On a Model for Mixture Flows: Derivation, Dissipation and Stability Properties

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Abstract. We propose a new model describing mixture flows. The non linear system couples heterogeneous Navier-Stokes equations with a constraint on the mean volume velocity of the flow. The PDE system is obtained from a more microscopic viewpoint, involving a Vlasov-like equation describing the disperse phase, through a certain hydrodynamic limit. The model has remarkable dissipation properties, inherited from the structure of the fluid-kinetic description. Based on these properties, together with additional estimates that can be obtained in the one-dimension framework, we establish the stability of weak solutions.

Key words. Mixture flows. Entropy dissipation. Stability of weak solutions.

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1 Introduction

This paper is devoted to the study of a non linear system of PDEs intended to describe mixture flows. The mixture is seen as a fluid phase carrying dilute "particles". The unknowns are the fluid volume fraction n, the bulk velocity u, the pressure p. All these quantities depend on the time and space variables, $t \geq 0$ and $x \in \mathbb{R}^N$ respectively. The PDE system depends on a (constant) parameter $0 \leq \bar{\phi} \leq 1$, which is interpreted as a typical volume fraction of the particles within the flow. It is convenient to introduce the auxiliary quantity

$$\bar{\phi}\phi(t,x) = 1 - n(t,x), \tag{1.1}$$

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which is nothing but the volume fraction of the disperse phase. The evolution of the flow is driven by the mass conservation

$$\partial_t n + \nabla_x \cdot (nu) = 0 \tag{1.2}$$

and the momentum balance

$$\partial_t(nu) + \nabla_x \cdot (nu \otimes u) + \nabla_x p - \nabla_x \cdot (\mu \mathbb{D}(u)) = -(\kappa n + \bar{\phi}\phi)\nabla_x \mathscr{E} - \nabla_x \phi, \tag{1.3}$$

where we denote $\mathbb{D}(u) = \nabla_x u + \nabla_x u^T$. The pressure p is defined by the relation

$$\nabla_x \cdot \left(u - \frac{\bar{\phi}\phi}{\mu\nu} \nabla_x (\ln(\phi) + \bar{\phi} p + \bar{\phi}\mathscr{E}) \right) = 0. \tag{1.4}$$

The system depends on a set of parameters the role of which will discussed in more detailed later on:

- $\mu : [0, \infty) \to [0, \infty)$ is the viscosity of the fluid.
- $\nu : [0, \infty) \to [0, \infty)$ enters in the definition of drag effects between the two phases.
- External forces are embodied into the potential $x \mapsto \mathscr{E}(x)$; both phases are subjected to its action, possibly with different amplitude and direction encoded into the parameter $\kappa \in \mathbb{R}$.

Both μ and ν are smooth functions of the density n, $\mathscr{E}$ is a smooth function of the space variable. The problem is completed by initial data

$$0 \leq n\Big|_{t=0} = n_{\text{Init}} \leq 1, \qquad u\Big|_{t=0} = u_{\text{Init}}. \tag{1.5}$$

When $\bar{\phi} = 0$, the fluid volume fraction is constant $n(t, x) = 1$ and the mixture is fully incompressible: $\nabla_x \cdot u = 0$. We recover in the case the standard homogeneous Incompressible Navier Stokes equations. When $\bar{\phi} \neq 0$, the behavior is hybrid: incompressibility ($\nabla_x \cdot u = 0$ defining implicitly the pressure p) holds only at positions that do not contain any particles ($\phi(t, x) = 0$). Note that models for two-phase flows with such a dual behavior have been discussed in [5, 6, 45], but with a completely different definition for the pressure in compressible domains.

The content of the paper can be summarized as follows. The aim of the paper is to bring out some remarkable properties of the system. The discussion remains mostly formal and can be seen as a preliminary step towards a proof of the existence of solutions or for a thorough numerical investigation of the model. We shall start in Section 2 by comparing the model (1.2)–(1.4) with other PDEs system describing fluid mixtures. We are particularly interested in the so–called Kazhikov-Smagulov systems where such flows can be described by mass conservation and momentum balance, together with a constraint that relates the divergence of the velocity to derivatives of the fluid mass density. This constraint defines implicitly the pressure like in the standard incompressible case. In Section 3 we investigate further the modeling issues by going back to a fluid-kinetic coupling, where the disperse phase is described through the particle distribution function in phase space. Then, we identify a relevant asymptotic regime by means of the physical parameters that characterize the flow. It allows us to obtain (1.2)–(1.4) as the limit of the microscopic description in a certain "hydrodynamic regime". The interest of this viewpoint relies on the identification of remarkable dissipation properties. We can define a suitable entropy functional for the fluid-kinetic model which satisfies the H-Theorem. The hydrodynamic model (1.2)–(1.4) naturally inherits of the dissipative structure, as detailed in Section 4. The discussion is mostly formal, nevertheless it provides a reasonable functional framework to contemplate the construction of weak solutions. Unfortunately, the estimates that can be deduced from the entropy dissipation do not give enough compactness to handle the non linearities of the system. In Section 5 we go one step further, at the price of restricting to the one-dimension framework. Indeed, in this case, further quantities involving derivatives of the fluid density can be shown to be dissipated. The derivation of such estimates is reminiscent to the techniques introduced in [13, 15, 48, 60, 61, 65] for more classical systems of fluid mechanics. In Section 6, these estimates are exploited to investigate the stability of the solutions of (1.2)–(1.4). It allows us to define a relevant functional framework for the definition of solutions. In particular we shall clarify the role of the pressure p and the meaning of the product $\phi\partial_x p$. It is clearly the most difficult term to deal with and, even in this simple framework, the arguments that permit us to pass to the limit in the product are quite intricate. The analysis, however, strongly relies on the restriction to the one-dimension framework.

2 Modeling of Mixture Flows

The mathematical description of mixture flows leads to non trivial variations about the standard PDEs systems of fluid mechanics. Typically, in the Navier-Stokes equations the incompressibility constraint on the velocity field $\nabla_x \cdot u = 0$ is replaced by a relation between the velocity and the gradient of a certain function of the density ρ, for instance

$$\nabla_x \cdot \big(u + \lambda \nabla_x \ln(\rho)\big) = 0 \tag{2.6}$$

where λ is a non negative coefficient. Of course, when $\lambda = 0$ the system reduces to the usual Incompressible Navier–Stokes system. The Fick law (3.13) has been introduced in [37, 41]. It is further discussed in [10, 31, 40, 54] for modeling flows where species (like salt or pollutant) are dissolved in a compressible or incompressible fluid. In such a modeling, the mixture is seen as an averaged continuum, described by a single density ρ and velocity u. In this situation, of course, the mean density ρ is naturally non homogeneous, and the constitutive law (3.13) is intended to model diffusion effects between the constituents of the mixture. According to Kazhikov and Smagulov [41] it is convenient to introduce the divergence free velocity

$$v = u + \lambda \nabla_x \ln(\rho),$$

and to write the equations satisfied by the pair (ρ, v). The mass conservation then contains a diffusion term, while in the momentum equation new (non–linear) terms involving derivatives of the density appear. Actually, a hierarchy of models can be derived this way, by getting rid of terms of order $\mathscr{O}(\lambda^2)$ or of the $\mathscr{O}(\lambda)$ terms in the momentum equation (which leads to the so–called Graffi model [31, 37]). The analysis of the simplified versions of the Kazhikov-Smagulov system dates back to [41] and [2, Chap. 3, Sect. 4, sp. Theorem 4.1] for the case where the dynamic viscosity of the mixture μ is assumed to be a positive constant. Further results on the existence-uniqueness and stability issues can be found in in [4, 41, 58, 59]. The numerical simulation of the Kazhigov-Smagulov system is discussed by different techniques in [18, 27, 28, 29] with specific applications to the simulation of powder-snow avalanches. In [27] a specific relation between the coefficient λ and the viscosity μ is postulated in order get rid of the higher order coupling term; this model also satisfies a remarkable balance law for the energy of the system, see [27, Section 2.2]. The mathematical analysis of this specific case is due to [17, 24, 42, 43].

