

GLOBAL WEAK SOLUTIONS TO COMPRESSIBLE QUANTUM NAVIER-STOKES EQUATIONS WITH DAMPING

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ABSTRACT. The global-in-time existence of weak solutions to the barotropic compressible quantum Navier-Stokes equations with damping is proved for large data in three dimensional space. The model consists of the compressible Navier-Stokes equations with degenerate viscosity, and a nonlinear third-order differential operator, with the quantum Bohm potential, and the damping terms. The global weak solutions to such system is shown by using the Faedo-Galerkin method and the compactness argument. This system is also a very important approximation to the compressible Navier-Stokes equations. It will help us to prove the existence of global weak solutions to the compressible Navier-Stokes equations with degenerate viscosity in three dimensional space.

1. INTRODUCTION

In this paper, we are interested in the existence of global weak solutions to the barotropic compressible quantum Navier-Stokes equations with damping terms

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho^\gamma - \operatorname{div}(\rho \mathbb{D} \mathbf{u}) &= -r_0 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u} + \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \end{aligned} \quad (1.1)$$

with initial data as follows

$$\rho(0, x) = \rho_0(x), \quad (\rho \mathbf{u})(0, x) = \mathbf{m}_0(x) \quad \text{in } \Omega, \quad (1.2)$$

where ρ is density, $\gamma > 1$, $\mathbf{u} \otimes \mathbf{u}$ is the matrix with components $\mathbf{u}_i \mathbf{u}_j$, $\mathbb{D} \mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the symmetric part of the velocity gradient, and $\Omega = \mathbb{T}^d$ is the d -dimensional torus, here $d = 2$ or 3 . The expression $\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$ is called as Bohm potential which can be interpreted as a quantum potential. The quantum Navier-Stokes equations have a lot of applications, in particular, quantum semiconductors [5], weakly interacting Bose gases [9] and quantum trajectories of Bohmian mechanics [15]. Recently some dissipative quantum fluid models have been derived by Jüngel, see [10]. The damping terms

$$-r_0 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u}$$

is motivated by the work of [1]. It allows us to recover the weak solutions to (1.1) by passing to the limits from the suitable approximation. The most importance is that the existence of solutions for the system (1.1) studied in the current paper is crucial to show

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the existence of weak solutions for the Navier Stokes equations with degenerate viscosity, see [14]. Models with these drag terms are also common in the literature, see [1, 2, 4].

When $r_0 = r_1 = \kappa = 0$ in (1.1), the system reduces to the compressible Navier-Stokes equations with degenerate viscosity $\mu(\rho) = \nu\rho$. The existence of global weak solutions of such system has been a long standing open problem. In the case $\gamma = 2$ in 2D, this corresponds to the shallow water equations, where $\rho(t, x)$ stands for the height of the water at position x , and time t , and $\mathbf{u}(t, x)$ is the 2D velocity at the same position, and same time. For the constant viscosity case, Lions in [12] established the global existence of renormalized solutions for $\gamma > \frac{9}{5}$, and Feireisl-Novotný-Petzeltová [6] and Feireisl [7] extended the existence results to $\gamma > \frac{3}{2}$, and even to Navier-Stokes-Fourier system. The first tool of handling the degenerate viscosity is due to Bresch, Desjardins and Lin, see [3], where the authors deduced a new mathematical entropy to show the structure of the diffusion terms providing some regularity for the density. It was later extended for the case with an additional quadratic friction term $r\rho|\mathbf{u}|^2$, see Bresch-Desjardins [1, 2]. Meanwhile, Mellet-Vasseur [13] deduced an estimate for proving the stability of smooth solutions for the compressible Navier-Stokes equations.

When $r_0 = r_1 = 0$ in (1.1), the system reduces to the so-called quantum Navier-Stokes equations. Up to our knowledge, there are no existence theorem of weak solutions for large data in any dimensional space. Compared to the degenerate compressible Navier-Stokes equations, we need to overcome the additional mathematical difficulty from the strongly nonlinear third- order differential operator. We have to mention that the Mellet-Vasseur type inequality does not hold for the quantum Navier-Stokes equations due to the quantum potential. Thus, there are short of the suitable a priori estimates for proving the weak stability. Jüngel [11] used the test function of the form $\rho\varphi$ to handle the convection term, thus he proved the existence of such a particular weak solution. In a very recent preprint, Gisclon-Violet [8] proved the existence of weak solutions to the quantum Navier-Stokes equations with singular pressure, where the authors adopt some arguments in [16] to make use of the cold pressure for compactness. Our methodology turns out to be very close to their paper. Actually, the authors of [8] mention that the existence can be obtained replacing the cold pressure by a drag force.

The existence of weak solutions to (1.1), with the uniform bounds of Theorem 1.1, is crucial for the existence of weak solutions to the compressible Navier-Stokes equations with degenerate viscosity in 3D, see [14]. In that work, we started from the weak solutions to (1.1), that is, the main result of this current paper. Unfortunately, the version with the cold pressure proved in [8], is not suitable for the result in [14]. On the approximation in [14], we need the terms $r_1\rho|\mathbf{u}|^2\mathbf{u}$ and $\kappa\rho(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})$ for proving a key lemma. In particular, inequality (1.6) is crucial to prove the existence of weak solutions to the compressible Navier-Stokes equations in 3D. This estimate is from the term $\kappa\rho\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})$.

We can deduce the following energy inequality for smooth solutions of (1.1)

$$E(t) + \int_0^T \int_{\Omega} \rho |\mathbb{D}\mathbf{u}|^2 dx dt + r_0 \int_0^T \int_{\Omega} |\mathbf{u}|^2 dx dt + r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 dx dt \leq E_0, \quad (1.3)$$

where

$$E(t) = E(\rho, \mathbf{u})(t) = \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{\gamma-1} \rho^\gamma + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 \right) dx,$$

and

$$E_0 = E(\rho, \mathbf{u})(0) = \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{1}{\gamma-1} \rho_0^\gamma + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 \right) dx.$$

However, we should point out that the above a priori estimate are not enough to show the stability of the solutions of (1.1), in particular, for the compactness of ρ^γ . we have the following Bresch-Desjardins entropy (see [1, 3]) for providing more regularity of the density

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u} + \nabla \ln \rho|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 - r_0 \log \rho \right) dx + \int_0^T \int_{\Omega} |\nabla \rho^{\frac{\gamma}{2}}|^2 dx dt \\ & + \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx dt + \kappa \int_0^T \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \\ & \leq \int_{\Omega} \left(\rho_0 |\mathbf{u}_0|^2 + |\nabla \sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 - r_0 \log_- \rho_0 \right) dx + C, \end{aligned} \quad (1.4)$$

where C is bounded by the initial energy, $\log_- g = \log \min(g, 1)$.

Thus, the initial data should be given in such a way

$$\begin{aligned} \rho_0 & \in L^\gamma(\Omega), \quad \rho_0 \geq 0, \quad \nabla \sqrt{\rho_0} \in L^2(\Omega), \quad -\log_- \rho_0 \in L^1(\Omega), \\ \mathbf{m}_0 & \in L^1(\Omega), \quad \mathbf{m}_0 = 0 \quad \text{if } \rho_0 = 0, \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega). \end{aligned} \quad (1.5)$$

We define the weak solution (ρ, \mathbf{u}) to the initial value problem (1.1) in the following sense: for any $t \in [0, T]$,

- (1.2) holds in $\mathcal{D}'(\Omega)$,
- (1.3) and (1.4) hold for almost every $t \in [0, T]$,
- (1.1) holds in $\mathcal{D}'((0, T) \times \Omega)$ and the following is satisfied

$$\begin{aligned} \rho & \in L^\infty(0, T; L^\gamma(\Omega)), & \sqrt{\rho} \mathbf{u} & \in L^\infty(0, T; L^2(\Omega)), \\ \nabla \sqrt{\rho} & \in L^\infty(0, T; L^2(\Omega)), & \nabla \rho^{\frac{\gamma}{2}} & \in L^2(0, T; L^2(\Omega)), \\ \sqrt{\rho} \mathbb{D} \mathbf{u} & \in L^2(0, T; L^2(\Omega)), & \sqrt{\rho} \nabla \mathbf{u} & \in L^2(0, T; L^2(\Omega)), \\ \rho^{\frac{1}{4}} \mathbf{u} & \in L^4(0, T; L^4(\Omega)), & \mathbf{u} & \in L^2(0, T; L^2(\Omega)), \\ \sqrt{\rho} |\nabla^2 \log \rho| & \in L^2(0, T; L^2(\Omega)). \end{aligned}$$

The following is our main result.

Theorem 1.1. *If the initial data satisfy (1.5), there exists a weak solution (ρ, \mathbf{u}) to (1.1)-(1.2) for any $\gamma > 1$, any $T > 0$, in particular, the weak solution (ρ, \mathbf{u}) satisfies energy inequality (1.3), BD-entropy (1.4) and the following inequality:*

$$\kappa^{\frac{1}{2}} \|\sqrt{\rho}\|_{L^2(0, T; H^2(\Omega))} + \kappa^{\frac{1}{4}} \|\nabla \rho^{\frac{1}{4}}\|_{L^4(0, T; L^4(\Omega))} \leq C, \quad (1.6)$$

where C only depends on the initial data. Moreover, the weak solution (ρ, \mathbf{u}) has the following properties

$$\rho \mathbf{u} \in C([0, T]; L_{weak}^{\frac{3}{2}}(\Omega)), \quad (\sqrt{\rho})_t \in L^2((0, T) \times \Omega); \quad (1.7)$$

If we use $(\rho_\kappa, \mathbf{u}_\kappa)$ to denote the weak solution for $\kappa > 0$, then

$$\sqrt{\rho_\kappa} \mathbf{u}_\kappa \rightarrow \sqrt{\rho} \mathbf{u} \text{ strongly in } L^2((0, T) \times \Omega), \text{ as } \kappa \rightarrow 0, \quad (1.8)$$

where (ρ, \mathbf{u}) in (1.8) is a weak solution to (1.1)-(1.2) with $\kappa = 0$. We remark the metric space $C([0, T]; L_{weak}^{\frac{3}{2}}(\Omega))$ of function $f : [0, T] \rightarrow L^\gamma(\Omega)$ which are continuous with respect to the weak topology.

Remark 1.1. We will use (1.6)- (1.7) in [14] to prove the weak solutions to (1.1) with $r_0 = r_1 = \kappa = 0$. In fact, inequality (1.6) is very crucial to prove a key lemma in [14].

Remark 1.2. The existence result contains the case with $\kappa = 0$, which can be obtained as the limit when $\kappa > 0$ goes to 0 in (1.1), by standard compactness analysis.

