# Partial Density Functions and Hele-Shaw Flow

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# Overview

# Initial Setup

Let $L$ denote a holomorphic line bundle over a complex manifold $X$ of dimension $n$.
We write $h$ for a hermitian metric on $L$. Now for some smooth potential function $\varphi$, it is the case that[1]

$$h^k = e^{-k\varphi} \tag{1}$$

where $k$ is some integer.

Let $\sigma_1, \sigma_2$ denote two sections of $L$. We define an inner product as follows:

$$\langle \sigma_1, \sigma_2 \rangle_\varphi = \int_X \sigma_1 \bar{\sigma}_2 \exp(-2\varphi(z)) \, \omega_\varphi^{[n]} \tag{2}$$

where $\omega^{[n]}$ is the curvature form corresponding to our potential $\varphi$.

[1] $\varphi$ should really be defined in terms of local holomorphic trivializations and is therefore not immediately globally defined.

# Partial Bergman Kernel

We let $H^{\epsilon k}_{k\varphi}$ be the subspace of holomorphic sections of $L^k$ vanishing to order $\epsilon k$ along $Y$.

## Definition: Partial Bergman Kernel (PBK)

Let $K^{\epsilon}_{k}$ be a sequence of sections of $L^k \bigotimes \bar{L}^k$ (over $X \times X$). This sequence is called a PBK if for all sections $s$ in $H^{\epsilon k}_{k\varphi}$, for any fixed $z$ in $X$, and for all $\zeta$, we have $K^{\epsilon}_{k}(z, \zeta) \in \bar{L}^k_z \bigotimes H^{\epsilon k}_{k\varphi}$ and

$$s(z) = \langle s, K^{\epsilon}_{k,z} \rangle_{\varphi k} = \int_X s(\zeta) \bar{K}^{\epsilon}_{k}(z, \zeta) e^{-2k\varphi(\zeta)} \omega^{[n]}_{\varphi\zeta} \tag{3}$$

Intuitively, sections can be thought of as functions on $X$. The PBK consists of sections with convenient properties with respect to our $L^2$ norm. Roughly, if two sections in the PBK are close in norm, they will be close in pointwise value.

# Partial Density Functions

## Definition: Partial Density Function (PDF)

The norm of the restriction of the PBK to the diagonal of some compact submanifold $Y$ is called the Partical Density Function. Therefore the PDF $\rho_k^\varepsilon$ is written

$$\rho_k^\epsilon(z) := \left|K_k^\epsilon(z,z)\right|_{k\varphi} \tag{4}$$

As a concrete example, consider the space of polynomials over $\mathbb{C}^n$ with the inner product,

$$(p,q) = \int_{\mathbb{C}^n} p(z)\bar{q}(z)e^{-k|z|^2} \tag{5}$$

For some orthonormal basis of polynomials $\{p_i\}$ the PDF is the sum of the inner products of these vectors.

$$\rho_k^\epsilon = \Sigma_i |p_i|^2 e^{-k|z|^2} \tag{6}$$

# Circle Invariant Case

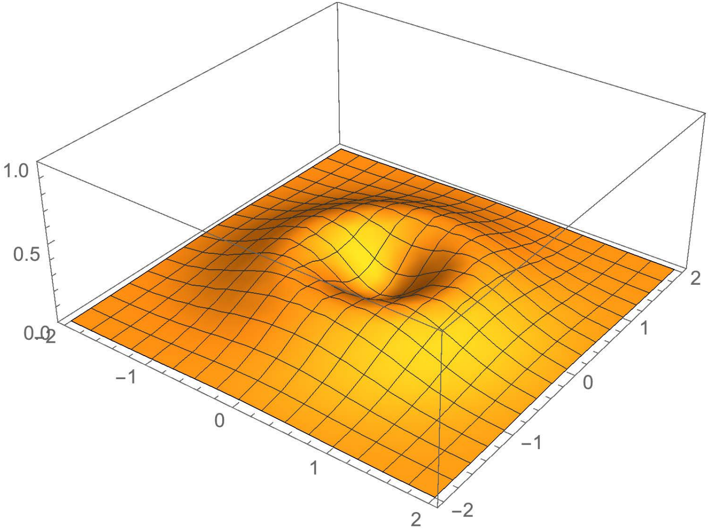
If we assume the potential function $\varphi$ is rotationally symmetric the situation becomes easier to analyze.

In particular, in this scenario there is a convenient orthonormal basis. This means we are in a very similar situation to the polynomial case from the last slide.