The relation (2.6) for mixture flows relies on the distinction between the *mean mass velocity* u, and the *mean volume velocity* v which is divergence free. Let us denote by $\bar{\rho}_d$ and $\bar{\rho}_p$ the mass density, supposed constant, of the carrier fluid and the disperse phase, respectively. Similarly, let u_d and u_p be the respective velocity fields. Considering each phase separately, the mass conservation relations read

$$\partial_t(\bar{\rho}_d(1-\phi)) + \nabla_x \cdot (\bar{\rho}_d(1-\phi)u_d) = 0 = \partial_t(\bar{\rho}_p\phi) + \nabla_x \cdot (\bar{\rho}_p\phi u_p).$$

Let $\rho = \bar{\rho}_d(1-\phi) + \bar{\rho}_p\phi$ stand for the mean mass density of the mixture. It satisfies

$$\partial_t\rho + \nabla_x \cdot (\rho u) = 0,$$

with $u = \bar{\rho}_d(1-\phi)u_d + \bar{\rho}_p\phi u_p$ the mean mass velocity. But, we also have

$$\partial_t(\phi + 1 - \phi) = 0 = -\nabla_x \cdot (\phi u_p + (1-\phi)u_d)$$

which means that the mean volume velocity $v = \phi u_p + (1-\phi)u_d$ is divergence free. This is a consequence of the incompressibility of each phase considered independently (by assumption their mass densities $\bar{\rho}_p, \bar{\rho}_d$ are constant). The relation (2.6) is obtained by postulating a simple closure relation between u and v and the density gradient namely $v = u + \lambda\nabla_x \ln(\rho)$, or, equivalently a relation between u_d and u_p:

$$(1-\phi)(1-\bar{\rho}_p/\rho)u_p = \phi(\bar{\rho}_d/\rho - 1)u_d + \lambda\nabla_x \ln(\rho).$$

To distinguish the two mean velocity fields might lead to consider corrections to the usual systems of fluid mechanics, even for single-phase flows, as pointed out by H. Brenner [8, 9, 10, 11]. It turns out that some of these corrected systems have a remarkable mathematical structure [30]. Detailed discussions on the closure relation describing diffuse volume flux can be found in Brenner's papers and the references therein, see e. g. [11, sp. Sect. 5]. In [18] a derivation of Kazhikov-Smagulov's systems through hydrodynamic limits has been proposed. We are going to revisit this approach to design new models for mixture flows. In particular, we shall obtain this way the system (1.2)–(1.4), that we interpret as a generalized Kazhikov–Smagulov system.

3 Fluid–Kinetic Model, Entropy Dissipation and Hydrodynamic Limit

In this Section we are interested in a microscopic description of the mixture, with disperse particles interacting with the carrier fluid through drag forces. For the disperse phase, we adopt a statistical description: it is seen as a large set of "particles" described by means of their distribution function in phase space. The drag force between the particles and the carrier fluid depends on their relative velocity. Of course, the PDE system remains highly non linear, with a Vlasov like equation for the disperse phase coupled to a hydrodynamic system for the fluid, together with algebraic constraints. We refer the reader to [1, 51, 52, 53, 64] for examples of such coupled fluid-kinetic, or Eulerian-Lagrangian, models. In the spirit of hydrodynamic regimes which make the connection between the Boltzmann equation and gas dynamics systems [56], our objective is to derive the system (1.2)–(1.4) from the fluid-kinetic model. This approach can be compared to the discussion and the derivation of macroscopic balance laws for particulate flows in [46], dealing with a different regime, though. See also [20, 21, 34, 35, 50] for the analysis of such asymptotic problem in the context of particulate flows. The interest of the discussion is two–fold. On the one hand, it places the model (1.2)–(1.4) in a certain hierarchy of equations, set up on asymptotic grounds. On the other hand, we shall bring out a certain entropy functional dissipated by the fluid-kinetic model. In turn, it allows us to identify some dissipation properties of the hydrodynamic model.

3.1 Eulerian-Lagrangian Framework

In this presentation, we place ourselves in the natural 3-dimensional framework, but the discussion can be adapted to any space dimension. The carrier phase is described by its mass density $\varrho_d(t,x) \geq 0$, the scalar pressure $p(t,x)$ and the velocity field $u_d(t,x)$. The fluid is also described by the viscosity $\mu \geq 0$, that might depend on ϱ_d. The disperse phase is described by means of the particle distribution function $F(t,x,\xi) \geq 0$. We assume that the particles are spherically shaped with fixed radius $a > 0$ and mass density $\bar\rho_p > 0$. Hence the integral

$$\frac{4}{3}\pi a^3 \bar\rho_p \int_{\Omega\times\mathscr{V}} F(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$

gives the mass of the particles which have, at time $t \geq 0$, their position x in the domain $\Omega \subset \mathbb{R}^3$, and their velocity in the domain $\mathscr{V} \subset \mathbb{R}^3$. Accordingly

$$\phi(t,x) = \frac{4}{3}\pi a^3 \int_{\mathbb{R}^3} F(t,x,\xi)\,\mathrm{d}\xi$$

is the volume fraction occupied by the particles, while $\bar\rho_p\phi(t,x)$ is the local mass density of the disperse phase.

The evolution of the particle distribution function is driven by the following PDE

$$\bar\rho_p\big(\partial_t F + \nabla_x\cdot(\xi F)\big) = \nabla_\xi\cdot(\mathscr{F}F + \mathscr{D}\nabla_\xi F). \tag{3.7}$$

It belongs to the class of Vlasov–Fokker–Planck equations. In (3.7), the force field $\mathscr{F}$ (which actually has the dimension of a force density, that is a force per unit volume) accounts for

- the effect of an external potential $x \mapsto \mathscr{E}(x)$,

- the action of the fluid on the particles which is two-fold:
  - the drag force proportional to the relative velocity $\xi - u_d(t,x)$. The precise form of the proportionality coefficient is essentially of phenomenological nature and it can be quite involved. The simplest model is based on the linear Stokes law

$$\frac{9\mu}{2a^2}\,\nu\,\big(\xi - u_d(t,x)\big).$$

    The parameter $\nu : \varrho \to \nu(\varrho) \geq 0$ is dimensionless. The standard Stokes law assumes $\nu(\varrho) = 1$. However, it can be natural to suppose that the drag coefficient depends on the fluid density: the smaller the density, the smaller the amplitude of the force, and in particular it vanishes in the absence of the fluid. We shall see that this coefficient might be important for the analysis of the problem (see Remark 5.1 below).
  - the pressure effect proportional to the pressure gradient $\nabla_x p$.

We finally arrive at

$$\mathscr{F} = \frac{9\mu}{2a^2}\,\nu(\varrho_d)\,\big(\xi - u_d(t,x)\big) + \nabla_x(p + \bar{\rho}_p\mathscr{E}). \tag{3.8}$$

In (3.7), the last term in the right hand side corresponds to a rough stochastic description of particles momentum fluctuation, see e. g. [32, 46]. The corresponding diffusion coefficient is proportional to the frictional coefficient [23, 46, Eq. (5)] as follows

$$\mathscr{D} = \frac{9\mu}{2a^2}\,\nu\,\frac{3k\theta}{4\pi a^3\bar{\rho}_p},$$

with $k > 0$ the Boltzmann constant and $\theta > 0$ the granular temperature of the flow, here assumed a given constant.