Remark 1.3. The weak formulation reads as

$$\begin{aligned} & \int_{\Omega} \rho \mathbf{u} \cdot \psi \, dx \Big|_{t=0}^{t=T} - \int_0^T \int_{\Omega} \rho \mathbf{u} \psi_t \, dx \, dt - \int_0^T \int_{\Omega} \rho \mathbf{u} \otimes \mathbf{u} : \nabla \psi \, dx \, dt \\ & - \int_0^T \int_{\Omega} \rho^\gamma \operatorname{div} \psi \, dx \, dt - \int_0^T \int_{\Omega} \rho \mathbb{D} \mathbf{u} : \nabla \psi \, dx \, dt \\ & = -r_0 \int_0^T \int_{\Omega} \mathbf{u} \psi \, dx \, dt - r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^2 \mathbf{u} \psi \, dx \, dt - 2\kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi \, dx \, dt \\ & - \kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \sqrt{\rho} \operatorname{div} \psi \, dx \, dt. \end{aligned} \quad (1.9)$$

for any test function ψ .

2. FAEDO-GALERKIN APPROXIMATION

In this section, we construct the solutions to the approximation scheme by Faedo-Galerkin method. Motivated by the work of Feireisl-Novotný-Petzelová [6] and Feireisl [7], we proceed similarly as in Jüngel [11]. We introduce a finite dimensional space $X_N = \operatorname{span}\{e_1, e_2, \dots, e_N\}$, where $N \in \mathbb{N}$, each e_i be an orthonormal basis of $L^2(\Omega)$ which is also an orthogonal basis of $H^2(\Omega)$. We notice that $\mathbf{u} \in C^0([0, T]; X_N)$ is given by

$$\mathbf{u}(t, x) = \sum_{i=1}^N \lambda_i(t) e_i(x), \quad (t, x) \in [0, T] \times \Omega,$$

for some functions $\lambda_i(t)$, and the norm of \mathbf{u} in $C^0([0, T]; X_N)$ can be written as

$$\|\mathbf{u}\|_{C^0([0, T]; X_N)} = \sup_{t \in [0, T]} \sum_{i=1}^N |\lambda_i(t)|.$$

And hence, \mathbf{u} can be bounded in $C^0([0, T]; C^k(\Omega))$ for any $k \geq 0$, thus

$$\|\mathbf{u}\|_{C^0([0, T]; C^k(\Omega))} \leq C(k) \|\mathbf{u}\|_{C^0([0, T]; L^2(\Omega))}.$$

For any given $\mathbf{u} \in C^0([0, T]; X_N)$, by the classical theory of parabolic equation, there exists a classical solution $\rho(t, x) \in C^1([0, T]; C^3(\Omega))$ to the following approximated system

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \quad \rho(0, x) = \rho_0(x) \quad \text{in } (0, T) \times \Omega \quad (2.1)$$

with the initial data

$$\rho(0, x) = \rho_0(x) \geq \nu > 0, \quad \text{and } \rho_0(x) \in C^\infty(\Omega), \quad (2.2)$$

where $\nu > 0$ is a constant.

We should remark that this solution $\rho(t, x)$ satisfies the following inequality

$$\inf_{x \in \Omega} \rho_0(x) \exp^{-\int_0^T \|\operatorname{div} \mathbf{u}\|_{L^\infty(\Omega)} ds} \leq \rho(t, x) \leq \sup_{x \in \Omega} \rho_0(x) \exp^{\int_0^T \|\operatorname{div} \mathbf{u}\|_{L^\infty(\Omega)} ds} \quad (2.3)$$

for all (t, x) in $(0, T) \times \Omega$. By (2.2) and (2.3), there exists a constant $\theta_0 > 0$ such that

$$0 < \theta_0 \leq \rho(t, x) \leq \frac{1}{\theta_0} \quad \text{for } (t, x) \in (0, T) \times \Omega. \quad (2.4)$$

Thus, we can introduce a linear continuous operator $S : C^0([0, T]; X_N) \rightarrow C^0([0, T]; C^k(\Omega))$ by $S(\mathbf{u}) = \rho$, and

$$\|S(\mathbf{u}_1) - S(\mathbf{u}_2)\|_{C^0([0, T]; C^k(\Omega))} \leq C(N, k) \|\mathbf{u}_1 - \mathbf{u}_2\|_{C^0([0, T]; L^2(\Omega))} \quad (2.5)$$

for any $k \geq 1$.

The Faedo-Galerkin approximation for the weak formulation of the momentum balance is as follows

$$\begin{aligned} & \int_{\Omega} \rho \mathbf{u}(T) \varphi \, dx - \int_{\Omega} \mathbf{m}_0 \varphi \, dx + \mu \int_0^T \int_{\Omega} \Delta \mathbf{u} \cdot \Delta \varphi \, dx \, dt - \int_0^T \int_{\Omega} (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi \, dx \, dt \\ & + \int_0^T \int_{\Omega} 2\rho \mathbb{D} \mathbf{u} : \nabla \varphi \, dx \, dt - \int_0^T \int_{\Omega} \rho^\gamma \nabla \varphi \, dx \, dt + \eta \int_0^T \int_{\Omega} \rho^{-10} \nabla \varphi \, dx \, dt \\ & + \varepsilon \int_0^T \int_{\Omega} \nabla \rho \cdot \nabla \mathbf{u} \varphi \, dx \, dt = -r_0 \int_0^T \int_{\Omega} \mathbf{u} \varphi \, dx \, dt - r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^2 \mathbf{u} \varphi \, dx \, dt \\ & - 2\kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi \, dx \, dt - \kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \sqrt{\rho} \operatorname{div} \psi \, dx \, dt + \delta \int_0^T \int_{\Omega} \rho \nabla \Delta^9 \rho \varphi \, dx \, dt, \end{aligned} \quad (2.6)$$

for any test function $\varphi \in X_N$. The extra terms $\eta \nabla \rho^{-10}$ and $\delta \rho \nabla \Delta^9 \rho$ are necessary to keep the density bounded, and bounded away from zero for all time. This enables us to take $\frac{\nabla \rho}{\rho}$ as a test function to derive the Bresch-Desjardins entropy.

To solve (2.6), we follow the same arguments as in [6, 7, 11] and introduce the following operators, given the density function $\rho(t, x) \in L^1(\Omega)$ with $\rho \geq \underline{\rho} > 0$, here we choose $\underline{\rho} = \theta_0$. We define

$$\mathfrak{M}[\rho(t), \cdot] : X_N \rightarrow X_N^*, \quad \langle \mathfrak{M}[\rho] \mathbf{u}, w \rangle = \int_{\Omega} \rho \mathbf{u} \cdot w \, dx, \quad \text{for } \mathbf{u}, w \in X_N.$$

We can show that $\mathfrak{M}[\rho]$ is invertible

$$\|\mathfrak{M}^{-1}(\rho)\|_{L(X_N^*, X_N)} \leq \underline{\rho}^{-1},$$

where $L(X_N^*, X_N)$ is the set of all bounded linear mappings from X_N^* to X_N . It is Lipschitz continuous in the following sense

$$\|\mathfrak{M}^{-1}(\rho_1) - \mathfrak{M}^{-1}(\rho_2)\|_{L(X_N^*, X_N)} \leq C(N, \rho) \|\rho_1 - \rho_2\|_{L^1(\Omega)} \quad (2.7)$$

for any ρ_1 and ρ_2 from the following set

$$N_\nu = \{\rho \in L^1(\Omega) \mid \inf_{x \in \Omega} \rho \geq \nu > 0.\}$$

For more details, we refer the readers to [6, 7, 11].

We are looking for $\mathbf{u}_n \in C([0, T]; X_n)$ solution of the following nonlinear integral equation

$$\mathbf{u}_N(t) = \mathfrak{M}^{-1}(S(\mathbf{u}_N))(t) \left(\mathfrak{M}[\rho_0](\mathbf{u}_0) + \int_0^t \mathfrak{N}(S(\mathbf{u}_N), \mathbf{u}_N)(s) ds \right), \quad (2.8)$$

where

$$\begin{aligned} \mathfrak{N}(S(\mathbf{u}_N), \mathbf{u}_N) = & -\operatorname{div}(\rho \mathbf{u}_N \otimes \mathbf{u}_N) + \operatorname{div}(\rho \mathbb{D} \mathbf{u}_N) + \mu \Delta^2 \mathbf{u}_N - \varepsilon \nabla \rho \cdot \nabla \mathbf{u}_N + \eta \nabla \rho^{-10} - \nabla \rho^\gamma \\ & - r_0 \mathbf{u}_N - r_1 \rho |\mathbf{u}_N|^2 \mathbf{u}_N + \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \delta \rho \nabla \Delta^9 \rho, \end{aligned}$$

$\rho = S(\mathbf{u}_N)$.

Thanks to (2.5) and (2.7), we can apply a fixed point argument to solve the nonlinear equation (2.8) on a short time interval $[0, T^*]$ for $T^* \leq T$, in the space $C^0([0, T^*]; X_N)$. Thus, there exists a local-in-time solution (ρ_N, \mathbf{u}_N) to (2.1), (2.8). Observe that L^2 -norm and C^2 -norm are equivalent on X_N .

