

Definition 6. Let $K\to M$ be a triangulation of a topological manifold. Then

$$c\left(K\right)=\sum_{\mathrm{Codim}\,\sigma=4}\left[\mathrm{Lk}\left(\sigma\right)\right]\sigma\in H_{n-4}\left(M,\Theta_{H}^{3}\right)\ . \tag{14}$$

Definition 7. $\Delta\left(M\right)=\mu\left(c\left(K\right)\right)$.

Note that K being combinatorial is equivalent to $c\left(K\right)=0$ since every link would be a sphere which is the identity element in this group.

References

[1] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

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[3] Ciprian Manolescu, *Lectures on the triangulation conjecture*, arXiv e-prints (2016Jul), arXiv:1607.08163, available at arXiv:1607.08163.

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