Remark 3.1 *Specializing to a single particle, equation* (3.7) *is equivalent to the equations of motion (Newton's law) which characterize the pair position/velocity* (X, V) *of the particle by means of the following stochastic differential equation*

$$\begin{aligned} &\mathrm{d}X = V\,\mathrm{d}t,\\ &\frac{4}{3}\pi a^3\bar{\rho}_p\,\mathrm{d}V = \mathscr{F}(t,X,V)\,\mathrm{d}t + \sqrt{2\mathscr{D}}\,\mathrm{d}B_t \end{aligned}$$

where $\mathrm{d}B_t$ *stands for the standard Brownian motion. Of course, a precise description of the mixture could be obtained by considering the differential system satisfied by the position and velocity of each particle present within the solution, coupled to the mass and momentum balance for the surrounding fluid, see e. g. [25, 57] for the analysis of this problem. In practice such a description might be of limited use due to the tiny size of the particles compared to the length scale of interest, and to the huge number of particles in the solution: both effect makes non affordable the direct simulation of the system. However, these equations can be suitably approximated or considered in the perspective of asymptotic analysis to derive relevant macroscopic models; we refer the reader on this aspect to [5, 26, 44].*

We turn to the evolution of the carrier fluid. On the one hand, the mass conservation holds

$$\partial_t \varrho_d + \nabla_x \cdot (\varrho_d u_d) = 0. \tag{3.9}$$

On the other hand, the momentum satisfies the balance relation

$$\partial_t(\varrho_d u_d) + \nabla_x \cdot (\varrho_d u_d \otimes u_d) + \nabla_x p - \nabla_x \cdot (\mu \mathbb{D}(u_d)) = -\kappa \varrho_d \nabla_x \mathscr{E} + \mathscr{C}. \tag{3.10}$$

In (3.10), $\kappa \in \mathbb{R}$ describes the effect of the external potential on the fluid that can differ both in direction and amplitude to the effect on the disperse phase. The coupling term $\mathscr{C}$ accounts for the back-reaction of the particles and it is given by

$$\mathscr{C} = \frac{4\pi a^3}{3} \int \Big(\frac{9\mu}{2a^2} \nu(\varrho_d)(\xi - u_d) + \nabla_x p \Big) F \, \mathrm{d}\xi.$$

We are out of an equation to determine the pressure field. Here, we adopt the *incompressible* framework: we suppose that the mass density of the carrier phase is constant in absence of particles, so that

$$\varrho_d(t,x) = \bar\rho_d(1 - \phi(t,x)), \tag{3.11}$$

with $\bar\rho_d > 0$ the given mass density of the carrier fluid. (We remind the reader that $(1 - \phi(t,x))$ is the volume fraction occupied by the carrier phase.) Therefore, the evolution of the mixture is governed by the set of equations (3.7)–(3.11). In what follows we shall work on the whole space. Hence, we need to prescribe the behavior at infinity of the unknowns. Throughout the paper, we shall suppose that F, and thus ϕ, and u vanish at infinity; due to the relation (3.11), it means that only the carrier phase is present at infinity and $\varrho_d(t,x) \to \bar\rho_d$ as $|x| \to \infty$.

For further purposes, let us define the bulk velocity of the disperse phase V as follows

$$\phi V(t,x) = \frac{4}{3}\pi a^3 \int_{\mathbb{R}^3} \xi F(t,x,\xi) \, \mathrm{d}\xi.$$

As a matter of fact, integrating (3.7) with respect to the velocity variable yields the particles mass conservation

$$\partial_t \phi + \nabla_x \cdot (\phi V) = 0. \tag{3.12}$$

Besides, with (3.11), (3.9) recasts as

$$\bar\rho_d\big(\partial_t(1-\phi) + \nabla_x \cdot \big((1-\phi)u_d\big)\big) = 0 = -\bar\rho_d\big(\partial_t \phi + \nabla_x(\phi u_d) - \nabla_x \cdot u_d\big).$$

We are thus led to the constraint

$$\nabla_x \cdot \big((1-\phi)u_d + \phi V\big) = 0, \tag{3.13}$$

which means that the *mean volume velocity* $(1-\phi)u_d + \phi V$ is solenoidal, by contrast to the *mean mass velocity* $\varrho_d u_d + \bar\rho_p \phi V = \bar\rho_d(1-\phi)u_d + \bar\rho_p \phi V$.

Note that we also have the following momentum balance equation

$$\partial_t(\phi V) + \nabla_x \cdot \mathbb{P} = \frac{9\mu}{2a^2\bar{\rho}_p}\nu(\varrho_d)\ \phi(u_d - V) - \frac{\phi}{\bar{\rho}_p}\nabla_x(p + \bar{\rho}_p\mathscr{E}). \tag{3.14}$$

where

$$\mathbb{P}(t,x) = \frac{4}{3}\pi a^3 \int_{\mathbb{R}^3} \xi \otimes \xi F(t,x,\xi)\,\mathrm{d}\xi.$$

Therefore, since $\mathscr{C} = \frac{9\mu}{2a^2}\nu(\varrho_d)\phi(V - u_d) + \phi\nabla_x p$, we can write the following relations for the mean density and the mean mass velocity

$$\begin{aligned}
&\partial_t(\varrho_d + \bar{\rho}_p\phi) + \nabla_x \cdot (\varrho_d u_d + \bar{\rho}_p\phi V) = 0,\\
&\partial_t(\varrho_d u_d + \bar{\rho}_p\phi V) + \nabla_x \cdot (\varrho_d u_d \otimes u_d + \bar{\rho}_p\mathbb{P}) + \nabla_x p - \nabla_x \cdot (\mu\mathbb{D}(u_d)) = -(\kappa\varrho_d + \bar{\rho}_p\phi)\nabla_x\mathscr{E}.
\end{aligned}$$

3.2 H-Theorem

Let us define the following free-energy

$$\mathscr{H} = \frac{4}{3}\pi a^3\bar{\rho}_p \int F\Big(\frac{\xi^2}{2} + \mathscr{E} + \frac{3k\theta}{4\pi a^3\bar{\rho}_p}\ln F\Big)\,\mathrm{d}\xi\,\mathrm{d}x + \int \varrho_d\frac{u_d^2}{2}\,\mathrm{d}x - \kappa\int(\bar{\rho}_d - \varrho_d)\mathscr{E}\,\mathrm{d}x.$$

(We remind the reader that ϱ_d tends to $\bar{\rho}_d$ at infinity, and, for the moment, $\kappa \in \mathbb{R}$ does not have a definite sign.) This quantity combines the entropy of the particles with the kinetic and potential energies of both phase. The remarkable fact relies on the dissipative properties of the problem, that cam be summarized in the following formal statement

Proposition 3.1 *Let* (F, ϱ_d, u_d, p) *be a smooth enough solution of* (3.7)–(3.11). *Then, we have* $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} \le 0$.