Differentiating (2.6) with respect to time t and taking $\varphi = \mathbf{u}_N$, we have the following energy balance

$$\begin{aligned} & \frac{d}{dt} E(\rho_N, \mathbf{u}_N) + \mu \int_{\Omega} |\Delta \mathbf{u}_N|^2 dx + \int_{\Omega} \rho_N |\mathbb{D} \mathbf{u}_N|^2 dx + \varepsilon \delta \int_{\Omega} |\Delta^5 \rho_N|^2 dx \\ & + \varepsilon \int_{\Omega} |\nabla \rho_N^{\frac{\gamma}{2}}|^2 dx + \varepsilon \eta \int_{\Omega} |\nabla \rho_N^{-5}|^2 dx + r_0 \int_{\Omega} |\mathbf{u}_N|^2 dx + r_1 \int_{\Omega} \rho_N |\mathbf{u}_N|^4 dx \\ & + \kappa \varepsilon \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx = 0, \end{aligned} \quad (2.9)$$

on $[0, T^*]$, where

$$E(\rho_N, \mathbf{u}_N) = \int_{\Omega} \left(\frac{1}{2} \rho_N |\mathbf{u}_N|^2 + \frac{\rho_N^\gamma}{\gamma - 1} + \frac{\eta}{10} \rho_N^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho_N}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho_N|^2 \right) dx,$$

and

$$E_0(\rho_N, \mathbf{u}_N) = \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} + \frac{\eta}{10} \rho_0^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho_0|^2 \right) dx.$$

Here we used the identity

$$2\rho_N \nabla \left(\frac{\Delta \sqrt{\rho_N}}{\sqrt{\rho_N}} \right) = \operatorname{div}(\rho_N \nabla^2(\log \rho_N))$$

to yield

$$\int_{\Omega} \frac{\Delta \sqrt{\rho_N}}{\sqrt{\rho_N}} \Delta \rho_N dx = - \int_{\Omega} \rho_N \nabla \log \rho_N \cdot \nabla \left(\frac{\Delta \sqrt{\rho_N}}{\sqrt{\rho_N}} \right) dx = \frac{1}{2} \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx.$$

Energy equality (2.9) yields

$$\int_0^{T^*} \|\Delta \mathbf{u}_N\|_{L^2}^2 dt \leq E_0(\rho_n, \mathbf{u}_N) < \infty. \quad (2.10)$$

Due to $\dim X_N < \infty$ and (2.3), there exists a constant $\theta_0 > 0$ such that

$$0 < \theta_0 \leq \rho_N(t, x) \leq \frac{1}{\theta_0} \quad (2.11)$$

for all $t \in (0, T^*)$. However, this θ_0 depends on N and it is the same to θ_0 in (2.4). Energy equality (2.9) gives us

$$\sup_{t \in (0, T^*)} \int_0^{T^*} \rho_N |\mathbf{u}_N|^2 dx \leq E_0(\rho_N, \mathbf{u}_N) < \infty,$$

and

$$\int_0^{T^*} \rho_N |\mathbb{D} \mathbf{u}_N|^2 dx dt \leq E_0(\rho_N, \mathbf{u}_N) < \infty,$$

which, together with (2.10), (2.11), implies

$$\sup_{0 \in (0, T^*)} (\|\mathbf{u}_N\|_{L^\infty} + \|\nabla \mathbf{u}_N\|_{L^\infty} + \|\Delta \mathbf{u}_N\|_{L^\infty}) \leq C(E_0(\rho_N, \mathbf{u}_N), N), \quad (2.12)$$

where we used a fact that the equivalence of L^2 and L^∞ on X_N . By (2.5), (2.7), (2.11) and (2.12), repeating our above arguments many times, we can extend T^* to T . Thus there exists a solution (ρ_N, \mathbf{u}_N) to (2.1), (2.8) for any $T > 0$.

Here we need to state the following lemma due to Jüngel [11]:

Lemma 2.1. *For any smooth positive function $\rho(x)$, we have*

$$\int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx \geq \frac{1}{7} \int_{\Omega} |\nabla^2 \sqrt{\rho}|^2 dx$$

and

$$\int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx \geq \frac{1}{8} \int_{\Omega} |\nabla \rho^{\frac{1}{4}}|^4 dx.$$

Proof. The above inequality of Lemma is firstly proved by Jüngel [11]. Here, we give a quick proof.

We notice

$$\sqrt{\rho} \cdot \nabla^2 \log \sqrt{\rho} = \sqrt{\rho} \cdot \nabla \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) = \nabla^2 \sqrt{\rho} - \frac{\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}}{\sqrt{\rho}}, \quad (2.13)$$

thus

$$\begin{aligned} \int_{\Omega} \rho |\nabla^2 \log \sqrt{\rho}|^2 dx &= \int_{\Omega} |\nabla^2 \sqrt{\rho}|^2 dx + \int_{\Omega} |2 \nabla \rho^{\frac{1}{4}}|^4 dx - 2 \int_{\Omega} \nabla^2 \sqrt{\rho} \cdot \frac{\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}}{\sqrt{\rho}}, \\ &= A + B - I, \end{aligned}$$

For I , we control it as follows

$$\begin{aligned} I &= 2 \int_{\Omega} \nabla^2 \sqrt{\rho} \cdot \left(\frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \otimes \nabla \sqrt{\rho} \right) dx \\ &= -2 \int_{\Omega} \frac{|\nabla \sqrt{\rho}|^2}{\sqrt{\rho}} \Delta \log \sqrt{\rho} dx - 2 \int_{\Omega} \nabla^2 \sqrt{\rho} \cdot \frac{\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}}{\sqrt{\rho}} dx. \end{aligned}$$

Hence:

$$2I = -2 \int_{\Omega} \frac{|\nabla \sqrt{\rho}|^2}{\sqrt{\rho}} \Delta \log \sqrt{\rho} dx \leq 2\sqrt{3BD},$$

where $D = \int_{\Omega} \rho |\nabla^2 \log \sqrt{\rho}|^2 dx$, and hence

$$A + B = D + I \leq (1 + 6)D + \frac{1}{8}B,$$

and thus,

$$\frac{1}{7}A + \frac{1}{8}B \leq D.$$

So we proved this lemma. \square

By (2.9), we have

$$E(\rho_N, \mathbf{u}_N) \leq E_0(\rho_N, \mathbf{u}_N),$$

this gives us

$$\|\rho_N\|_{L^\infty(0,T;H^9(\Omega))} \leq C(E_0(\rho_N, \mathbf{u}_N), \delta),$$

this, together with (2.11), gives us that the density $\rho(t, x)$ is a positive smooth function for all (t, x) . We also notice that

$$\kappa\varepsilon \int_0^T \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx dt \leq E_0(\rho_N, \mathbf{u}_N) < \infty.$$

By Lemma 2.1, we have the following uniform estimate:

$$(\kappa\varepsilon)^{\frac{1}{2}} \|\sqrt{\rho_N}\|_{L^2(0,T;H^2(\Omega))} + (\kappa\varepsilon)^{\frac{1}{4}} \|\nabla \rho_N^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \leq C, \quad (2.14)$$

where the constant $C > 0$ is independent of N .

To conclude this part, we have the following lemma on the approximate solutions (ρ_N, \mathbf{u}_N) :

Proposition 2.1. *Let (ρ_N, \mathbf{u}_N) be the solution of (2.1), (2.8) on $(0, T) \times \Omega$ constructed above, then we have the following energy inequality*

$$\begin{aligned} & \sup_{t \in (0, T)} E(\rho_N, \mathbf{u}_N) + \mu \int_0^T \int_{\Omega} |\Delta \mathbf{u}_N|^2 dx dt + \int_0^T \int_{\Omega} \rho_N |\mathbb{D} \mathbf{u}_N|^2 dx dt + \varepsilon \delta \int_0^T \int_{\Omega} |\Delta^5 \rho_N|^2 dx dt \\ & + \varepsilon \int_0^T \int_{\Omega} |\nabla \rho_N^{\frac{\gamma}{2}}|^2 dx dt + \varepsilon \eta \int_0^T \int_{\Omega} |\nabla \rho_N^{-5}|^2 dx dt + r_0 \int_0^T \int_{\Omega} |\mathbf{u}_N|^2 dx dt \\ & + r_1 \int_0^T \int_{\Omega} \rho_N |\mathbf{u}_N|^4 dx dt + \kappa\varepsilon \int_0^T \int_{\Omega} \rho_N |\nabla^2 \log \rho_N|^2 dx dt \leq E_0(\rho_N, \mathbf{u}_N), \end{aligned} \quad (2.15)$$

where

$$E(\rho_N, \mathbf{u}_N) = \int_{\Omega} \left(\frac{1}{2} \rho_N |\mathbf{u}_N|^2 + \frac{\rho_N^{\frac{\gamma}{2}}}{\gamma - 1} + \frac{\eta}{10} \rho_N^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho_N}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho_N|^2 \right) dx.$$

Moreover, we have the following uniform estimate:

$$(\kappa\varepsilon)^{\frac{1}{2}} \|\sqrt{\rho_N}\|_{L^2(0,T;H^2(\Omega))} + (\kappa\varepsilon)^{\frac{1}{4}} \|\nabla \rho_N^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \leq C, \quad (2.16)$$

where the constant $C > 0$ is independent of N .

In particular, we have the following estimates

$$\sqrt{\rho_N} \mathbf{u}_N \in L^\infty(0, T; L^2(\Omega)), \sqrt{\rho_N} \mathbb{D} \mathbf{u}_N \in L^2((0, T) \times \Omega), \sqrt{\mu} \Delta \mathbf{u}_N \in L^2((0, T) \times \Omega), \quad (2.17)$$

$$\sqrt{\varepsilon \delta} \Delta^5 \rho_N \in L^2((0, T) \times \Omega), \sqrt{\delta} \rho_N \in L^\infty(0, T; H^9(\Omega)), \sqrt{\kappa} \sqrt{\rho_N} \in L^\infty(0, T; H^1(\Omega)), \quad (2.18)$$

$$\nabla \rho_N^{\frac{7}{2}} \in L^2((0, T) \times \Omega), \quad \rho_N^{-1} \in L^\infty(0, T; L^{10}(\Omega)), \sqrt{\varepsilon \eta} \nabla \rho_N^{-5} \in L^2((0, T) \times \Omega), \quad (2.19)$$

$$\mathbf{u}_N \in L^2((0, T) \times \Omega), \quad \rho_N^{\frac{1}{4}} \mathbf{u}_N \in L^4((0, T) \times \Omega). \quad (2.20)$$

Based on above estimates, we have the following estimates uniform on N :

Lemma 2.2. *The following estimates hold for any fixed positive constants ε , μ , η and δ :*

$$\|(\sqrt{\rho_N})_t\|_{L^2((0, T) \times \Omega)} + \|\sqrt{\rho_N}\|_{L^2(0, T; H^2(\Omega))} \leq K, \quad (2.21)$$

$$\|(\rho_N)_t\|_{L^2((0, T) \times \Omega)} + \|\rho_N\|_{L^2(0, T; H^{10}(\Omega))} \leq K, \quad (2.22)$$

$$\|(\rho_N \mathbf{u}_N)_t\|_{L^2(0, T; H^{-9}(\Omega))} + \|\rho_N \mathbf{u}_N\|_{L^2((0, T) \times \Omega)} \leq K, \quad (2.23)$$

$$\nabla(\rho_N \mathbf{u}_N) \text{ is uniformly bounded in } L^4(0, T; L^{\frac{6}{5}}(\Omega)) + L^2(0, T; L^{\frac{3}{2}}(\Omega)). \quad (2.24)$$

$$\|\rho_N^\gamma\|_{L^{\frac{5}{3}}((0, T) \times \Omega)} \leq K, \quad (2.25)$$

$$\|\rho_N^{-10}\|_{L^{\frac{5}{3}}((0, T) \times \Omega)} \leq K, \quad (2.26)$$

where K is independent of N , depends on ε , μ , η and δ .

