FINITE DIMENSIONAL APPROXIMATION (CONT.) AND CONLEY INDEX

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Recall from last time that the Seiberg-Witten operator $\mathrm{SW} = \ell + c : V \to V$ where b is a self-adjoint Fredholm operator and c is a compact, quadratic map defined by

$$\begin{aligned} \ell(a, \varphi) &= \left(*da, \not{\partial}\varphi\right) \\ c(a, \varphi) &= \pi_{V^0}\left(-2\rho^{-1}\left(\varphi \otimes \varphi^*\right)_0, \rho(a)\varphi\right) . \end{aligned}$$

Define the space $V_{(k)} = W^{2,k}$ $(k \geq 5)$ to be the Sobolev completion of V.

Theorem 1. *Fix $k \geq 5$. Then there exists $R \geq 0$ such that all critical points and the flow lines of* SW *are contained in $B(R) \subset V_{(k)}$.*

We should think of a flow line for SW as a critical point of SW on $Y \times \mathbb{R}$.

1. Finite-dimensional approximation

Recall our aim is to use the SW flow as a Morse flow on V. The problem is we need to deal with things like transversality, and these things are more complicated in infinite-dimensions. So the idea is to simply consider some finite-dimensional subspace. But we need to make sure this gives a coherent answer that doesn't "depend" on this approximation.

We will follow the approach of Bauer and Furuta [1]. Define

$$V_\lambda^\mu = \bigoplus \left(\text{Eigenspaces of } \ell \text{ with eigenvalues in } (\lambda, \mu), \lambda \ll 0 \ll \mu\right)$$

and

$$\mathrm{SW}_\lambda^\mu = \mathrm{SW}|_{V_\lambda^\mu} = \ell + P_\lambda^\mu c .$$

This should again be thought of as a vector field, this time on V_λ^μ./

Theorem 2. *There exists some $R > 0$ such that for all $\mu \gg 0 \gg \lambda$ all critical points of SW_λ^μ are in the ball $B(2R)$ and the flow lines between the flow lines which lie in $B(2R)$ actually are contained in $B(R)$.*

Recall the general setup of usual Morse theory. The idea is that we have a closed manifold M as in fig. 1 and a Morse function f. The complex is generated by critical points of f, and the differential is given by

$$\partial x = \sum_y m_{x,y} y$$

where $m_{x,y}$ is the number of index 1 graded flow lines starting at x and ending at y. These should be visualized as in fig. 1. Then it is a nice fact that this agrees with the singular homology.

Date: October 4, 2019; Notes by Jackson Van Dyke. All errors introduced are my own.

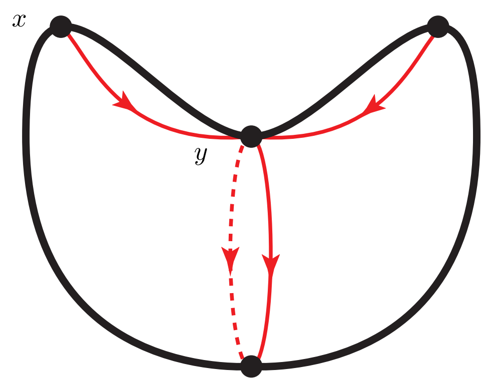


FIGURE 1. Morse theory of a closed manifold where the Morse function is just the height function.

Let's consider what happens in the non-compact case. If we remove a neighborhood of the flow lines, we get that

$$\partial^2 x = \partial(\partial x) = \partial(\pm y) = \pm 1 \neq 0$$

so we don't get an honest complex.

Now define $S \subseteq M$ to be the set of all points or flow lines limiting to a critical point. As it turns out, this gives us:

$$H(C_*(S, f)) = H(I(S))$$

where $I(S)$ is the Conley index, which we define as follows.

Definition 1. Let $S \subseteq M$, we define $\{\varphi_t\}_t \circlearrowright M$. Then we define the invariant set to be:

$$\text{inv}\, A = \{x \in M \mid \varphi_t(x) \in A \forall t \in \mathbb{R}\} \ .$$

Definition 2. A compact subset $S \subset M$ is called an isolated invariant if there exists A a compact neighborhood of S such that

$$S = \text{inv}(A) \subseteq \text{int}(A) \ .$$

Definition 3. For an isolated invariant S, the Conley index is defined as the pointed homotopy type

$$I(S) = (N/L, [L]) \ ,$$

where $L \subseteq N \subseteq M$, where L and N are both compact, and they satisfy:

- $\text{inv}(N - L) = S \subseteq \int(N - L)$,
- L is an *exit set* for N.
- L is positively invariant in N, if $x \in L$, at $t > 0$ $\varphi_t(x) \in N$ implies $\varphi_t(x) \in L$. In this case (N, L) is called an index pair.

The picture is as in fig. 2.