Proof. We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi a^3\bar{\rho}_p\int F\Big(\frac{\xi^2}{2} + \frac{3k\theta}{4\pi a^3\bar{\rho}_p}\ln F\Big)\,\mathrm{d}\xi\,\mathrm{d}x\right)\\
&\quad = -\frac{4}{3}\pi a^3\int\left(\left(\frac{9\mu}{2a^2}\nu(\xi - u_d) + \nabla_x p + \bar{\rho}_p\nabla_x\mathscr{E}\right)F + \frac{9\mu}{2a^2}\nu\frac{3k\theta}{4\pi a^3\bar{\rho}_p}\nabla_\xi F\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\left(\xi + \frac{3k\theta}{4\pi a^3\bar{\rho}_p}\frac{\nabla_\xi F}{F}\right)\,\mathrm{d}\xi\,\mathrm{d}x
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int F\mathscr{E}\,\mathrm{d}\xi\,\mathrm{d}x = \int \xi F \cdot \nabla_x\mathscr{E}\,\mathrm{d}\xi\,\mathrm{d}x.$$

For the kinetic energy associated to the carrier fluid, we have[1]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho_d\frac{u_d^2}{2}\,\mathrm{d}x \;=\; &-\frac{1}{2}\int \mu|\nabla u_d|^2\,\mathrm{d}x + \frac{4}{3}\pi a^3\int\left(\int\left(\frac{9\mu}{2a^2}\nu\Big(\xi - u_d\Big) + \nabla_x p\right)F\,\mathrm{d}\xi\right)\cdot u\,\mathrm{d}x\\
&-\int(\nabla_x p + \kappa\varrho_d\nabla_x\mathscr{E})\cdot u\,\mathrm{d}x
\end{aligned}$$

[1] Here and below, for a vector field $u : \mathbb{R}^N \to \mathbb{R}^N$, we denote $|\nabla_x u|^2 = \sum_{i,j}(\partial_{x_i}u_j + \partial_{x_j}u_i)^2$.

while for the potential energy we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\bar\rho_d-\varrho_d)\mathscr{E}\,\mathrm{d}x = -\int \kappa\varrho_d u_d\cdot\nabla_x\mathscr{E}\,\mathrm{d}x.$$

Hence, in the time evolution of $\mathscr{H}$, the terms that involve the external potential $\mathscr{E}$ cancel out. Similarly, by using integration by parts, we observe that the terms involving the pressure combine as follows

$$\begin{aligned}-\frac{4}{3}\pi a^3\int \nabla_x p\cdot\xi F\,\mathrm{d}\xi\,\mathrm{d}x+\frac{4}{3}\pi a^3\int\nabla_x p\cdot u_d F\,\mathrm{d}\xi\,\mathrm{d}x-\int\nabla_x p\cdot u_d\,\mathrm{d}x\\ = -\int(\phi V+(1-\phi)u_d)\cdot\nabla_x p\,\mathrm{d}x=\int\nabla_x\cdot(\phi V+(1-\phi)u_d)\;p\,\mathrm{d}x=0\end{aligned}$$

by virtue of the volume constraint (3.13). Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} = -\frac{1}{2}\int\mu|\nabla u_d|^2\,\mathrm{d}x-\frac{4}{3}\pi a^3\int\frac{9\mu}{2a^2}\nu\Big((\xi-u_d)\sqrt{F}+2\frac{3k\theta}{4\pi a^3\bar\rho_p}\nabla_\xi\sqrt{F}\Big)^2\,\mathrm{d}\xi\,\mathrm{d}x\le 0.$$

■

3.3 Hydrodynamic Regime

We are going to derive a hydrodynamic model from the coupled system (3.7)-(3.11) through asymptotic arguments. Proceeding this way, we propose a derivation of the macroscopic system (1.2)–(1.4). To this end, we introduce time and length scales of reference, denoted T and L, respectively. Accordingly, we set $\mathrm{U}=\mathrm{L}/\mathrm{T}$ as the velocity unit. We denote $\bar\mu$ a typical value of the viscocity. For the particles, we define the thermal velocity $\mathrm{V}_{th}=\sqrt{\frac{3k\theta}{4\pi a^3\bar\rho_p}}$, and we need a typical value $0<\bar\phi<1$ of the particle volume fraction. The asymptotic regimes are driven by the ratio of mass densities $\frac{\bar\rho_p}{\bar\rho_d}$ and the so–called Stokes settling time $\frac{2\bar\rho_p a^2}{9\bar\mu}$, which measures the influence of the drag force. Let us define the following dimensionless parameters

$$\varepsilon=\frac{2\bar\rho_p a^2}{9\bar\mu\mathrm{T}},\qquad \eta=\frac{\mathrm{V}_{th}}{\mathrm{U}}.$$

We define dimensionless variables and unknowns as follows

$$\begin{aligned}&t=\mathrm{T}t_\star,\qquad x=\mathrm{L}x_\star,\qquad \xi=\mathrm{V}_{th}\xi_\star,\\ &F(t,x,\xi)=\frac{3}{4\pi a^3}\frac{1}{\mathrm{V}_{th}^3}\;\bar\phi\;F_\star(t_\star,x_\star,\xi_\star),\\ &\phi(t,x)=\bar\phi\;\phi_\star(t_\star,x_\star)=\bar\phi\int_{\mathbb{R}^3}F_\star(t_\star,x_\star,\xi_\star)\,\mathrm{d}\xi_\star,\\ &n_\star=(1-\bar\phi\phi_\star),\qquad u_d(t,x)=\mathrm{U}\;u_\star(t_\star,x_\star),\\ &\mu=\bar\mu\;\mu_\star(n_\star),\qquad \nu=\nu_\star(n_\star).\end{aligned}$$

We rescale the pressure as follows

$$p(t,x) = \mathrm{P}p_\star(t_\star,x_\star), \qquad \mathrm{P} = \bar\rho_d \mathrm{U}^2.$$

For the external potential, we set

$$\bar\rho_p \mathscr{E}(x) = \mathrm{P}\mathscr{E}_\star(x_\star).$$

In what follows we assume the following scaling relation

$$\eta = \frac{1}{\sqrt{\varepsilon}}, \qquad \bar\phi\frac{\bar\rho_p}{\bar\rho_d} = \varepsilon,$$

while the diffusion coefficient scales as $\frac{\bar\mu \mathrm{T}}{\bar\rho_d \mathrm{L}^2} = 1$. (The keypoint is to prescribe the behavior of these quantities with respect to the scaling parameter ε; in these relations, 1 can be replaced by any positive constant.) We thus rewrite the system in the following dimensionless form

$$\begin{aligned}
&\partial_{t_\star} F_\star + \frac{1}{\sqrt{\varepsilon}}\xi_\star\cdot\nabla_{x_\star}F_\star = \frac{\mu_\star(n_\star)}{\varepsilon}\nu_\star(n_\star)\nabla_{\xi_\star}\cdot\big((\xi_\star-\sqrt{\varepsilon}u_\star)F_\star + \nabla_{\xi_\star}F_\star\big) + \frac{\bar\phi}{\sqrt{\varepsilon}}\nabla_{\xi_\star}\cdot(F_\star\nabla_{x_\star}(p_\star+\mathscr{E}_\star)),\\
&\partial_{t_\star} n_\star + \nabla_{x_\star}\cdot(n_\star u_\star) = 0,\\
&\partial_{t_\star}(n_\star u_\star) + \nabla_{x_\star}\cdot(n_\star u_\star\otimes u_\star) + \nabla_{x_\star}p_\star\\
&\qquad = \nabla_{x_\star}\cdot\big(\mu_\star\mathbb{D}(u_\star)\big) - \bar\kappa n\nabla_{x_\star}\mathscr{E} + \mu_\star(n_\star)\nu_\star(n_\star)\int_{\mathbb{R}^3}\Big(\frac{1}{\sqrt{\varepsilon}}\xi_\star - u_\star\Big)F_\star\,\mathrm{d}\xi_\star + \bar\phi\nabla_{x_\star}p_\star\int F_\star\,\mathrm{d}\xi_\star\\
&\qquad = \nabla_{x_\star}\cdot\big(\mu_\star\mathbb{D}(u_\star)\big) - \bar\kappa n\nabla_{x_\star}\mathscr{E} + \mu_\star(n_\star)\nu_\star(n_\star)\,(\phi_\star V_\star - \phi_\star u_\star) + \bar\phi\phi_\star\nabla_{x_\star}p_\star,
\end{aligned} \tag{3.15}$$

