$\operatorname{Pin}(2)$-EQUIVARIANCE OF SWF HOMOTOPY TYPE

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1. The story so far

Consider an integral homology three sphere Y^3, with a Riemannian metric g. This means there is a unique spin^c structure with a spinor bundle:

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{S}^1 \\ & & \downarrow \\ & & Y \end{array} .$$

Now consider $\mathcal{A} \times \Gamma\left(\mathbb{S}^1\right)$ where $\mathcal{A}$ is the space of spin^c connections, which is the same as $\Omega^1_Y(i\mathbb{R})$.

Then we have the gauge group which we think of as automorphisms of this $\mathrm{U}(1)$ bundle:

$$\mathcal{G} = C^\infty\left(Y, S^1\right) \circlearrowright \mathcal{A} \times \Gamma .$$

The action is given by:

$$g(a,\varphi) = \left(a - g^{-1}\, dg\, , g\varphi\right) .$$

So we have stabilizers:

$$\operatorname{Stab}_{(a,\varphi)} = \operatorname{const}\left(Y, S^1\right) = S^1 \subset \mathcal{G} .$$

So we have

$$\ker\left(\mathcal{G} \circlearrowright \mathcal{A} \times \Gamma\right) = S^1$$

and if we mod out we get

$$\mathcal{G}/S^1 \simeq \mathcal{G}_0 = \left\{ g \in \mathcal{G} \,\middle|\, g = e^\xi, \int \xi = 0 \right\} .$$

This acts freely on γ. When we mod out by this action we get

$$\mathcal{A} \times \Gamma / \mathcal{G}_0 \simeq V = \ker\left(d^*\right) \oplus \Gamma\left(\mathbb{S}^1\right) .$$

Recall $\operatorname{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}^\times$. We have an action $\operatorname{Pin}(2) \circlearrowright \mathcal{A} \times \Gamma$ given by

$$e^{i\theta}(a,\varphi) = \left(a, e^{i\theta}\varphi\right) \qquad j(a,\varphi) = (-a, \varphi j) .$$

Then we have the Chern-Simons-Dirac functional:

$$\mathcal{A} \times \Gamma \xrightarrow{\quad \mathrm{CSD} \quad} \mathbb{R}$$

$$(A,\varphi) \longmapsto \tfrac{1}{2}\left(\int \left\langle \varphi, \partial\!\!\!/\varphi + \rho(a)\varphi \right\rangle d\operatorname{Vol} - \int a \wedge da\right) .$$

Notice that this is $\mathcal{G}$ and $\operatorname{Pin}(2)$ equivariant.

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Define $\tilde{\mathcal{G}} = \mathcal{G}_0 \times \mathrm{Pin}(2)$. Now the idea is to do equivariant Morse theory with these actions, with CSD as the Morse function. Naively we want to take the Borel quotient:

$$\begin{array}{ccc} \mathcal{A} \times \Gamma \times_{\tilde{\mathcal{G}}} \widetilde{E\mathcal{G}} & \xrightarrow{\text{CSD}} & \mathbb{R} \\ \downarrow \simeq & & \\ (\mathcal{A} \times \Gamma / \mathcal{G}_0) \times_{\mathrm{Pin}(2)} E\,\mathrm{Pin}(2) & & \\ \downarrow \simeq & & \\ V \times_{\mathrm{Pin}_2} E\,\mathrm{Pin}(2) & & \end{array} .$$

First we have that

$$\mathrm{CSD}\left(a + t\dot{a}, \varphi + tp\dot{h}i\right) = \mathrm{CSD}(a,\varphi) + \underbrace{td\,\mathrm{CSD}(\dot{a},\dot{\varphi})}_{\langle \nabla\,\mathrm{CSD}, (\dot{a},\dot{\varphi})\rangle} + t^2 Q((\dot{a},\dot{\varphi})) + \dots$$

where explicitly we have

$$\nabla\,\mathrm{CSD}_{(a,\varphi)} = \left(* da - 2\rho^{-1}\left((\varphi \otimes \varphi)_0\right), \not{\partial}\varphi + \rho(a)\varphi\right) = \mathrm{SW}(a,\varphi) \ .$$

Then we have the Hessian

$$\mathrm{Hess}_{\mathrm{CSD}}(X,Y) = \frac{1}{2}\left(Q(X+Y) - Q(X) - Q(Y)\right) \ .$$

Then

$$\nabla\left(\mathrm{CSD}|_V\right) = \pi_V(\nabla\,\mathrm{CSD}) \ .$$

To do Floer theory we need:

(1) $\mathcal{M}_{\mathrm{crit}}$ to be cut out transversely, or equivalently Hess_a is surjective at $a \in$ crit. This implies that we have isolated non-degenerate critical points.
(2) $\mathcal{M}_{(a,b)}$ to be cut out transversely. This means ∂ is well-defined.