Proof. By (2.16), we have

$$\|\sqrt{\rho_N}\|_{L^2(0, T; H^2(\Omega))} \leq C.$$

We notice that

$$\begin{aligned} (\rho_N)_t &= -\rho_N \operatorname{div} \mathbf{u}_N - \nabla \rho_N \cdot \mathbf{u}_N \\ &= -(4 \nabla \rho_N^{\frac{1}{4}})(\rho_N^{\frac{1}{4}} \mathbf{u}_N)(\rho_N^{\frac{1}{2}}) - \sqrt{\rho_N} \sqrt{\rho_N} \operatorname{div} \mathbf{u}_N, \end{aligned}$$

which gives us

$$\begin{aligned} \|(\rho_N)_t\|_{L^2((0, T) \times \Omega)} &\leq 4 \|\nabla \rho_N^{\frac{1}{4}}\|_{L^4((0, T) \times \Omega)} \|\rho_N^{\frac{1}{4}} \mathbf{u}_N\|_{L^4((0, T) \times \Omega)} \|\rho_N^{\frac{1}{2}}\|_{L^\infty((0, T) \times \Omega)} \\ &\quad + \|\sqrt{\rho_N}\|_{L^\infty((0, T) \times \Omega)} \|\sqrt{\rho_N} \nabla \mathbf{u}_N\|_{L^2((0, T) \times \Omega)}, \end{aligned}$$

thanks to (2.17)-(2.20) and Sobelov inequality.

Meanwhile, we have

$$\begin{aligned} 2(\sqrt{\rho_N})_t &= -\sqrt{\rho_N} \operatorname{div} \mathbf{u}_N - 2 \nabla \sqrt{\rho_N} \cdot \mathbf{u}_N \\ &= -\sqrt{\rho_N} \operatorname{div} \mathbf{u}_N - 8 \nabla \rho_N^{\frac{1}{4}} \rho_N^{\frac{1}{4}} \mathbf{u}_N, \end{aligned}$$

which yields $(\sqrt{\rho_N})_t$ is bounded in $L^2((0, T) \times \Omega)$.

Here we claim that $(\rho_N \mathbf{u}_N)_t$ is bounded in $L^2(0, T; H^{-9}(\Omega))$. By

$$\begin{aligned} (\rho_N \mathbf{u}_N)_t &= -\operatorname{div}(\rho_N \mathbf{u}_N \otimes \mathbf{u}_N) - \nabla \rho_N^\gamma + \eta \nabla \rho_N^{-10} + \mu \Delta^2 \mathbf{u}_N + \operatorname{div}(\rho_N \mathbb{D} \mathbf{u}_N) - r_0 \mathbf{u}_N \\ &\quad - r_1 \rho_N |\mathbf{u}_N|^2 \mathbf{u}_N + \varepsilon \nabla \rho_N \cdot \nabla \mathbf{u}_N + \kappa \rho_N \nabla \left(\frac{\Delta \sqrt{\rho_N}}{\sqrt{\rho_N}} \right) + \delta \rho_N \nabla \Delta^9 \rho_N, \end{aligned}$$

we can show the claim by the above estimates.

And

$$\|\rho_N \mathbf{u}_N\|_{L^2((0,T)\times\Omega)} \leq \|\rho_N^{\frac{3}{4}}\|_{L^\infty(0,T;L^4(\Omega))} \|\rho_N^{\frac{1}{4}} \mathbf{u}_N\|_{L^4((0,T)\times\Omega)} \leq K,$$

where we used Sobelov inequality and (2.16). Thus we have (2.22).

We calculate

$$\nabla(\rho_N \mathbf{u}_N) = \nabla \sqrt{\rho_N} \rho_N^{\frac{1}{4}} \mathbf{u}_N \rho^{\frac{1}{4}} + \sqrt{\rho_N} \sqrt{\rho_N} \nabla \mathbf{u}_N,$$

it allows us to have (2.23). For any given $\varepsilon > 0$, we have

$$\|\nabla \rho_N^{\frac{\gamma}{2}}\|_{L^2((0,T)\times\Omega)} \leq K,$$

which gives us

$$\|\rho_N^\gamma\|_{L^1(0,T;L^3(\Omega))} \leq K.$$

Notice

$$\rho_N^\gamma \in L^\infty(0,T;L^1(\Omega)),$$

we apply Hölder inequality to have

$$\|\rho_N^\gamma\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq \|\rho_N^\gamma\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{2}{5}} \|\rho_N^\gamma\|_{L^1(0,T;L^3(\Omega))}^{\frac{3}{5}} \leq K.$$

Similarly, we can show (2.25). □

Applying Aubin-Lions Lemma and Lemma 2.2, we conclude

$$\rho_N \rightarrow \rho \quad \text{strongly in } L^2(0,T;H^9(\Omega)), \quad \text{weakly in } L^2(0,T;H^{10}(\Omega)), \quad (2.26)$$

$$\sqrt{\rho_N} \rightarrow \sqrt{\rho} \quad \text{strongly in } L^2(0,T;H^1(\Omega)), \quad \text{weakly in } L^2(0,T;H^2(\Omega))$$

and

$$\rho_N \mathbf{u}_N \rightarrow \rho \mathbf{u} \quad \text{strongly in } L^2((0,T)\times\Omega). \quad (2.27)$$

We notice that $\mathbf{u}_N \in L^2((0,T)\times\Omega)$, thus,

$$\mathbf{u}_N \rightarrow \mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega).$$

Thus, we can pass into the limits for term $\rho_N \mathbf{u}_N \otimes \mathbf{u}_N$ as follows

$$\rho_N \mathbf{u}_N \otimes \mathbf{u}_N \rightarrow \rho \mathbf{u} \otimes \mathbf{u}$$

in the distribution sense.

Here we state the following lemma on the convergence of $\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N$.

Lemma 2.3. *When $N \rightarrow \infty$, we have*

$$\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \rightarrow \rho |\mathbf{u}|^2 \mathbf{u} \quad \text{strongly in } L^1(0,T;L^1(\Omega)).$$

Proof. Fatou's lemma yields

$$\int_{\Omega} \rho |\mathbf{u}|^4 dx \leq \int_{\Omega} \liminf \rho_N |\mathbf{u}_N|^4 dx \leq \liminf \int_{\Omega} \rho_N |\mathbf{u}_N|^4 dx,$$

and hence $\rho |\mathbf{u}|^4$ is in $L^1(0,T;L^1(\Omega))$.

By (2.26) and (2.27), we have, up to a subsequence, such that

$$\rho_N \rightarrow \rho(t,x) \quad \text{a.e.}$$

and

$$\rho_N \mathbf{u}_N \rightarrow \rho \mathbf{u} \quad \text{a.e.}$$

Thus, for almost every (t, x) such that when $\rho_N(t, x) \neq 0$, we have

$$\mathbf{u}_N = \frac{\rho_N \mathbf{u}_N}{\rho_N} \rightarrow \mathbf{u}.$$

For almost every (t, x) such that $\rho_N(t, x) = 0$, then

$$\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M} \leq M^3 \rho_N = 0 = \rho |\mathbf{u}|^2 \mathbf{u} \chi_{|\mathbf{u}| \leq M}.$$

Hence, $\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M}$ converges to $\rho |\mathbf{u}|^2 \mathbf{u} \chi_{|\mathbf{u}| \leq M}$ almost everywhere for (t, x) . Meanwhile, $\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ thanks to (2.18).

The dominated convergence theorem gives us

$$\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M} \rightarrow \rho |\mathbf{u}|^2 \mathbf{u} \chi_{|\mathbf{u}| \leq M} \quad \text{strongly in } L^1(0, T; L^1(\Omega)). \quad (2.28)$$

For any $M > 0$, we have

$$\begin{aligned} & \int_0^T \int_\Omega |\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N - \rho |\mathbf{u}|^2 \mathbf{u}| \, dx \, dt \\ & \leq \int_0^T \int_\Omega |\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M} - \rho |\mathbf{u}|^2 \mathbf{u} \chi_{|\mathbf{u}| \leq M}| \, dx \, dt \\ & \quad + 2 \int_0^T \int_\Omega \rho_N |\mathbf{u}_N|^3 \chi_{|\mathbf{u}_N| \geq M} \, dx \, dt + 2 \int_0^T \int_\Omega \rho |\mathbf{u}|^3 \chi_{|\mathbf{u}| \geq M} \, dx \, dt \\ & \leq \int_0^T \int_\Omega |\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \chi_{|\mathbf{u}_N| \leq M} - \rho |\mathbf{u}|^2 \mathbf{u} \chi_{|\mathbf{u}| \leq M}| \, dx \, dt \\ & \quad + \frac{2}{M} \int_0^T \int_\Omega \rho_N |\mathbf{u}_N|^4 \, dx \, dt + \frac{2}{M} \int_0^T \int_\Omega \rho |\mathbf{u}|^4 \, dx \, dt. \end{aligned} \quad (2.29)$$

Thanks to (2.28), we have

$$\limsup_{\varepsilon, \mu \rightarrow 0} \|\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N - \rho |\mathbf{u}|^2 \mathbf{u}\|_{L^1(0, T; L^1(\Omega))} \leq \frac{C}{M}$$

for fixed $C > 0$ and all $M > 0$. Letting $M \rightarrow \infty$, we have

$$\rho_N |\mathbf{u}_N|^2 \mathbf{u}_N \rightarrow \rho |\mathbf{u}|^2 \mathbf{u} \quad \text{strongly in } L^1(0, T; L^1(\Omega)).$$

□

By (2.24) and ρ_N^γ converges almost everywhere to ρ^γ , we have

$$\rho_N^\gamma \rightarrow \rho^\gamma \quad \text{strongly in } L^1((0, T) \times \Omega).$$

Meanwhile, we have to mention the following Sobolev inequality, see [2, 16],

$$\|\rho^{-1}\|_{L^\infty(\Omega)} \leq C(1 + \|\rho\|_{H^{k+2}(\Omega)})^2(1 + \|\rho^{-1}\|_{L^3(\Omega)})^3,$$

for $k \geq \frac{3}{2}$. Thus the estimates on density from (2.17)-(2.19) enable us to use the above Sobolev inequality to have

$$\|\rho\|_{L^\infty((0, T) \times \Omega)} \geq C(\delta, \eta) > 0, \quad \text{a. e. in } (0, T) \times \Omega. \quad (2.30)$$

This enables us to have ρ_N^{-10} converges almost everywhere to ρ^{-10} . Thanks to (2.25), we have

$$\rho_N^{-10} \rightarrow \rho^{-10} \quad \text{strongly in } L^1((0, T) \times \Omega).$$