with $\bar\kappa = \kappa\frac{\bar\rho_d}{\bar\rho_p}$. Let us only detail where the expression of the coefficient in front of the pressure term in the kinetic equation comes from. Passing to dimensionless quantities, this coefficient reads

$$\frac{\mathrm{P}}{\mathrm{L}}\frac{\mathrm{T}}{\mathrm{V}_{th}}\frac{1}{\bar\rho_p} = \frac{\mathrm{P}}{\bar\rho_d}\frac{1}{\mathrm{U}\mathrm{V}_{th}}\frac{\bar\rho_d}{\bar\rho_p} = \frac{\mathrm{U}}{\mathrm{V}_{th}}\frac{\bar\rho_d}{\bar\rho_p} = \bar\phi\frac{1}{\varepsilon}\sqrt{\varepsilon} = \frac{\bar\phi}{\sqrt{\varepsilon}}.$$

A similar computation holds for the external force.

The moment equations, see (3.12) and (3.14) respectively, become

$$\begin{aligned}
&\partial_{t_\star}\phi_\star + \nabla_{x_\star}\cdot\phi_\star V_\star = 0,\\
&\varepsilon\partial_{t_\star}\phi_\star V_\star + \nabla_{x_\star}\cdot\mathbb{P}_\star = \mu_\star(n_\star)\nu_\star(n_\star)(\phi_\star u_\star - J_\star) - \bar\phi\phi_\star\nabla_{x_\star}(p_\star+\mathscr{E}_\star).
\end{aligned} \tag{3.16}$$

The volume conservation

$$\bar\phi\phi_\star + n_\star = \bar\phi\phi_\star + (1-\bar\phi\phi_\star) = 1$$

yields the constraint

$$\nabla_x\cdot(\bar\phi\phi_\star V_\star + (1-\bar\phi\phi_\star)u_\star) = 0.$$

Of course, the dissipation property exhibited in the previous Section can be rephrased in dimensionless form.

We are interested in the asymptotics

$$\varepsilon \to 0, \qquad \bar\phi \to \bar\phi_\ell \in [0,1].$$

We shall obtain this way a set of macroscopic equations — with unknowns depending only on the time-space variables t, x — that describe the mixture flow. We shall see that assuming $\bar{\phi}_\ell > 0$ leads to interesting features, with a complex pressure law, while $\bar{\phi}_\ell = 0$ describes a fully incompressible regime. Let us assume that all macroscopic quantities admit limits

$$\phi_\star,\ \phi_\star V_\star,\ u_\star,\ p_\star \xrightarrow[\varepsilon\to 0,\ \bar{\phi}\to\bar{\phi}_\ell]{} \phi_\ell,\ J_\ell,\ u_\ell,\ p_\ell, \tag{3.17}$$

while $n_\star$ tends to $n_\ell = 1 - \bar{\phi}_\ell \phi_\ell$. As ε tends to 0, one expects that the particle distribution function tends to an element of the kernel of the leading (Fokker-Planck) operator

$$\nabla_{\xi_\star} \cdot (\xi_\star F + \nabla_{\xi_\star} F) = \nabla_{\xi_\star} \cdot \left(e^{-\xi^2/2} \nabla_\xi \left(\frac{F}{e^{-\xi^2/2}} \right) \right)$$

which means

$$F_\star(t_\star, x_\star, \xi_\star) \simeq \phi_\star(t_\star, x_\star) \left(\frac{1}{2\pi}\right)^{3/2} \exp\Big(- \frac{|\xi_\star|^2}{2} \Big).$$

According to this ansatz, the second moment looks like

$$\mathbb{P}_\star(t_\star, x_\star) \simeq \frac{\phi_\star(t_\star, x_\star)}{(2\pi)^{3/2}} \int \xi_\star \otimes \xi_\star \exp\Big(- \frac{|\xi_\star|^2}{2} \Big) \,\mathrm{d}\xi = \phi_\star(t_\star, x_\star)\mathbb{I}.$$

With the moment equations, we are thus led to

$$\begin{aligned}
&\partial_{t_\star} \phi_\ell + \nabla_{x_\star} \cdot J_\ell = 0, \\
&J_\ell = \phi_\ell \left(u_\ell - \frac{\bar{\phi}_\ell}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star}(p_\ell + \mathscr{E}_\star) \right) - \frac{1}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star} \phi_\ell.
\end{aligned}$$

Therefore the particles density obeys

$$\partial_{t_\star} \phi_\ell + \nabla_{x_\star} \cdot \left(\phi_\ell \Big(u_\ell - \frac{\bar{\phi}_\ell}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star}(p_\ell + \mathscr{E}_\star) - \frac{1}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star} \ln(\phi_\ell) \Big) \right) = 0.$$

The incompressibility of the mean volume velocity becomes

$$\nabla_{x_\star} \cdot \Big(u_\ell - \frac{\bar{\phi}_\ell^2 \phi_\ell}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star}(p_\ell + \mathscr{E}_\star) - \frac{\bar{\phi}_\ell}{\mu_\star(n_\ell)\nu_\star(n_\ell)} \nabla_{x_\star} \phi_\ell \Big) = 0.$$

Observe that $\bar{\phi}_\ell = 0$ simply means that the limit velocity field u_ℓ is solenoidal; $\bar{\phi}_\ell > 0$ leads to an hybrid behavior: the fluid behaves as incompressible only when the disperse phase is absent ($\phi_\ell = 0$). Note that the mean mass density of the mixture reads $\bar{\rho}_d(1 - \phi) + \bar{\rho}_p \phi$, which becomes in rescaled variables $1 - \bar{\phi}\phi_\star + \bar{\phi}\frac{\bar{\rho}_p}{\bar{\rho}_d}\phi_\star = 1 - \bar{\phi}\phi_\star + \varepsilon\phi_\star \to (1 - \bar{\phi}_\ell \phi_\ell) = n_\ell$. Passing to the limit in the mass conservation yields

$$\partial_t n_\ell + \nabla_x \cdot (n_\ell u_\ell) = 0,$$

and the momentum equation leads to

$$\begin{aligned}
\partial_{t_\star}(n_\ell u_\ell) + \nabla_{x_\star} \cdot (n_\ell u_\ell \otimes u_\ell) + \nabla_{x_\star} p_\ell - \nabla_{x_\star} \cdot (\mu_\star(n_\ell)\mathbb{D}(u_\ell)) \\
= -\bar{\kappa} n_\ell \nabla_{x_\star} \mathscr{E} + \mu_\star(n_\ell)\nu_\star(n_\ell)(J_\ell - \phi_\ell u_\ell) + \bar{\phi}_\ell \phi_\ell \nabla_{x_\star} p_\ell \\
= -(\bar{\kappa} n_\ell + \bar{\phi}_\ell \phi_\ell) \nabla_{x_\star} \mathscr{E} - \nabla_{x_\star} \phi_\ell.
\end{aligned}$$

Up to a change of notation, we have thus derived (1.2)–(1.4).