Remark 1. This is called the Conley index because it generalizes the Morse index. Suppose φ is the downward gradient flow of a Morse function, and $S = \{x\}$ consists

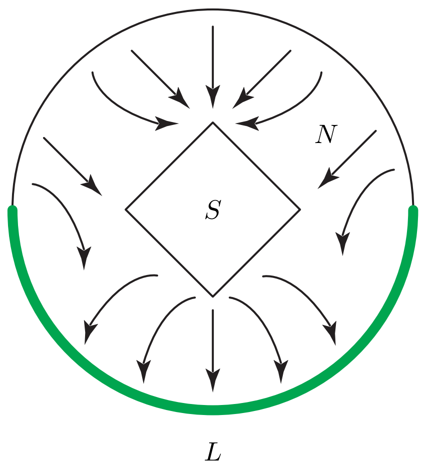
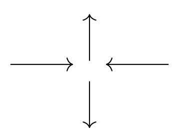


FIGURE 2. A canonical example of the Conley index. L is highlighted in green.

of a single critical point of Morse index k:



We can find an isolating neighborhood N for $\{x\}$ of the form $D^k \times D^{n-k}$ with $L = \partial D^k \times D^{n-k}$ being the exist set. We deduce that the Conley index of $\{x\}$ is the homotopy type of S^k, so k can be recovered from $I\left(\varphi, S\right)$.

In the case we are interested in we have $A = B\left(2R\right)$ and $S = \operatorname{inv}\left(A\right)$ is the union of critical points of SW and flow lines in $B\left(2R\right)$. So as we said above, we get

$$H^*\left(C\left(S, \mathrm{SW}\right)\right) = H\left(I\left(S\right)\right)$$

Definition 4. Define $\mathrm{SWFH}_*^{S^1}\left(Y\right) = \tilde{H}_{*+s}^{S^1}\left(I_\lambda^\mu\right)$ where $I_\lambda^\mu = S \cap V_\lambda^\mu$, s is some shift, and the RHS is the Borel homology.

Now we want to see the dependence on μ and λ. Write $\dot{x} = -\mathrm{SW}\left(x\left(t\right)\right)$. For $\lambda' > \lambda$ we want to compact $I_l^{\mu'} am$ and I_λ^μ. First we decompose

$$V_\lambda^{\mu'} = V_\lambda^\mu \oplus V_\mu^{\mu'}$$

which induces

$$l + P_\lambda^{\mu'} c \rightsquigarrow l + P_\lambda^\mu c \oplus l + P_\mu^{\mu'} c\ .$$

The Conley index is invariant under deformations, i.e. if we have a family of flows $\varphi\left(s\right)$ where $s \in [0,1]$ such that

$$S\left(s\right) = \operatorname{inv}\left(B\left(R\right) \in \varphi\left(s\right)\right) \subset \int B\left(R\right)$$

where $s \in [0,1]$ the $I\left(S\left(0\right)\right) \simeq I\left(S\left(1\right)\right)$.

Then we claim that the flow of $l+P_{\mu}^{\mu'}$ is isotopic to the flow of l on $V_{\mu}^{\mu'}$. The idea is to isotope to the identity by rescaling the eigenvalues.

In our case, let $\varphi(0)$ be the flow of $l+P_{\lambda}^{\mu'}c$ and deform it into $\varphi(1)$, the direct sum of the flow of $l+p_{\lambda}^{\mu}$ and the linear flow l on $V_{\mu}^{\mu'}$. We get

$$I_{\lambda}^{\mu'}=I\left(S\left(\right)\right)=I\left(S\left(1\right)\right)=I_{\lambda}^{\mu}\wedge I_{\mu}^{\mu'}\left(l\right) \ .$$

$I_{\mu}^{\mu'}(l)$ is the Conley index for the linear flow $\dot{x}=-l\left(x\right)$ on $V_{\mu}^{\mu'}$. Since the restriction of l to that subspace has only positive eigenvalues, we see that

$$I_{\mu}^{\mu'}\left(l\right)=S^{\text{Morse index}}=S^{0} \ ,$$

so we obtain

$$I_{\lambda}^{\mu'}=I_{\lambda}^{\mu}$$

for $\mu,\mu'\gg 0$.

Now we also need to argue for λ. This time the Conley index changes by the formula:

$$I_{\lambda'}^{\mu}=I_{\lambda}^{\mu}\wedge\left(V_{\lambda'}^{\lambda}\right) \ .$$

We conclude that

$$\tilde{H}_{*+\dim V_{\lambda}^{0}}\left(I_{\lambda}^{\mu}\right)$$

is independent of λ and μ, provided $\mu\gg 0\gg\lambda$. The same is true for the $\mathrm{Pin}\,(2)$-equivariant homology. This suggests including a degree shift $\dim V_{\lambda}^{0}$ in the definitions.

References

[1] Stefan Bauer and Mikio Furuta, *A stable cohomotopy refinement of seiberg-witten invariants: I*, 2002.