By the above compactness, we are ready to pass into the limits as $N \rightarrow \infty$ in the approximation system (2.1), (2.8). Thus, we have shown that (ρ, \mathbf{u}) solves

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho \quad \text{pointwise in } (0, T) \times \Omega,$$

and for any test function φ such that the following integral hold

$$\begin{aligned} & \int_{\Omega} \rho \mathbf{u}(T) \varphi \, dx - \int_{\Omega} \mathbf{m}_0 \varphi \, dx + \mu \int_0^T \int_{\Omega} \Delta \mathbf{u} \cdot \Delta \varphi \, dx \, dt - \int_0^T \int_{\Omega} (\rho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi \, dx \, dt \\ & + \int_0^T \int_{\Omega} 2\rho \mathbb{D} \mathbf{u} : \nabla \varphi \, dx \, dt - \int_0^T \int_{\Omega} \rho^\gamma \nabla \varphi \, dx \, dt + \eta \int_0^T \int_{\Omega} \rho^{-10} \nabla \varphi \, dx \, dt \\ & + \varepsilon \int_0^T \int_{\Omega} \nabla \rho \cdot \nabla \mathbf{u} \varphi \, dx \, dt = -r_0 \int_0^T \int_{\Omega} \mathbf{u} \varphi \, dx \, dt - r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^2 \mathbf{u} \varphi \, dx \, dt \\ & - 2\kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi \, dx \, dt - \kappa \int_0^T \int_{\Omega} \Delta \sqrt{\rho} \sqrt{\rho} \operatorname{div} \psi \, dx \, dt + \delta \int_0^T \int_{\Omega} \rho \nabla \Delta^9 \rho \varphi \, dx \, dt. \end{aligned} \tag{2.31}$$

Thanks to the weak lower semicontinuity of convex functions, we can pass into the limits in the energy inequality (2.15), by the strong convergence of the density and velocity, we have the following energy inequality in the sense of distributions on $(0, T)$

$$\begin{aligned} & \sup_{t \in (0, T)} E(\rho, \mathbf{u}) + \mu \int_0^T \int_{\Omega} |\Delta \mathbf{u}|^2 \, dx \, dt + \int_0^T \int_{\Omega} \rho |\mathbb{D} \mathbf{u}|^2 \, dx \, dt + \varepsilon \delta \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 \, dx \, dt \\ & + \varepsilon \int_0^T \int_{\Omega} |\nabla \rho^{\frac{7}{2}}|^2 \, dx \, dt + \varepsilon \eta \int_0^T \int_{\Omega} |\nabla \rho^{-5}|^2 \, dx \, dt + r_0 \int_0^T \int_{\Omega} |\mathbf{u}|^2 \, dx \, dt \\ & + r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 \, dx \, dt + \kappa \varepsilon \int_0^T \int_{\Omega} \rho |\nabla^2 \log \rho|^2 \, dx \, dt \leq E_0, \end{aligned} \tag{2.32}$$

where

$$E(\rho, \mathbf{u}) = \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{\eta}{10} \rho^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho|^2 \right) \, dx.$$

Thus, we have the following Lemma on the existence of weak solutions at this level approximation system.

Proposition 2.2. *There exists a weak solution (ρ, \mathbf{u}) to the following system*

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= \varepsilon \Delta \rho, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho^\gamma - \eta \nabla \rho^{-10} - \operatorname{div}(\rho \mathbb{D} \mathbf{u}) - \mu \Delta^2 \mathbf{u} + \varepsilon \nabla \rho \cdot \nabla \mathbf{u} \\ &= -r_0 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u} + \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \delta \rho \nabla \Delta^9 \rho, \end{aligned}$$

with suitable initial data, for any $T > 0$. In particular, the weak solutions (ρ, \mathbf{u}) satisfies the energy inequality (2.32) and (2.30).

3. BRESCH-DESJARDINS ENTROPY AND VANISHING LIMITS

The goal of this section is to deduce the Bresch-Desjardins Entropy for the approximation system in Proposition 2.2, and to rely on it to pass into the limits as ε , μ , η , δ go to zero. By (2.18) and (2.30), we have

$$\rho(t, x) \geq C(\delta, \eta) > 0 \quad \text{and} \quad \rho \in L^2(0, T; H^{10}(\Omega)) \cap L^\infty(0, T; H^9(\Omega)). \quad (3.1)$$

3.1. BD entropy. Thanks to (3.1), we can use $\varphi = \nabla(\log \rho)$ to test the momentum equation to derive the Bresch-Desjardins entropy. Thus, we have

Lemma 3.1.

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u} + \frac{\nabla \rho}{\rho}|^2 + \frac{\delta}{2} |\nabla^9 \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\rho^{-10}}{10} \right) dx + \eta \int_{\Omega} |\nabla \rho^{-5}|^2 dx \\ & + \int_{\Omega} |\nabla \rho^{\frac{7}{2}}|^2 dx + \delta \varepsilon \int_{\Omega} |\Delta^5 \rho|^2 dx + 2\delta \int_{\Omega} |\Delta^5 \rho|^2 dx + \frac{1}{2} \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx \\ & + \mu \int_{\Omega} |\Delta \mathbf{u}|^2 dx + \kappa \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx + \varepsilon \int_{\Omega} \frac{|\Delta \rho|^2}{\rho} dx \\ & = \varepsilon \int_{\Omega} \nabla \rho \cdot \nabla \mathbf{u} \cdot \nabla \log \rho dx + \varepsilon \int_{\Omega} \Delta \rho \frac{|\nabla \log \rho|^2}{2} dx - \varepsilon \int_{\Omega} \operatorname{div}(\rho \mathbf{u}) \frac{1}{\rho} \Delta \rho dx \\ & - \mu \int_{\Omega} \Delta \mathbf{u} \cdot \nabla \Delta \log \rho dx - r_1 \int_{\Omega} |\mathbf{u}|^2 \mathbf{u} \nabla \rho dx - r_0 \int_{\Omega} \frac{\mathbf{u} \cdot \nabla \rho}{\rho} dx \\ & = R_1 + R_2 + R_3 + R_4 + R_5 + R_6. \end{aligned}$$

We can follow the same way as in [16] to deduce the above equality, and control terms R_i for $i = 1, 2, 3, 4$, and they approach to zero as $\varepsilon \rightarrow 0$ or $\mu \rightarrow 0$. We estimate R_5 as follows

$$|R_5| \leq C \int_{\Omega} \rho |\mathbf{u}|^2 |\nabla \mathbf{u}| dx \leq C \int_{\Omega} \rho |\mathbf{u}|^4 dx + \frac{1}{8} \int_{\Omega} \rho |\nabla \mathbf{u}|^2 dx,$$

and for R_6 we have

$$R_6 = r_0 \int_{\Omega} \frac{\rho_t + \rho \operatorname{div} \mathbf{u} - \varepsilon \Delta \rho}{\rho} dx = r_0 \int_{\Omega} (\log \rho)_t dx - \varepsilon r_0 \int_{\Omega} \frac{\Delta \rho}{\rho} dx.$$

since ρ is uniformly bounded in $L^\infty(0, T; L^\gamma(\Omega))$, we have

$$r_0 \int_{\Omega} \log_+ \rho dx \leq C, \quad \text{where} \quad \log_+ g = \log \max(g, 1).$$

Thus, we need to assume that $-r_0 \int_{\Omega} \log_- \rho_0 dx$ is uniformly bounded in $L^1(\Omega)$. Also we can control

$$\left| \varepsilon r_0 \int_{\Omega} \frac{\Delta \rho}{\rho} dx \right| \leq \varepsilon \|\rho\|_{H^2(\Omega)} \|\rho^{-1}\|_{L^\infty(\Omega)},$$

and it goes to zero as $\varepsilon \rightarrow 0$.

Thus, we have the following inequality

$$\begin{aligned}
& \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u} + \frac{\nabla \rho}{\rho}|^2 + \frac{\delta}{2} |\nabla^9 \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\rho^{-10}}{10} - r_0 \log \rho \right) dx + \eta \int_0^T \int_{\Omega} |\nabla \rho^{-5}|^2 dx dt \\
& + \int_0^T \int_{\Omega} |\nabla \rho^{\frac{\gamma}{2}}| dx dt + \delta \varepsilon \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 dx dt + 2\delta \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 dx dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx dt + \mu \int_0^T \int_{\Omega} |\Delta \mathbf{u}|^2 dx + \kappa \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \\
& \leq \sum_{i=1}^4 R_i + \varepsilon \|\rho\|_{H^2(\Omega)} \|\rho^{-1}\|_{L^\infty(\Omega)} + C \int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 dx dt + \frac{1}{8} \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u}|^2 dx dt + \\
& \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0 + \frac{\nabla \rho_0}{\rho_0}|^2 + \frac{\delta}{2} |\nabla^9 \rho_0|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\rho_0^{-10}}{10} - r_0 \log_- \rho_0 \right) dx,
\end{aligned} \tag{3.2}$$

where $\int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 dx dt$ is bounded by the initial energy, and $\frac{1}{8} \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u}|^2 dx dt$ can be controlled by

$$\int_0^T \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx dt, \quad \text{and} \quad \int_0^T \int_{\Omega} \rho |\mathbb{D} \mathbf{u}|^2 dx dt.$$

In-deed, it can be controlled by

$$\int_0^T \int_{\Omega} \rho |\nabla \mathbf{u}|^2 dx dt \leq \int_{\Omega} \left(\rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla \sqrt{\rho_0}|^2 - r_0 \log_- \rho_0 \right) dx + 2E_0.$$

Thus, (3.2) gives us

$$\begin{aligned}
& \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u} + \frac{\nabla \rho}{\rho}|^2 + \frac{\delta}{2} |\nabla^9 \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\rho^{-10}}{10} - r_0 \log \rho \right) dx + \eta \int_0^T \int_{\Omega} |\nabla \rho^{-5}|^2 dx dt \\
& + \int_0^T \int_{\Omega} |\nabla \rho^{\frac{\gamma}{2}}| dx dt + \delta \varepsilon \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 dx dt + 2\delta \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 dx dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx dt + \mu \int_0^T \int_{\Omega} |\Delta \mathbf{u}|^2 dx + \kappa \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \\
& \leq 2 \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0 + \frac{\nabla \rho_0}{\rho_0}|^2 + \frac{\delta}{2} |\nabla^9 \rho_0|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\rho_0^{-10}}{10} - r_0 \log_- \rho_0 \right) dx \\
& + \sum_{i=1}^4 R_i + \varepsilon \|\rho\|_{H^2(\Omega)} \|\rho^{-1}\|_{L^\infty(\Omega)} + 2E_0.
\end{aligned} \tag{3.3}$$

Thus, we infer the following estimate from the Bresch-Desjardins entropy

$$\kappa \int_0^T \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \leq C,$$

where C is independent on ε , η , μ , δ .