Proposition 3.2 *Let $(F_\star, n_\star, u_\star, p_\star)$ be a sequence of solutions of (3.15). As ϵ goes to 0, we suppose that $F_\star(t_\star, x_\star, \xi_\star) \to \phi_\ell(t_\star, x_\star)\ (2\pi)^{-3/2}\exp(-|\xi_\star|^2/2)$ and (3.17) hold in a strong enough sense. Then, the limit defines a solution of (1.2)–(1.4).*

Let us end this Section with a few comments on the technical issues. First of all, the fluid-kinetic system (3.7)-(3.11) is a highly non linear system, which couples the evolution of unknowns that do not depend on the same set of variables. Hence, the existence theory, even for smooth solutions locally in time, is certainly far from obvious. We can mention [3] (local existence of smooth solutions for an inviscid fluid), [19] (global existence of solutions close to equilibrium for an inviscid fluid), [36] (global existence of solutions close to equilibrium for a viscous fluid), [7, 38, 47] (global existence of weak solutions for a viscous fluid) for results in this direction with a simpler coupling, but we are not aware of analysis in the framework of thick sprays. Second of all, even in one assumes the existence of solutions satisfying the uniform estimates that will be discussed in the forthcoming Section, justifying the asymptotic behavior remains a tough piece of analysis, due to the passage to the limits in nonlinearities, see e. g. [22, 34, 35, 49] for the study of related questions.

4 Dissipation Properties and a priori Estimates

Now, we are going to discuss a priori estimates satisfied by solutions of (1.2)–(1.4). The discussion remains formal; it assumes the existence of smooth enough solutions. But the study provides natural estimates that solutions of the problem can be expected to satisfy.

In view of its physical interpretation as a volume fraction, $\bar{\phi}\phi$ is expected to belong to the interval $[0, 1]$. By combining (1.2) and (1.4), we can check that the particle volume fraction satisfies

$$\partial_t \phi = -\nabla_x(\phi u) + \frac{1}{\bar{\phi}}\nabla_x \cdot u = -\nabla_x \cdot \left(\phi\Big(u - \frac{1}{\mu\nu}\nabla_x(\bar{\phi}p + \bar{\phi}\mathscr{E})\Big)\right) + \nabla_x \cdot \left(\frac{1}{\mu\nu}\nabla_x\phi\right). \quad (4.18)$$

Therefore, $n = 1 - \bar{\phi}\phi$ and ϕ are solutions of the convection equation (1.2) and the convection-diffusion equation (4.18), respectively, so that starting from non negative initial data, they both remain non negative. This statement remains to be justified in the construction of solutions through a suitable approximation procedure, but the first basic a priori estimate is

$$0 \leq n = 1 - \bar{\phi}\phi \leq 1. \quad (4.19)$$

As already said above, for the behavior at infinity, we assume that ϕ vanishes and the mixture is only made of the carrier phase at infinity. Thus, integrating (4.18) we are led to

$$\int \phi \,\mathrm{d}x = \int \phi_{\mathrm{Init}} \,\mathrm{d}x.$$

4.1 Entropy Dissipation

The hydrodynamic model (1.2)–(1.4) inherits a dissipative structure from the microscopic model. Indeed, let us start by computing

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \Big(\phi\ln(\phi)+n\frac{u^2}{2}\Big)\,\mathrm{d}x &= \int \phi\Big(u-\frac{\bar\phi}{\mu\nu}\nabla_x(p+\mathscr{E})-\frac{1}{\mu\nu}\nabla_x\ln(\phi)\Big)\cdot\frac{\nabla_x\phi}{\phi}\,\mathrm{d}x\\
&\quad -\int\nabla_x(p+\phi)\cdot u\,\mathrm{d}x-\frac12\int\mu|\nabla_x u|^2\,\mathrm{d}x-\int(\kappa n+\bar\phi\phi)\nabla_x\mathscr{E}\cdot u\,\mathrm{d}x\\
&= -\int\nabla_x p\cdot\Big(\frac{\bar\phi}{\mu\nu}\nabla_x\phi+u\Big)\,\mathrm{d}x-\int\frac{1}{\mu\nu}\frac{|\nabla_x\phi|^2}{\phi}\,\mathrm{d}x\\
&\quad -\frac12\int\mu|\nabla_x u|^2\,\mathrm{d}x-\int(\kappa n+\bar\phi\phi)\nabla_x\mathscr{E}\cdot u\,\mathrm{d}x-\int\frac{\bar\phi}{\mu\nu}\nabla_x\mathscr{E}\cdot\nabla_x\phi\,\mathrm{d}x.
\end{aligned}$$

The terms which involve the pressure p can be rewritten as follows

$$\begin{aligned}
\int p\nabla_x\cdot\Big(\frac{\bar\phi}{\mu\nu}\nabla_x\phi+u\Big)\,\mathrm{d}x &= \int\Big\{p\nabla_x\cdot\Big(u-\frac{\bar\phi}{\mu\nu}\nabla_x\phi\Big)+2p\nabla_x\cdot\frac{\bar\phi}{\mu\nu}\nabla_x\phi\Big\}\,\mathrm{d}x\\
&= -\int\frac{\bar\phi^2}{\mu\nu}\phi|\nabla_x p|^2\,\mathrm{d}x-\int\frac{\bar\phi^2}{\mu\nu}\phi\nabla_x p\cdot\nabla_x\mathscr{E}\,\mathrm{d}x-2\int\nabla_x p\cdot\frac{\bar\phi}{\mu\nu}\nabla_x\phi\,\mathrm{d}x
\end{aligned}$$

by using (1.4). Besides, we have, on the one hand

$$\bar\phi\frac{\mathrm{d}}{\mathrm{d}t}\int\phi\mathscr{E}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int(1-n)\mathscr{E}\,\mathrm{d}x=-\int nu\cdot\nabla_x\mathscr{E}\,\mathrm{d}x.$$

and, on the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\phi\mathscr{E}\,\mathrm{d}x=\int\Big(\phi u-\frac{\bar\phi\phi}{\mu\nu}\nabla_x(p+\mathscr{E})-\frac{1}{\mu\nu}\nabla_x\phi\Big)\cdot\nabla_x\mathscr{E}\,\mathrm{d}x.$$

Combining these relations, we are led to the following statement.

Proposition 4.1 *Let* (n,u,p) *be a smooth enough solution of* (1.2)–(1.4)*. Let us introduce the following free-energy*

$$\mathscr{G}=\int\Big(\phi\ln(\phi)+(1-\kappa)\bar\phi\ \phi\mathscr{E}+n\frac{u^2}{2}\Big)\,\mathrm{d}x.$$

Then, we have $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G}\le 0$.

