INVOLUTIVE HEEGAARD-FLOER HOMOLOGY

LECTURE: KAI NAKAMURA

1. Refresher

Let Y be a 3-manifold. In particular, take Y to be a $\mathbb{Q}HS^3$. Let Σ be a surface of genus g. Write the α curves as $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$ and similarly for the β-curves $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$. Also pick a base point not in the union $\alpha \cup \beta$.

Then CF^+ is generated over $\mathbb{F}[u]$ by the intersection points of

$$\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g \tag{1}$$

$$\mathbb{T}_b = \beta_1 \times \ldots \beta_g \ . \tag{2}$$

The idea is that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangian submanifolds of

$$\mathrm{Sym}^g \Sigma \coloneqq \Sigma^{\times g} / S_g \tag{3}$$

where S_g is the symmetric group. Write this data as

$$\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \ . \tag{4}$$

We have a decomposition

$$\mathrm{CF}^+ = \bigoplus_{\mathfrak{s} \text{ spin}^c} \mathrm{CF}^+ (Y, \mathfrak{s}) \tag{5}$$

over spin^c structures. There is a differential on this complex, which we won't talk much about, but it counts J-holomorphic disks.

2. Involutive Heegaard-Floer homology

Write

$$\bar{\mathcal{H}} = \left(\bar{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha}, z\right) \tag{6}$$

for the tuple where Σ has opposite orientation and the α-curves and β-curves are switched. Then we get a map

$$\eta = \mathrm{CF}^+ (\mathcal{H}, \mathfrak{s}) \to \mathrm{CF}^+ \left(\bar{\mathcal{H}}, \bar{\mathfrak{s}}\right) \ . \tag{7}$$

Write

$$\varphi\left(\bar{\mathcal{H}}, \mathcal{H}\right) : \mathrm{CF}^+ \left(\bar{\mathcal{H}}, \bar{\mathfrak{s}}\right) \to \mathrm{CF}^+ (\mathcal{H}, \bar{\mathfrak{s}}) \ . \tag{8}$$

Now the involution map is:

$$\iota = \varphi\left(\bar{\mathcal{H}}, \mathcal{H}\right) \circ \eta : \mathrm{CF}^+ (\mathcal{H}, \mathfrak{s}) \to \mathrm{CF}^+ (\mathcal{H}, \bar{\mathfrak{s}}) \ . \tag{9}$$

Lemma 1. *ι^2 is chain homotopic to the identity.*

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This implies that

$$\iota_* : \mathrm{HF}^+(Y, \mathfrak{s}) \to \mathrm{HF}^+(Y, \bar{\mathfrak{s}}) \tag{10}$$

is an involution.

Now we define CFI, the complex for involutive Heegaard Floer. Recall the mapping cone complex of

$$\mathrm{CF}^+(\mathcal{H}, \mathfrak{s}) \xrightarrow{1+\iota} \mathrm{CF}^+\left(\bar{\mathcal{H}}, \mathfrak{s}\right) \tag{11}$$

the direct sum (up to a -1 shift in the first factor) and for any

$$(x, y) \in \mathrm{CF}^+(\mathcal{H}, \mathfrak{s})[-1] \oplus \mathrm{CF}^+\left(\bar{\mathcal{H}}, \mathfrak{s}\right) \tag{12}$$

the differential is given by

$$\begin{bmatrix} \partial & 0 \\ 1+\iota & \partial \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} . \tag{13}$$

We want an $R = \mathbb{F}[Q, v] / \left(v^2\right)$ module where Q is a formal variable of degree -1 and $Q^2 = 0$. So we take the mapping cone of

$$Q(1+\iota) : \mathrm{CF}^+(\mathcal{H}, \mathfrak{s}) \to Q\,\mathrm{CF}^+(\mathcal{H}, \mathfrak{s})[-1] \tag{14}$$

which gives us

$$\mathrm{CFI}^+(\mathcal{H}, \mathfrak{s}) = \left(\mathrm{CF}^+(\mathcal{H}, \mathfrak{s})[-1] \oplus \otimes \mathbb{F}[Q] / \left(Q^1\right), \partial + Q(1+\iota)\right) . \tag{15}$$

Write $\mathrm{HFI}^+(Y, \mathfrak{s})$ for the homology of this complex.