In short, there is a forbidden region where this density function is exponentially small. These regions corresponds to circles of varying radii. In addition, this density function is shown to have a gaussian error function behavior along the boundary of these regions. [3]

The natural question to be asked next, is what happens if we don't assume this symmetry?

# Forbidden Region



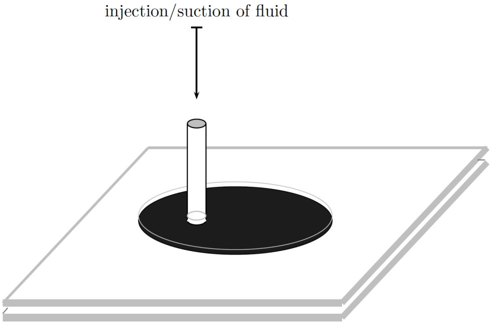

The forbidden region for $\epsilon = 1$

# Summary of Key Ideas so far

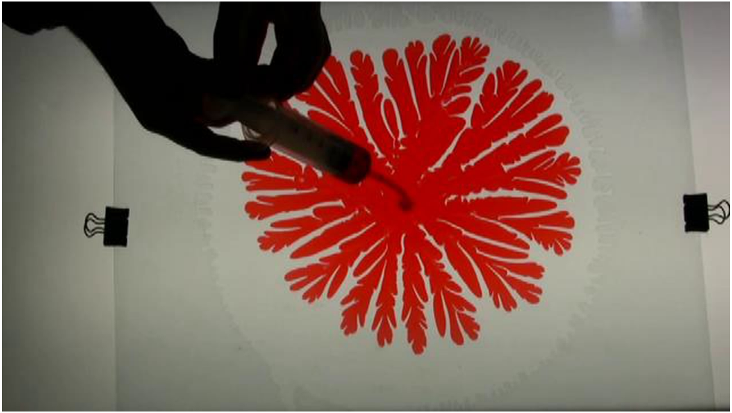
- Take the illustrative example of holomorphic functions on the complex plane,
- Consider the functions that vanish to some order $\epsilon k$.
- Assume the whole situation is circularly symmetric.
- Therefore take the convenient orthonormal basis of polynomials, and express our PDF as a sum of their inner products
- This sum of products is exponentially close to 0 near the origin.
- At a radius of approximately $\epsilon$, the PDF exhibits gaussian behavior.
- This means as we increase our choice of $\epsilon$, the PDF is exponentially small within a larger and larger circle.
- Without symmetry, this boundary will also change, but it will not simply increase as a circle...

# Physical Setup of Hele-Shaw Flow

Hele-Shaw flow is a certain type of two-dimensional fluid flow introduced by Henry Selby Hele-Shaw in 1898. [1]



# Viscous Fingering



Video [4]

# Applications of Hele-Shaw Flow

- Gaseous boundary interactions
- Wave modeling
- Tumor growth
- Worldsheet of a bosonic string
- The PDF question from earlier

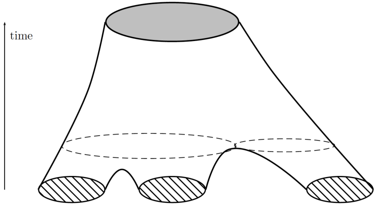



Figure 7.1: Hele-Shaw worldsheet

[1]

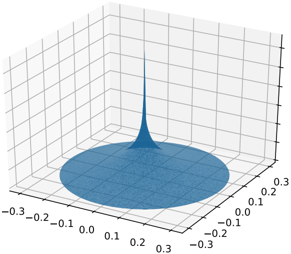
# The Dirichlet Problem

Given a certain region $\Omega$, the solution to the canonical dirichlet problem is a function which satisfies some differential equation (usually in terms of the laplacian) inside the region, and is equal to some value on the boundary. For example,

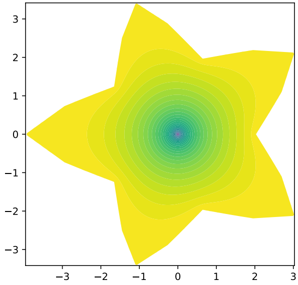
$$\Delta u = (\partial_x^2 + \partial_y^2) u \tag{7}$$

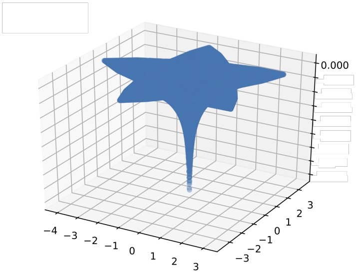
$$= -\delta_0 \text{ in } \Omega \tag{8}$$

$$u_\nu | \partial D = 0 \tag{9}$$



# Star Domain



# The Neumann Problem

Given a domain $\Omega$, the Neumann problem prescribes a condition inside the region similar to the Dirichlet problem, but instead of a constant value, it requires a normal derivative.