To get these things we perturb as follows:

$$\mathrm{CSD}_\rho(a,\varphi) = \mathrm{CSD}(a,\varphi) + \int a \wedge *\rho$$

where $\rho = dv \in \Omega^1_Y(i\mathbb{R})$. Now we can ask the question of which perturbations are $\mathrm{Pin}(2)$ invariant. This is hard, so instead we do the following procedure.

1.1. **Conley index.** Let N^{cpt} be an isolating neighborhood for a flow φ. This gives us $S = \mathrm{inv}(N,\varphi)$. Write $L \subset N$ for the exit set. Then the *Conley index* is

$$I(\varphi,S) \simeq (N/L, [L]) \ .$$

This has the following properties:

(1) $I(\varphi,S)$ is robust under perturbations. I.e. for a family of flows $\{\varphi_\tau\}_{\tau\in[0,1]}$, and writing $S_\tau = \mathrm{inv}(N,\varphi_\tau)$, we have

$$I(\varphi,S_0) \simeq I(\varphi,S_1) \ .$$

(2) The following is due to Floer. Let S be an isolating invariant set for the flow of a Morse-Smale function (on a finite-dimensional manifold with a Riemannian metric). Then the morse homology is:

$$H_*^{\mathrm{Morse}}\left(\dots \to C_i^S \to C_{i-1}^S \to \dots\right) = \tilde{H}_*^{\mathrm{Sing}}(I(\varphi,S))$$

(3) Let $G \circlearrowright M$, and φ be a G invariant flow. Let S be an isolated invariant set for φ. Then there exists a G invariant (N, L) such that we have a G-homotopy equivalence:

$$I_G(\varphi, S) \simeq (N/L, [L])$$

and this is in fact a finite G-CW complex.

Now if we could (e.g. if we weren't in infinite dimensions) then we would take φ_t to be the ∇ flow of $\mathrm{CSD}_{\tau\cdot\rho}$. Then the third property gives us:

$$I_{\mathrm{Pin}(2)}\left(\varphi_0, S_0^{\mathrm{cpt}} \subset V\right) \simeq I\left(\varphi_1, S_1^{\mathrm{cpt}} \subset V\right) \text{ by property 1}$$
$$\implies H_*^{\mathrm{Morse}}(S_1 \subset V) = \tilde{H}_*(I(\varphi_1, S_1)) = \tilde{H}_*(I(\varphi_0, S_0)) \text{ by property 2.}$$

The upshot is that we can just work with the unperturbed thing.

But we can't literally do this since we are infinite-dimensional. So we take the finite dimensional approximation V_λ^μ which we define to be the $[\lambda, \mu]$ eigenspace of $l = \mathrm{Hess}_{(0,0)}$. Then we have the following:

Theorem 1 (Manolescu)**.** *There exists $R > 0$ such that*

- $\mathrm{crit}(\mathrm{CSD})$ *and flow lines of* CSD *are contained in $B(R) \subset V$.*
- *The intersection of the critical points and flow-lines intersected with $B(2R)$ (which we call S_λ^μ) is actually contained in $B(R)$.*

Write φ_λ^μ to be the ∇ flow on V_λ^μ. Then

- $I_{\mathrm{Pin}(2)}\left(\varphi_{\lambda'}^{\mu'}, S_{\lambda'}^{\mu'}\right) = I_{\mathrm{Pin}(2)}(\varphi_\lambda^\mu, S_\lambda^\mu) \wedge S^{\left|V_\lambda^{\lambda'}\right|}$ for $\lambda' < \lambda < 0$.
- Metric independence: For metric g_τ we have $n(Y, g) \in 2\mathbb{Z}$ and then we have
$$I_\lambda^\mu(g_1) = I_\lambda^\mu(g_0) \wedge S^{n(Y,g)} .$$

2. Construction of suspension spectrum

We have the following $\mathbb{R}$-irreps of $\mathrm{Pin}(2)$:

$$\mathbb{R} \text{ (trivial)}$$
$$\tilde{\mathbb{R}}\left(j = -1, S^1 = -\mathrm{id}\right)$$
$$\mathbb{H} \text{ (on the right)}$$

The objects are $\left(X, n_{\mathbb{R}}, n_{\tilde{\mathbb{R}}}, n_{\mathbb{H}}\right)$ and the morphisms are:

$$\left[\left(X, n_{\mathbb{R}}, n_{\tilde{\mathbb{R}}}, n_{\mathbb{H}}\right), \left(X, m_{\mathbb{R}}, m_{\tilde{\mathbb{R}}}, m_{\mathbb{H}}\right)\right] = \begin{cases} \lim_{N\to\infty}\left[\Sigma^{N-n_{\mathbb{R}}}\Sigma^{N-n_{\tilde{\mathbb{R}}}}\Sigma^{N-n_{\mathbb{H}}}X, \Sigma\Sigma\Sigma Y\right] & m_p - n_p \in \mathbb{Z} \\ 0 & \text{o/w} \end{cases} \tag{1}$$

Then we can define

$$\mathrm{SWF}(Y) = \Sigma^{H\cdot(-n(Y,g)/4)}\Sigma^{-V_\lambda^0} I_\lambda^\mu(g) .$$