More precisely, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{G} &= -\int\mu|\nabla_x u|^2\,\mathrm{d}x\\
&\quad-\int\Big(\frac{\bar\phi^2\phi}{\mu\nu}|\nabla_x p|^2+\frac{\bar\phi^2\phi}{\mu\nu}|\nabla_x\mathscr{E}|^2+\frac{1}{\mu\nu}\frac{|\nabla_x\phi|^2}{\phi}\\
&\qquad\quad+2\nabla_x p\cdot\frac{\bar\phi}{\mu\nu}\nabla_x\phi+2\frac{\bar\phi}{\mu\nu}\nabla_x\mathscr{E}\cdot\nabla_x\phi+2\frac{\bar\phi^2\phi}{\mu\nu}\nabla_x\mathscr{E}\cdot\nabla_x p\Big)\,\mathrm{d}x\\
&= -\int\mu|\nabla_x u|^2\,\mathrm{d}x-\int\Bigg(\bar\phi\sqrt{\frac{\phi}{\mu\nu}}\nabla_x(p+\mathscr{E})+2\sqrt{\frac{1}{\mu\nu}}\nabla_x\sqrt{\phi}\Bigg)^2\,\mathrm{d}x.
\end{aligned}$$

4.2 A priori Estimates

In order to make this property a useful a priori estimate, we need further assumptions on the external potential. Roughly speaking, $\mathscr{E}$ should be confining on the disperse phase. Technically, we require

$$
(\mathbf{H1}) \qquad \begin{cases} 1-\kappa>0, \\ x\mapsto \mathscr{E}(x) \text{ is a Lipschitz function on } \mathbb{R}^N, \\ \text{for a. e. } x\in\mathbb{R}^N,\ \mathscr{E}(x)\geq 0, \\ \text{there exists } 0<r\leq (1-\kappa)/4 \text{ such that } x\mapsto e^{-r\mathscr{E}(x)}\in L^1(\mathbb{R}^N). \end{cases}
$$

As a matter of fact note that $x\mapsto e^{-r\mathscr{E}(x)}$ is Lipschitz and integrable, which implies that it tends to 0 as $|x|\to\infty$, and thus we have $\lim_{|x|\to\infty}\mathscr{E}(x)=+\infty$.

Lemma 4.1 *Assume (H1). Let $\phi:\mathbb{R}^N\to[0,\infty)$ satisfy $\phi\in L^1(\mathbb{R}^N)$. We assume that*

$$\int \big(\phi\ln(\phi)+(1-\kappa)\phi\mathscr{E}\big)\,\mathrm{d}x\leq C$$

holds for some $0<C<\infty$. Then, there exists $0<M<\infty$ such that

$$\int \big(\phi|\ln(\phi)|+\phi\mathscr{E}\big)\,\mathrm{d}x\leq M.$$

Proof. The trick consists in writing

$$
\begin{aligned}
\phi|\ln(\phi)| &= \phi\ln(\phi)-2\phi\ln(\phi)\mathbf{1}_{0\leq\phi\leq 1}=\phi\ln(\phi)-2\phi\ln(\phi)\mathbf{1}_{e^{-2r\mathscr{E}}\leq\phi\leq 1}-2\phi\ln(\phi)\mathbf{1}_{0\leq\phi\leq e^{-2r\mathscr{E}}} \\
&\leq \phi\ln(\phi)+4r\mathscr{E}\phi+\frac{4}{e}\,e^{-r\mathscr{E}}.
\end{aligned}
$$

by using the elementary inequality $-z\ln(z)=2\sqrt{z}(-\sqrt{z})\ln(\sqrt{z})\leq\frac{2}{e}\sqrt{z}$ for any $z\in[0,1]$. We conclude by integrating over $\mathbb{R}^N$. ∎

We can therefore deduce from the dissipation property the following a priori estimate.

Proposition 4.2 *Assume (H1). Let the initial data satisfy*

$$
\begin{aligned}
&0\leq n_{\text{Init}}=1-\bar{\phi}\phi_{\text{Init}}\leq 1, \\
&\mathscr{M}_0=\int\Big(\phi_{\text{Init}}+\phi_{\text{Init}}|\ln(\phi_{\text{Init}})|+(1-\kappa)\bar{\phi}\ \phi_{\text{Init}}\mathscr{E}+n_{\text{Init}}\frac{u_{\text{Init}}^2}{2}\Big)\,\mathrm{d}x<\infty.
\end{aligned}
\tag{4.20}
$$

Then, the solutions of (1.2)–(1.4) *satisfy*

$$
\begin{aligned}
&\sup_{t\geq 0}\int\Big(\phi+\phi|\ln(\phi)|+(1-\kappa)\bar{\phi}\ \phi\mathscr{E}+n\frac{u^2}{2}\Big)\,\mathrm{d}x<\infty, \\
&\int_0^\infty\int \mu|\nabla_x u|^2\,\mathrm{d}x<\infty, \\
&\int_0^\infty\int\Big(\bar{\phi}\sqrt{\frac{\phi}{\mu\nu}}\nabla_x(p+\mathscr{E})+2\sqrt{\frac{1}{\mu\nu}}\nabla_x\sqrt{\phi}\Big)^2\,\mathrm{d}x<\infty.
\end{aligned}
$$

Remark 4.1 *It is worth remarking that Proposition 4.2 combined to* (1.3) *provides an estimate on* $\nabla_x p$*, in a negative Sobolev space (with respect to time and space variables), for instance* $\nabla_x p$ *lies in* $W^{-1,1}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$*, say.*

This statement provides the basic estimate that can be used to analyze the system. We will be interested in the stability of solutions: we are wondering whether a sequence of solutions of (1.2)–(1.4) which satisfies uniformly the estimates in Proposition 4.2 (admits a subsequence which) converges to a solution of (1.2)–(1.4). Of course, we have in mind to construct approximated solutions to the problem, and to justify the existence of solutions to (1.2)–(1.4) by extending the stability argument to such approximated solutions. Clearly, due to the nonlinearities we need to justify the compactness of the sequences in a strong enough sense, and Proposition 4.2 seems insufficient for this purpose.

5 The One Dimension Case: Further Dissipation Properties and Uniform Estimates

Restricting to the framework of dimension one, we are going to exhibit further dissipative properties of the system, which involve the space derivative of the density n. The idea consists in considering the modified energy

$$\frac{1}{2}\int n\Big|u+\frac{\partial_x\psi(n)}{n}\Big|^2\,\mathrm{d}x$$

where the function $\psi : [0,\infty)\to[0,\infty)$ has to be suitably chosen, depending on the viscosity μ. This method has been used successfully for classical systems of fluid mechanics, see [60, 61, 65] and [48], including in higher dimensions [12, 13, 16]. In what follows we assume that μ and ν are continuous functions, with continuous derivatives. The viscosity is supposed to satisfy

(**H2**) There exists $0\le r<1/2$, $\underline{\mu}>0$ such that for any $0\le n\le 1$, $\mu(n)\ge\underline{\mu}n^r$.

For the drag coefficient, we assume

(**H3**) There exists $\overline{\nu}>0$ such that for any $0<n\le 1$, $\dfrac{\mu(n)^2}{n^3}\,\nu(n)\le\overline{\nu}$.

Finally, we also need the following property on the behavior of the potential

$$(\mathbf{H1'})\qquad \partial_x\mathscr{E}\in L^2(\mathbb{R}).$$

Remark 5.1 *Clearly, we can assume that* $\mu=\bar{\mu}$ *is constant. In this case, the drag coefficient satisfies* $\nu(n)\le(\overline{\nu}/\underline{\mu})n^3$*. More generally, with (H2), we get* $\nu(n)\le(\overline{\nu}/\underline{\mu})n^{3-2r}$*. At first sight, this modification of the usual Stokes' law might be considered as surprising. However, the assumption essentially modifies the Stokes law for small values of* n*, which means that the carrier fluid is rarefied. In such situations the derivation of Stokes' formula itself becomes questionable, because slip on the particles cannot be neglected. Note also that the definition of the viscosity itself when* *the density tends to 0 is the object of more or less empirical extension. We refer the reader to the discussion in [63] on relevant corrections to the Stokes law; in particular the formula proposed by Cunningham for the drag coefficient makes a density dependent term appear. Finally, we shall see that the role of hypothesis (H2) and (H3) is precisely to avoid the formation of vacuum! The situation parallels with the analysis of standard fluid mechanics equations, and we can just recopy the motivation from [39]:* vacuum states cannot occur at positive times if none are present initially. We recall in this regard that the physical derivation of the Navier–Stokes system presupposes that the fluid in question is nondilute. Our result therefore establishes an important self consistency for this model.