Because of uniqueness for both of these problems, their solution does not coincide in general.

However, if you do find a boundary which gives a solution to both of these problems, it has some special characteristics.
Notably, the Hele-Shaw flow can be defined as the domains which allow for these solutions to coexist. [2]

# Pluri-Subharmonic Functions and Extremal Envelopes

Roughly speaking, a plurisubharmonic function is upper-semicontinuous, and satisfies the mean value inequality for every open ball within the relevant domain. For the purposes here, it is sufficient to describe pluri-subharmonic functions has having laplacian greater than 0.

## Extremal Envelope

Given a plurisubharmonic potential on a region, the extremal envelope for a certain constant $\epsilon$ is defined as:

$$\varphi_\epsilon = \sup\{\lambda \text{ defined on } \Omega : \Delta\gamma \geq 0, \gamma \geq \varphi, \nu(\gamma) \geq \epsilon\} \tag{10}$$

where $\nu$ is the lelong number at the origin.

Essentially this means $\gamma$ has to be less than $\varphi$, it has to have a singularity at the origin, and it has to be subharmonic.

# Envelopes, Dirichlet and Neumann Problems

It can be shown, that if we define a region

$$F_\nu = \{z : \varphi_\epsilon(z) < \varphi(z)\} \tag{11}$$

The function $\varphi_\epsilon$ solves the dirichlet problem and the Neumann problem on this region.

Furthermore, these methods can be applied to the PDF problem posed earlier. As it turns out, we can define a clever basis in terms of these envelope functions and get some of the same convenient behavior, without assuming any symmetry.

# PDF and Hele-Shaw Flow

Recall the notion of the PDF from the beginning of the presentation. When we dropped the symmetry it is not obvious what shape the new domains will take.

It turns out, in the general case, there exists a basis, for which the Hele-Shaw flow gives us these vanishing domains.

In particular, consider the sections defined as:

## Peaked Sections

Consider the extremal envelope $u_\epsilon$. If we are working in a simply connected domain, we always have a function $g_\epsilon$ such that $\Re[g_\epsilon] = u_\epsilon - \epsilon \log|z|$ Our sections will then be defined as:

$$\sigma_n = z^n \exp(kg) \tag{12}$$

# Peak Along the Boundary

We can write the inner products for these sections as follow:

$$|\sigma_n|^2 = \exp(2kg_\epsilon - 2k\varphi)) \tag{13}$$
$$= \exp(2k(u_\epsilon - \varphi)) \tag{14}$$

Now we can taylor expand this. For brevity I jump to the eventual expression:

$$u_\epsilon - \varphi = -1/2(\Delta\varphi(0))x^2 + \mathcal{O}(x^3) + \mathcal{O}(yx^2) \tag{15}$$

We can therefore write,

$$\exp(2k(u_\epsilon - \varphi)) \approx \exp(-k\Delta\varphi(0)x^2) \tag{16}$$

# The Fordbidden Region for the General Case

As before, we can write:

$$|\sigma_n|^2 = \exp(2k(u_\epsilon - \varphi)) \tag{17}$$

Now recall, that the function $u_\epsilon$ goes like $\log|z|^2$ near the origin.

In addition, we know that $\varphi$ goes as $\log(1 + |z|^2)$ so we have that their difference is roughly negative infinity at the origin, and 0 at the boundary.

Therefore, in the expression for the inner product of the sections goes roughly as $\exp(-\log(1 + z^{-2}))$ which is exponentially small at the origin.

Intuitively these sections allow the envelope to jump into the exponential and play with the potential in a manner reminiscent of the invariant case.