Applying Lemma 2.1, we have the following uniform estimate:

$$\kappa^{\frac{1}{2}} \|\sqrt{\rho}\|_{L^2(0,T;H^2(\Omega))} + \kappa^{\frac{1}{4}} \|\nabla \rho^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \leq C,$$

where the constant $C > 0$ is independent on $\varepsilon, \eta, \mu, \delta$.

3.2. Passing to the limits as $\varepsilon, \mu \rightarrow 0$. We use $(\rho_{\varepsilon, \mu}, \mathbf{u}_{\varepsilon, \mu})$ to denote the solutions at this level of approximation. It is easy to find that $(\rho_{\varepsilon, \mu}, \mathbf{u}_{\varepsilon, \mu})$ has the following uniform estimates

$$\sqrt{\rho_{\varepsilon, \mu}} \mathbf{u}_{\varepsilon, \mu} \in L^\infty(0, T; L^2(\Omega)), \sqrt{\rho_{\varepsilon, \mu}} \mathbb{D} \mathbf{u}_{\varepsilon, \mu} \in L^2((0, T) \times \Omega), \sqrt{\mu} \Delta \mathbf{u}_{\varepsilon, \mu} \in L^2((0, T) \times \Omega), \quad (3.4)$$

$$\sqrt{\varepsilon} \Delta^5 \rho_{\varepsilon, \mu} \in L^2((0, T) \times \Omega), \sqrt{\delta} \rho_{\varepsilon, \mu} \in L^\infty(0, T; H^9(\Omega)), \sqrt{\kappa} \sqrt{\rho_{\varepsilon, \mu}} \in L^\infty(0, T; H^1(\Omega)), \quad (3.5)$$

$$\rho_{\varepsilon, \mu}^{-1} \in L^\infty(0, T; L^{10}(\Omega)), \sqrt{\varepsilon \eta} \nabla \rho_{\varepsilon, \mu}^{-5} \in L^2((0, T) \times \Omega), \quad (3.6)$$

$$\mathbf{u}_{\varepsilon, \mu} \in L^2((0, T) \times \Omega), \rho_{\varepsilon, \mu}^{\frac{1}{4}} \mathbf{u}_{\varepsilon, \mu} \in L^4((0, T) \times \Omega). \quad (3.7)$$

By the Bresch-Desjardins entropy, we also have the following additional estimates

$$\nabla \sqrt{\rho_{\varepsilon, \mu}} \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\delta} \Delta^5 \rho_{\varepsilon, \mu} \in L^2(0, T; L^2(\Omega)), \quad (3.8)$$

and

$$\nabla \rho_{\varepsilon, \mu}^{\frac{\gamma}{2}} \in L^2((0, T) \times \Omega), \quad \sqrt{\eta} \nabla \rho_{\varepsilon, \mu}^{-5} \in L^2((0, T) \times \Omega). \quad (3.9)$$

Also we have the following uniform estimate

$$\kappa^{\frac{1}{2}} \|\sqrt{\rho_{\varepsilon, \mu}}\|_{L^2(0, T; H^2(\Omega))} + \kappa^{\frac{1}{4}} \|\nabla \rho_{\varepsilon, \mu}^{\frac{1}{4}}\|_{L^4(0, T; L^4(\Omega))} \leq C, \quad (3.10)$$

where the constant $C > 0$ is independent of $\varepsilon, \eta, \mu, \delta$.

By Lemma 3.1, one deduces

$$\int_0^T \int_\Omega \rho_{\varepsilon, \mu} |\nabla \mathbf{u}_{\varepsilon, \mu} - \nabla^T \mathbf{u}_{\varepsilon, \mu}|^2 dx dt \leq C,$$

which together with (3.4), yields

$$\int_0^T \int_\Omega \rho_{\varepsilon, \mu} |\nabla \mathbf{u}_{\varepsilon, \mu}|^2 dx dt \leq C, \quad (3.11)$$

where the constant $C > 0$ is independent of $\varepsilon, \eta, \mu, \delta$. Based on above estimates, we have the following estimates uniform in ε :

Lemma 3.2. *The following estimates holds:*

$$\begin{aligned} & \|(\sqrt{\rho_{\varepsilon, \mu}})_t\|_{L^2(0, T; L^2(\Omega))} + \|\sqrt{\rho_{\varepsilon, \mu}}\|_{L^2(0, T; H^2(\Omega))} \leq K, \\ & \|(\rho_{\varepsilon, \mu})_t\|_{L^2(0, T; L^{\frac{3}{2}}(\Omega))} + \|\rho_{\varepsilon, \mu}\|_{L^\infty(0, T; H^9(\Omega))} + \|\rho_{\varepsilon, \mu}\|_{L^2(0, T; H^{10}(\Omega))} \leq K, \end{aligned} \quad (3.12)$$

$$\|(\rho_{\varepsilon, \mu} \mathbf{u}_{\varepsilon, \mu})_t\|_{L^2(0, T; H^{-9}(\Omega))} + \|\rho_{\varepsilon, \mu} \mathbf{u}_{\varepsilon, \mu}\|_{L^2((0, T) \times \Omega)} \leq K, \quad (3.13)$$

$$\nabla(\rho_{\varepsilon, \mu} \mathbf{u}_{\varepsilon, \mu}) \text{ is uniformly bounded in } L^4(0, T; L^{\frac{6}{5}}(\Omega)) + L^2(0, T; L^{\frac{3}{2}}(\Omega)). \quad (3.14)$$

$$\|\rho_{\varepsilon, \mu}^\gamma\|_{L^{\frac{5}{3}}((0, T) \times \Omega)} \leq K, \quad (3.15)$$

$$\|\rho_{\varepsilon, \mu}^{-10}\|_{L^{\frac{5}{3}}((0, T) \times \Omega)} \leq K, \quad (3.16)$$

where K is independent of ε, μ .

Proof. By (3.4)-(3.11), following the same way as in the proof of Lemma 2.2, we can prove the above estimates.

Applying Aubin-Lions Lemma and Lemma 3.2, we conclude

$$\rho_{\varepsilon,\mu} \rightarrow \rho \quad \text{strongly in } C(0, T; H^9(\Omega)), \quad \text{weakly in } L^2(0, T; H^{10}(\Omega)), \quad (3.17)$$

$$\sqrt{\rho_{\varepsilon,\mu}} \rightarrow \sqrt{\rho} \quad \text{strongly in } L^2(0, T; H^1(\Omega)), \quad \text{weakly in } L^2(0, T; H^2(\Omega))$$

and

$$\rho_{\varepsilon,\mu} \mathbf{u}_{\varepsilon,\mu} \rightarrow \rho \mathbf{u} \quad \text{strongly in } L^2((0, T) \times \Omega). \quad (3.18)$$

We notice that $\mathbf{u}_{\varepsilon,\mu} \in L^2((0, T) \times \Omega)$, thus,

$$\mathbf{u}_{\varepsilon,\mu} \rightarrow \mathbf{u} \quad \text{weakly in } L^2((0, T) \times \Omega).$$

Thus, we can pass into the limits for term $\rho_{\varepsilon,\mu} \mathbf{u}_{\varepsilon,\mu} \otimes \mathbf{u}_{\varepsilon,\mu}$ as follows

$$\rho_{\varepsilon,\mu} \mathbf{u}_{\varepsilon,\mu} \otimes \mathbf{u}_{\varepsilon,\mu} \rightarrow \rho \mathbf{u} \otimes \mathbf{u}$$

in the distribution sense.

We can show

$$\rho_{\varepsilon,\mu} |\mathbf{u}_{\varepsilon,\mu}|^2 \mathbf{u}_{\varepsilon,\mu} \rightarrow \rho |\mathbf{u}|^2 \mathbf{u} \quad \text{strongly in } L^1(0, T; L^1(\Omega))$$

as the same to Lemma 2.3.

Here we state the following lemma on the strong convergence of $\sqrt{\rho_n} \mathbf{u}_n$, which will be used later again. The proof is essential same to [13].

Lemma 3.3. *If $\rho_n^{\frac{1}{4}} \mathbf{u}_n$ is bounded in $L^4(0, T; L^4(\Omega))$, ρ_n almost everywhere converges to ρ , $\rho_n \mathbf{u}_n$ almost everywhere converges to $\rho \mathbf{u}$, then*

$$\sqrt{\rho_n} \mathbf{u}_n \rightarrow \sqrt{\rho} \mathbf{u} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

Proof. Fatou's lemma yields

$$\int_{\Omega} \rho |\mathbf{u}|^4 dx \leq \int_{\Omega} \liminf \rho_n |\mathbf{u}_n|^4 dx \leq \liminf \int_{\Omega} \rho_n |\mathbf{u}_n|^4 dx,$$

and hence $\rho |\mathbf{u}|^4$ is in $L^1(0, T; L^4(\Omega))$.

For almost every (t, x) such that when $\rho_n(t, x) \neq 0$, we have

$$\mathbf{u}_n = \frac{\rho_n \mathbf{u}_n}{\rho_n} \rightarrow \mathbf{u}.$$

For almost every (t, x) such that $\rho_n(t, x) = 0$, then

$$\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M} \leq M \sqrt{\rho_n} = 0 = \sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \leq M}.$$

Hence, $\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M}$ converges to $\sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \leq M}$ almost everywhere for (t, x) . Meanwhile, $\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M}$ is uniformly bounded in $L^\infty(0, T; L^3(\Omega))$.