Now let Y be a $\mathbb{Q}HS^3$ with $c_1(\mathfrak{s})$ torsion. This induces a $\mathbb{Q}$ grading on CF^+, and CFI^+.

$\mathrm{HF}^+(Y, \mathfrak{s})$ will decompose as $\mathbb{F}\left[u, u^{-1}\right]/u$ and $\mathrm{HF}^+_{\mathrm{red}}(Y, \mathfrak{s})$. Define $\partial(Y, \mathfrak{s})$ to be the lowest degree of an element in this $\mathbb{F}\left[u, u^{-1}\right] / (u)$ summand. Now we have u-towers:

$$\begin{tikzcd}
\cdots \\
\mathbb{F} \arrow[d, bend left, "Q"] \arrow[dd, bend right, "u"'] \\
\mathbb{F} \arrow[dd, bend right, "u"'] \\
\mathbb{F} \arrow[d, bend left, "Q"] \arrow[dd, bend right, "u"'] \\
\mathbb{F} & \bar{\partial} \arrow[l] \\
\mathbb{F} \arrow[dd, bend right, "u"'] \\
0 \\
\mathbb{F} & \underline{\partial}+1 \arrow[l]
\end{tikzcd} . \tag{16}$$

Define $\underline{\partial}+1$ and $\bar{\partial}$ to be the lowest degrees of the u-towers. $\bar{\partial}$ is defined to lie in the image of q. So we get this inequality

$$\underline{\partial}(Y,\mathfrak{s}) \leq \partial(Y,\mathfrak{s}) \leq \bar{\partial}(Y,\mathfrak{s}) \ . \tag{17}$$

Note that these are maps

$$\partial : \Theta^3_{\mathbb{Z}} \to \mathbb{Z} \qquad \underline{\partial} : \Theta^3_{\mathbb{Z}} \to \mathbb{Z} \qquad \bar{\partial} : \Theta^3_{\mathbb{Z}} \to \mathbb{Z} \tag{18}$$

but neither of the $\underline{\partial}$ and $\bar{\partial}$ are group homomorphisms. Note that we cannot use these for the triangulation conjecture.

Example 1. We have

$$\bar{\partial}(\Sigma(2,3,7)) = \partial(\Sigma(2,3,7)) = 0 \tag{19}$$

and

$$\underline{\partial}(\Sigma(2,3,7)) = -2 \ . \tag{20}$$

This implies that $\Sigma(2,3,7)$ is not homology-cobordant to any L-space.

3. Why HFI?

Basically they are trying to transfer this $\mathrm{Pin}(2)$ symmetry to the Heegaard-Floer setting. HF^+ corresponds to S^1-equivariant SWF. Since $\mathrm{Pin}(2) = S^1 \cup jS^1$, a $\mathrm{Pin}(2)$ version of HF would be $\mathbb{Z}/2\mathbb{Z}$ equivariant HF^+. But we can't really do this. $\mathbb{Z}/2\mathbb{Z}$ equivariant Lagrangian Floer homology would look like the following. The chain complex would look like

$$\mathrm{CF}^{\mathrm{Pin}(2)} = \begin{tikzcd} \mathrm{CF}^+ & \mathrm{CF}^+ \arrow[l, "Q(1+\iota)"'] & \mathrm{CF}^+ \arrow[l, "Q(1+\iota)"'] \arrow[ll, bend left, "Q^2-w"] & \cdots \arrow[l] \arrow[lll, bend left, "Q^2-z"] \end{tikzcd} \tag{21}$$

where w is a chain homotopy from ι^2 to the identity and z is a higher homotopy. But we don't know how to compute it, so we quotient out by Q^2 to kill this.

For X with a $\mathrm{Pin}(2)$ action we have

$$H_*\left(C^{\mathrm{Pin}(2)}(X)/\left(Q^2\right),\mathbb{F}\right) \cong H_*^{\mathbb{Z}/4}(X,\mathbb{F}) \ . \tag{22}$$

So conjecturally, HFI^+ should correspond to $\mathbb{Z}/4$ equivariant SWF.

So in summary, from SWFH^{S^1} (or HF^+) we get ∂, from $\mathrm{SWFH}^{\mathrm{Pin}(2)}$ we get α, β, and γ, and from $\mathrm{SWFH}^{\mathbb{Z}/4}$ we get $\underline{\partial}$ and $\bar{\partial}$.