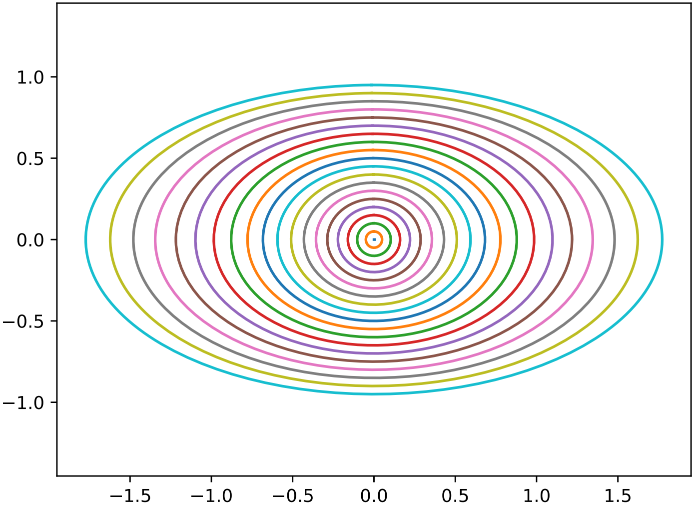
# Orthonormality of this Basis

In order to get sections peaked conveniently at the boundary, we have given up complete orthonormality.

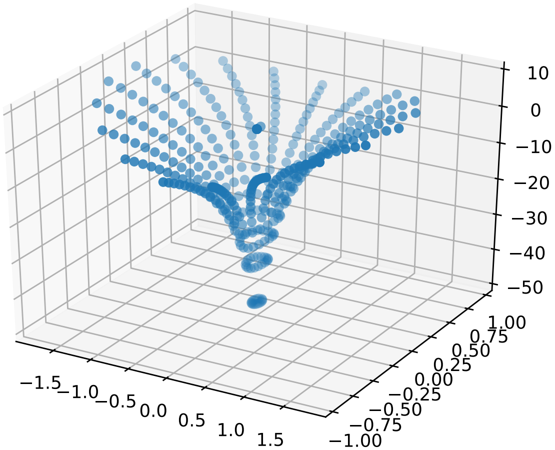
Instead there is a decay of inner products. As the chosen values of $\epsilon$ move farther apart, the inner products decay as well.

This is expected because the gaussian bumps will overlap less and less, as the boundaries get further from one another.

# Domains



# Envelope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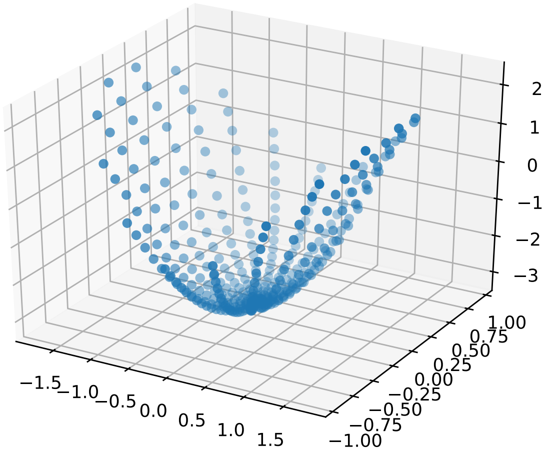
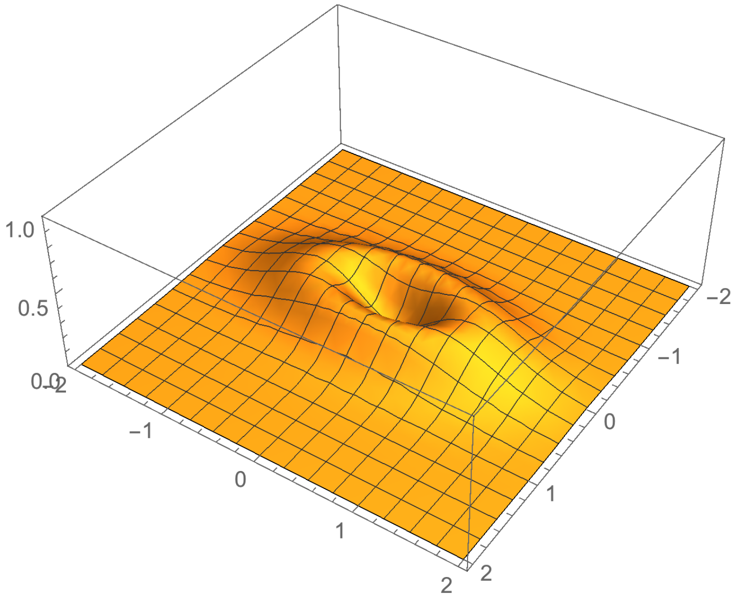
# Potential



# Section Inner Product



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Nervous System, *Hele-shaw cell experiments*.

# The End

Questions?