The dominated convergence theorem gives us

$$\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M} \rightarrow \sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \leq M} \quad \text{strongly in } L^2(0, T; L^2(\Omega)). \quad (3.19)$$

For any $M > 0$, we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} |\sqrt{\rho_n} \mathbf{u}_n - \sqrt{\rho} \mathbf{u}|^2 dx dt \\
& \leq \int_0^T \int_{\Omega} |\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M} - \sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \leq M}|^2 dx dt \\
& + 2 \int_0^T \int_{\Omega} |\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \geq M}|^2 dx dt + 2 \int_0^T \int_{\Omega} |\sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \geq M}|^2 dx dt \quad (3.20) \\
& \leq \int_0^T \int_{\Omega} |\sqrt{\rho_n} \mathbf{u}_n \chi_{|\mathbf{u}_n| \leq M} - \sqrt{\rho} \mathbf{u} \chi_{|\mathbf{u}| \leq M}|^2 dx dt \\
& + \frac{2}{M^2} \int_0^T \int_{\Omega} \rho_n |\mathbf{u}_n|^4 dx dt + \frac{2}{M^2} \int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 dx dt.
\end{aligned}$$

Thanks to (3.19), we have

$$\limsup_{\varepsilon, \mu \rightarrow 0} \|\sqrt{\rho_n} \mathbf{u}_n - \sqrt{\rho} \mathbf{u}\|_{L^2(0, T; L^2(\Omega))} \leq \frac{C}{M}$$

for fixed $C > 0$ and all $M > 0$. Letting $M \rightarrow \infty$, we have

$$\sqrt{\rho_n} \mathbf{u}_n \rightarrow \sqrt{\rho} \mathbf{u} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

□

Applying Lemma 3.3 with (3.17), (3.18) and

$$\int_0^T \int_{\Omega} \rho_{\varepsilon, \mu} |\mathbf{u}_{\varepsilon, \mu}|^4 dx dt \leq C < \infty,$$

we have

$$\sqrt{\rho_{\varepsilon, \mu}} \mathbf{u}_{\varepsilon, \mu} \rightarrow \sqrt{\rho} \mathbf{u} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

By (3.15) and $\rho_{\varepsilon, \mu}^{\gamma}$ converges almost everywhere to ρ^{γ} , we have

$$\rho_{\varepsilon, \mu}^{\gamma} \rightarrow \rho^{\gamma} \quad \text{strongly in } L^1((0, T) \times \Omega).$$

Thanks to (3.1), we have $\rho_{\varepsilon, \mu}^{-10}$ converges almost everywhere to ρ^{-10} . Thus, with (3.16), we obtain

$$\rho_{\varepsilon, \mu}^{-10} \rightarrow \rho^{-10} \quad \text{strongly in } L^1((0, T) \times \Omega).$$

By previous estimates we can extract subsequences, such that

$$\varepsilon \nabla \rho_{\varepsilon, \mu} \rightarrow 0 \quad \text{strongly in } L^2((0, T) \times \Omega),$$

and

$$\varepsilon \nabla \rho_{\varepsilon, \mu} \nabla \mathbf{u}_{\varepsilon, \mu} \rightarrow 0 \quad \text{strongly in } L^1((0, T) \times \Omega).$$

For the convergence of term $\mu \Delta^2 \mathbf{u}_{\mu}$, for any test function $\varphi \in L^2(0, T; H^2(\Omega))$, we have

$$\left| \int_0^T \int_{\Omega} \mu \Delta^2 \mathbf{u}_{\varepsilon, \mu} \varphi dx dt \right| \leq \sqrt{\mu} \|\sqrt{\mu} \Delta \mathbf{u}_{\varepsilon, \mu}\|_{L^2(0, T; L^2(\Omega))} \|\Delta \varphi\|_{L^2(0, T; L^2(\Omega))} \rightarrow 0$$

as $\mu \rightarrow 0$, thanks to (3.4).

Due to weak lower semicontinuity of convex functions we can pass into the limits in energy inequality (2.32), we have the following Lemma.

Lemma 3.4.

$$\begin{aligned}
& \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\eta}{10} \rho^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho|^2 \right) dx \\
& + \int_0^T \int_{\Omega} \rho |\mathbb{D}\mathbf{u}|^2 dx dt + r_0 \int_0^T \int_{\Omega} |\mathbf{u}|^2 dx dt + r_1 \int_0^T \int_{\Omega} \rho |\mathbf{u}|^4 dx dt \\
& \leq \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\eta}{10} \rho_0^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + \frac{\delta}{2} |\nabla \Delta^4 \rho_0|^2 \right) dx,
\end{aligned} \tag{3.21}$$

Passing to the limits in (3.3) as $\varepsilon \rightarrow 0$ and $\mu \rightarrow 0$, we have the following BD entropy.

Lemma 3.5.

$$\begin{aligned}
& \int_{\Omega} \left(\frac{1}{2} \rho |\mathbf{u} + \frac{\nabla \rho}{\rho}|^2 + \frac{\delta}{2} |\nabla^9 \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{\rho^{-10}}{10} - r_0 \log \rho \right) dx + \\
& \eta \int_0^T \int_{\Omega} |\nabla \rho^{-5}|^2 dx dt + \int_0^T \int_{\Omega} |\nabla \rho^{\frac{7}{2}}| dx dt + 2\delta \int_0^T \int_{\Omega} |\Delta^5 \rho|^2 dx dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho |\nabla \mathbf{u} - \nabla^T \mathbf{u}|^2 dx dt + \kappa \int_0^T \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \\
& \leq 2 \int_{\Omega} \left(\frac{1}{2} \rho_0 |\mathbf{u}_0 + \frac{\nabla \rho_0}{\rho_0}|^2 + \frac{\delta}{2} |\nabla^9 \rho_0|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\rho_0^{-10}}{10} - r_0 \log_- \rho_0 \right) dx \\
& + 2E_0.
\end{aligned} \tag{3.22}$$

Thus, letting $\varepsilon \rightarrow 0$ and $\mu \rightarrow 0$, we have shown that the following existence on the approximation system.

Proposition 3.1. *There exists the weak solutions (ρ, \mathbf{u}) to the following system*

$$\begin{aligned}
\rho_t + \operatorname{div}(\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho^\gamma - \eta \nabla \rho^{-10} - \operatorname{div}(\rho \mathbb{D}\mathbf{u}) \\
&= -r_0 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u} + \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \delta \rho \nabla \Delta^9 \rho,
\end{aligned}$$

with suitable initial data, for any $T > 0$. In particular, the weak solutions (ρ, \mathbf{u}) satisfies the BD entropy (3.22) and the energy inequality (3.21).

3.3. Pass to limits as $\eta, \delta \rightarrow 0$. At this level, the weak solutions (ρ, \mathbf{u}) satisfies the BD entropy (3.22) and the energy inequality (3.21), thus we have the following regularities:

$$\sqrt{\rho} \mathbf{u} \in L^\infty(0, T; L^2(\Omega)), \sqrt{\rho} \mathbb{D}\mathbf{u} \in L^2((0, T) \times \Omega), \tag{3.23}$$

$$\sqrt{\delta} \rho \in L^\infty(0, T; H^9(\Omega)), \sqrt{\kappa} \sqrt{\rho} \in L^\infty(0, T; H^1(\Omega)), \tag{3.24}$$

$$\eta^{1/10} \rho^{-1} \in L^\infty(0, T; L^{10}(\Omega)), \sqrt{\eta} \nabla \rho^{-5} \in L^2((0, T) \times \Omega), \tag{3.25}$$

$$\mathbf{u} \in L^2(0, T; L^2(\Omega)), \rho^{\frac{1}{4}} \mathbf{u} \in L^4((0, T) \times \Omega),$$

$$\nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\delta} \Delta^5 \rho \in L^2(0, T; L^2(\Omega)). \tag{3.26}$$

In particular, we have

$$\kappa \int_0^T \int_{\Omega} \rho |\nabla^2 \log \rho|^2 dx dt \leq C,$$

which yields

$$\kappa^{\frac{1}{2}} \|\sqrt{\rho}\|_{L^2(0,T;H^2(\Omega))} + \kappa^{\frac{1}{4}} \|\nabla \rho^{\frac{1}{4}}\|_{L^4(0,T;L^4(\Omega))} \leq C, \quad (3.27)$$

where the constant $C > 0$ is independent of η , δ . That is, this inequality is still true after $\eta \rightarrow 0$ and $\delta \rightarrow 0$.

Thus, we have the same estimates as in Lemma 3.2 at the levels with η and δ . Thus, we deduce the same compactness for $(\rho_\eta, \mathbf{u}_\eta)$ and $(\rho_\delta, \mathbf{u}_\delta)$. Here, we focus on the convergence of the terms $\eta \nabla \rho^{-10}$ and $\delta \rho \nabla \Delta^9 \rho$. Here we pass to the limits with respect to η first, and then with respect to δ . Here we state the following two lemmas.

Lemma 3.6. *For any ρ_η defined as in Proposition 3.1, we have*

$$\eta \int_0^T \int_{\Omega} \rho_\eta^{-10} dx dt \rightarrow 0$$

as $\eta \rightarrow 0$.

Proof. By (3.22), we have

$$\int_{\Omega} (\ln(\frac{1}{\rho_\eta}))_+ dx \leq C(r_0) < \infty.$$

We notice that

$$y \in \mathbb{R}^+ \rightarrow \ln(\frac{1}{y})_+$$

is a convex continuous function. Moreover, Fatou's Lemma yields

$$\begin{aligned} \int_{\Omega} (\ln(\frac{1}{\rho}))_+ dx &\leq \int_{\Omega} \liminf (\ln(\frac{1}{\rho_\eta}))_+ dx \\ &\leq \liminf_{\eta \rightarrow 0} \int_{\Omega} (\ln(\frac{1}{\rho_\eta}))_+ dx, \end{aligned}$$

and hence $(\ln(\frac{1}{\rho}))_+$ is in $L^\infty(0, T; L^1(\Omega))$. It allows us to conclude that

$$|\{x : |\rho(t, x) = 0\}| = 0 \quad \text{for almost every } t, \quad (3.28)$$

where $|A|$ denotes the measure of set A .

By $(\rho_\eta)_t = -\nabla \rho_\eta \mathbf{u}_\eta - \rho_\eta \operatorname{div} \mathbf{u}_\eta$, and thanks to (3.23)-(3.27), we have

$$(\rho_\eta)_t \in L^2(0, T; L^3(\Omega)) + L^2((0, T; L^2(\Omega))).$$

This, together with (3.24), up to a subsequence and the Aubin-Lions Lemma gives us that ρ_η converges to ρ in $L^2(0, T; L^1(\Omega))$, and hence $\rho_\eta \rightarrow \rho$ a.e..

Thanks to (3.28), we deduce

$$\eta \rho_\eta^{-10} \rightarrow 0 \quad \text{a.e.} \quad (3.29)$$

By (3.25) and Poincaré's inequality, we have a uniform bound, with respect to η , of

$$\eta \rho_\eta^{-10} \in L^\infty(0, T; L^1(\Omega)) \cap L^1(0, T; L^3(\Omega)).$$

The $L^p - L^q$ interpolation inequality gives

$$\|\eta \rho_\eta^{-10}\|_{L^{\frac{5}{3}}(0,T;L^{\frac{5}{3}}(\Omega))} \leq \|\eta \rho_\eta^{-10}\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{2}{5}} \|\eta \rho_\eta^{-10}\|_{L^1(0,T;L^3(\Omega))}^{\frac{3}{5}} \leq C,$$

and hence $\eta\rho_\eta^{-10}$ is uniformly bounded in $L^{\frac{5}{3}}(0, T; L^{\frac{5}{3}}(\Omega))$. This, with (3.29), yields

$$\eta\rho_\eta^{-10} \rightarrow 0 \quad \text{strongly in } L^1(0, T; L^1(\Omega)).$$

□

Lemma 3.7. *For any ρ_δ defined as in Proposition 3.1, we have, for any test function φ ,*

$$\delta \int_0^T \int_\Omega \rho_\delta \nabla \Delta^9 \rho_\delta \varphi \, dx \, dt \rightarrow 0$$

as $\delta \rightarrow 0$.

Proof. By (3.24) and (3.26), We have uniform bounds with respect to δ of

$$\rho_\delta \in L^\infty(0, T; L^3(\Omega)), \quad \sqrt{\delta}\rho_\delta \in L^\infty(0, T; H^9(\Omega)), \quad \sqrt{\delta}\rho_\delta \in L^2(0, T; H^{10}(\Omega)).$$

This, with Gagliardo-Nirenberg interpolation inequality, yields

$$\|\nabla^9 \rho_\delta\|_{L^3} \leq C \|\nabla^{10} \rho_\delta\|_{L^2}^{\frac{18}{19}} \|\rho_\delta\|_{L^3}^{\frac{1}{19}}.$$

Thus, we have

$$\int_0^T \delta \left(\int_\Omega |\nabla^9 \rho_\delta|^3 \, dx \right)^{\frac{19}{27}} \, dt \leq C \sup_{t \in (0, T)} (\|\rho_\delta\|_{L^3(\Omega)})^{\frac{1}{9}} \int_0^T \int_\Omega \delta |\nabla^{10} \rho_\delta|^2 \, dx \, dt,$$

which implies

$$\delta^{\frac{9}{19}} |\nabla^9 \rho_\delta| \in L^{\frac{19}{9}}(0, T; L^3(\Omega)). \quad (3.30)$$

For the term

$$\delta \int_0^T \int_\Omega \rho_\delta \nabla \Delta^9 \rho_\delta \varphi \, dx \, dt = -\delta \int_0^T \int_\Omega \Delta^4 \operatorname{div}(\rho_\delta \varphi) \Delta^5 \rho_\delta \, dx \, dt,$$

we focus on the most difficulty term

$$\begin{aligned} \left| \delta \int_0^T \int_\Omega \Delta^4 (\nabla \rho_\delta) \Delta^5 \rho_\delta \varphi \, dx \, dt \right| &\leq C(\varphi) \int_0^T \int_\Omega \sqrt{\delta} |\nabla^{10} \rho_\delta| \delta^{\frac{9}{19}} |\nabla^9 \rho_\delta| \delta^{\frac{1}{38}} \, dx \, dt \\ &\leq C(\varphi) \delta^{\frac{1}{38}} \|\sqrt{\delta} \nabla^{10} \rho_\delta\|_{L^2(0, T; L^2(\Omega))} \|\delta^{\frac{9}{19}} \nabla^9 \rho_\delta\|_{L^{\frac{19}{9}}(0, T; L^3(\Omega))} \\ &\rightarrow 0 \end{aligned}$$

as $\delta \rightarrow 0$, where we used (3.30).

We can apply the same arguments to handle the other terms from

$$\delta \int_0^T \int_\Omega \Delta^4 \operatorname{div}(\rho_\delta \varphi) \Delta^5 \rho_\delta \, dx \, dt.$$

Thus we have

$$\delta \int_0^T \int_\Omega \rho_\delta \nabla \Delta^9 \rho_\delta \varphi \, dx \, dt \rightarrow 0$$

as $\delta \rightarrow 0$. □

Here we have to remark that (3.27) is still true even after vanishing η and δ . Thus, letting $\eta \rightarrow 0$ and $\delta \rightarrow 0$, we have shown that (ρ, \mathbf{u}) solves (1.1).

Meanwhile, due to weak lower semicontinuity of convex functions, we have (1.3) by vanishing η and δ in energy inequality (3.21). Similarly, we can obtain BD-entropy (1.4) by passing into the limits in (3.22) as $\eta \rightarrow 0$ and $\delta \rightarrow 0$.

3.4. Other Properties. The time evolution of the integral averages

$$t \in (0, T) \mapsto \int_{\Omega} (\rho \mathbf{u})(t, x) \cdot \psi(x) dx$$

is defined by

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (\rho \mathbf{u})(t, x) \cdot \psi(x) dx &= \int_{\Omega} \rho \mathbf{u} \otimes \mathbf{u} : \nabla \psi dx + \int_{\Omega} \rho^\gamma \operatorname{div} \psi dx + \int_{\Omega} \rho \mathbb{D} \mathbf{u} \nabla \psi dx \\ &+ r_0 \int_{\Omega} \mathbf{u} \psi dx + r_1 \int_{\Omega} \rho |\mathbf{u}|^2 \mathbf{u} \psi dx + 2\kappa \int_{\Omega} \Delta \sqrt{\rho} \nabla \sqrt{\rho} \psi dx + \kappa \int_{\Omega} \Delta \sqrt{\rho} \sqrt{\rho} \operatorname{div} \psi dx. \end{aligned} \quad (3.31)$$

All estimates from (1.3) and (1.4) imply (3.31) is continuous function with respect to $t \in [0, T]$. On the other hand, we have

$$\rho \mathbf{u} \in L^\infty(0, T; L^{\frac{3}{2}}(\Omega)) \cap L^4(0, T; L^2(\Omega)),$$

and hence

$$\rho \mathbf{u} \in C([0, T]; L^{\frac{3}{2}}_{weak}(\Omega)).$$

We notice

$$(\sqrt{\rho})_t = -\frac{1}{2} \sqrt{\rho} \operatorname{div} \mathbf{u} - \nabla \sqrt{\rho} \cdot \mathbf{u},$$

thus

$$\|(\sqrt{\rho})_t\|_{L^2((0, T) \times \Omega)} \leq C \|\sqrt{\rho} \operatorname{div} \mathbf{u}\|_{L^2((0, T) \times \Omega)} + C \|\nabla \rho^{\frac{1}{4}}\|_{L^4((0, T) \times \Omega)} \|\rho^{\frac{1}{4}} \mathbf{u}\|_{L^4((0, T) \times \Omega)}.$$

This, with $\nabla \sqrt{\rho_\kappa} \in L^\infty(0, T; L^2(\Omega))$, we have

$$\sqrt{\rho_\kappa} \rightarrow \sqrt{\rho} \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (3.32)$$

Because

$$(\rho \mathbf{u})_t = -\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \nabla \rho^\gamma + \operatorname{div}(\rho \mathbb{D} \mathbf{u}) - r_0 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u} + \kappa \rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right),$$

thus we have $(\rho \mathbf{u})_t$ is bounded in $L^4(0, T; W^{-1,4}(\Omega))$. Meanwhile, we have

$$\nabla(\rho \mathbf{u}) = (\rho^{\frac{1}{4}} \mathbf{u}) \cdot \nabla \sqrt{\rho} \rho^{\frac{1}{4}} + \sqrt{\rho} \sqrt{\rho} \nabla \mathbf{u},$$

which yields $\nabla(\rho \mathbf{u}) \in L^4(0, T; L^{\frac{6}{5}}(\Omega)) + L^2(0, T; L^{\frac{3}{2}}(\Omega))$. The Aubin-Lions Lemma gives us

$$\rho_\kappa \mathbf{u}_\kappa \rightarrow \rho \mathbf{u} \text{ strongly in } L^2((0, T) \times \Omega). \quad (3.33)$$

Applying Lemma 3.3 with (3.32), (3.33), and

$$\int_0^T \int_{\Omega} \rho_\kappa |\mathbf{u}_\kappa|^4 dx dt \leq C < \infty,$$

we have

$$\sqrt{\rho_\kappa} \mathbf{u}_\kappa \rightarrow \sqrt{\rho} \mathbf{u} \text{ strongly in } L^2(0, T; L^2(\Omega)).$$

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REFERENCES

- [1] D. Bresch, B. Desjardins, *Existence of global weak solutions for 2D viscous shallow water equations and convergence to the quasi-geostrophic model*. Comm. Math. Phys., 238 (2003), no.1-3, 211-223.
- [2] D. Bresch and B. Desjardins, *On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models*. J. Math. Pures Appl. (9) 86 (2006), no. 4, 362-368.
- [3] D. Bresch, B. Desjardins, Chi-Kun Lin, *On some compressible fluid models: Korteweg, lubrication, and shallow water systems*. Comm. Partial Differential Equations 28 (2003), no. 3-4, 843-868.
- [4] P.Fabrie, F. Marche, *Another proof of stability for global weak solutions of 2D degenerated shallow water models*. J. Math. Fluid Mech. 11 (2009), no. 4, 536-551.
- [5] D. Ferry, J.-R. Zhou, *Form of the quantum potential for use in hydrodynamic equations for semiconductor device modeling*. Phys. Rev. B, Vol(48) (1993), 7944-7950.
- [6] E. Feireisl, A. Novotný, H. Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations*. J. Math. Fluid Mech. **3** (2001), 358-392.
- [7] E. Feireisl, *Dynamics of Viscous Compressible Fluids*. Oxford Lecture Series in Mathematics and its Applications, 26. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2004.
- [8] M. Gisclon, I. Lacroix-Violet, *About the barotropic compressible quantum Navier-Stokes equations*. <http://arxiv.org/abs/1412.1332>
- [9] J. Grant, *Pressure and stress tensor expressions in the fluid mechanical formulation of the Bose condensate equations*. J. Phys. A: Math., Nucl. Gen., Vol 6 (1973), L151-L153.
- [10] A. Jüngel, *Dissipative quantum fluid models*. Revista Mat. Univ. Parma 3 (2012), 217-290.
- [11] A. Jüngel, *Global weak solutions to compressible Navier-Stokes equations for quantum fluids*. SIAM J. Math. Anal. 42 (2010), no. 3, 1025-1045.
- [12] P.-L. Lions, *Mathematical topics in fluid mechanics*. Vol. 2. Compressible models. Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.
- [13] A. Mellet, A. Vasseur, *On the barotropic compressible Navier-Stokes equations*. Comm. Partial Differential Equations 32 (2007), no. 1-3, 431-452.
- [14] A. Vasseur, C. Yu, *Global Weak Solutions for 3D Degenerate Compressible Navier-Stokes equations*. 2015, <http://arxiv.org/abs/1501.06803>.
- [15] R. Wyatt, *Quantum Dynamics with Trajectories*. Springer, New York, (2005).
- [16] E. Zatorska, *On the flow of chemically reacting gaseous mixture*. J. Diff. Eqns. 253 (2012) 3471-3500.

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