Remark 5.2 *An example of a potential which satisfies (H1)-(H1') is* $\mathscr{E}(x) = (1 + x^2)^\alpha$, *with* $0 < \alpha < 1/4$.

The computation relies on the following observations. Firstly, we have

$$\partial_t \psi(n) + \partial_x(\psi(n)u) + (n\psi'(n) - \psi(n))\partial_x u = 0.$$

Secondly, for any quantity h which satisfies $\partial_t h + \partial_x(hu) = H$, owing to the mass conservation (1.2), we get

$$\partial_t(nh) + \partial_x(nhu) = nH - nh\partial_x u = n\partial_t h + nu\partial_x h.$$

Then, we recall that

$$n(\partial_t u + u\partial_x u) + \partial_x p = \partial_x(\mu\partial_x u) - \partial_x\phi - (\kappa n + \bar{\phi}\phi)\partial_x\mathscr{E}.$$

We check that

$$\begin{aligned}
&\partial_t\Big(n\frac{\partial_x\psi(n)}{n}\Big) + \partial_x\Big(nu\frac{\partial_x\psi(n)}{n}\Big) + \partial_x(n\psi'(n)\partial_x u)\\
&\quad= n\partial_t\Big(\frac{\partial_x\psi(n)}{n}\Big) + nu\partial_x\Big(\frac{\partial_x\psi(n)}{n}\Big) + \partial_x(n\psi'(n)\partial_x u) = 0
\end{aligned}$$

holds. Accordingly, we obtain

$$\begin{aligned}
&n\partial_t\Big(u + \frac{\partial_x\psi(n)}{n}\Big) + nu\partial_x\Big(u + \frac{\partial_x\psi(n)}{n}\Big) + \partial_x(p+\phi)\\
&\quad= \partial_x\big((\mu - n\psi'(n))\partial_x u\big) - (\kappa n + \bar{\phi}\phi)\partial_x\mathscr{E}\\
&\quad= \partial_t\Big(n\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\Big) + \partial_x\Big(nu\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\Big) + \partial_x(p+\phi).
\end{aligned}$$

These observations allow us to compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int n\Big|u + \frac{\partial_x\psi(n)}{n}\Big|^2\,\mathrm{d}x\\
&\quad= \frac{1}{2}\int n\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\partial_t\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\,\mathrm{d}x + \frac{1}{2}\int\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\partial_t\Big(n\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\Big)\,\mathrm{d}x\\
&\quad= -\frac{1}{2}\int\partial_x\Big(nu\Big|u + \frac{\partial_x\psi(n)}{n}\Big|^2\Big)\,\mathrm{d}x\\
&\qquad+ \int\Big(u + \frac{\partial_x\psi(n)}{n}\Big)\Big(-\partial_x(p+\phi) + \partial_x\big((\mu(n) - n\psi'(n))\partial_x u\big) - (\kappa n + \bar{\phi}\phi)\partial_x\mathscr{E}\Big)\,\mathrm{d}x.
\end{aligned}$$

From now on, we select the function ψ in order to get rid of the viscous term; namely ψ is required to satisfy

$$n\psi'(n) = \mu(n). \tag{5.21}$$

We are going to combine the previous relations to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \big(\phi\ln(\phi) + \bar\phi(1-\kappa)\phi\mathscr{E}\big)\,\mathrm{d}x &= \kappa\int nu\partial_x\mathscr{E}\,\mathrm{d}x\\ &+\int\Big(\phi u - \frac{\bar\phi}{\mu\nu}\phi\partial_x(p+\mathscr{E}) - \frac{1}{\mu\nu}\partial_x\phi\Big)\Big(\frac{\partial_x\phi}{\phi} + \bar\phi\partial_x\mathscr{E}\Big)\,\mathrm{d}x.\end{aligned}$$

Let us set

$$\widetilde{\mathscr{H}} = \int\Big(\phi\ln(\phi) + \bar\phi(1-\kappa)\phi\mathscr{E} + \frac{1}{2}n\Big|u + \frac{\partial_x\psi(n)}{n}\Big|^2\Big)\,\mathrm{d}x.$$

We are led to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{H}} &= \int p\partial_x u\,\mathrm{d}x - \int\frac{1}{\mu\nu}\frac{|\partial_x\phi|^2}{\phi}\,\mathrm{d}x - \int\frac{\bar\phi}{\mu\nu}\partial_x\phi\partial_x\mathscr{E}\,\mathrm{d}x\\ &-\int\frac{\partial_x\psi(n)}{n}\big(\partial_x(p+\phi) + (\kappa n + \bar\phi\phi)\partial_x\mathscr{E}\big)\,\mathrm{d}x\\ &-\int\frac{\bar\phi}{\mu\nu}\partial_x(p+\mathscr{E})\big(\partial_x\phi + \bar\phi\phi\partial_x\mathscr{E}\big)\,\mathrm{d}x.\end{aligned}$$

The constraint (1.4) yields

$$\int p\partial_x u\,\mathrm{d}x = -\int\frac{\bar\phi}{\mu\nu}\partial_x p\big(\partial_x\phi + \bar\phi\phi\partial_x(p+\mathscr{E})\big)\,\mathrm{d}x.$$

Let us set

$$\Pi = \ln(\phi) + \bar\phi(p+\mathscr{E}).$$

Hence, by using $1-\bar\phi\phi = n$ and $\partial_x\phi = -\partial_x n/\bar\phi$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{H}} = -\int\frac{\phi}{\mu\nu}\big|\partial_x\Pi\big|^2\,\mathrm{d}x - \int\frac{\psi'(n)}{\bar\phi^2\phi}|\partial_x n|^2\,\mathrm{d}x + \int(1-\kappa)\partial_x\psi(n)\partial_x\mathscr{E}\,\mathrm{d}x - \int\frac{\partial_x\psi(n)}{n\bar\phi}\partial_x\Pi\,\mathrm{d}x. \tag{5.22}$$

We aim at estimating the last two terms by either the dissipated quantities or the dissipation terms. This is where we appeal to (H2)-(H3). Indeed, we simply use the Cauchy-Schwarz and Young inequalities to estimate

$$\begin{aligned}\Big|\int\frac{\partial_x\psi(n)}{n\bar\phi}\partial_x\Pi\,\mathrm{d}x\Big| &\le \Big(\int\frac{\phi}{\mu\nu}|\partial_x\Pi|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int\frac{|\psi'(n)|^2}{\bar\phi^2 n^2}\frac{\mu\nu}{\phi}|\partial_x n|^2\,\mathrm{d}x\Big)^{1/2}\\ &\le \frac{1}{2}\int\frac{\phi}{\mu\nu}|\partial_x\Pi|^2\,\mathrm{d}x + \frac{\overline{\nu}}{2}\int\frac{\psi'(n)}{\bar\phi^2\phi}|\partial_x n|^2\,\mathrm{d}x,\end{aligned}$$

which can be absorbed in the first two terms in the right hand side of (5.22) provided $0 < \overline{\nu} < 2$. We treat similarly the penultimate term in (5.22), by making use of (H1') and $0 \le n \le 1$

$$\begin{aligned}\Big|\int(1-\kappa)\partial_x\psi(n)\partial_x\mathscr{E}\,\mathrm{d}x\Big| &\le (1-\kappa)\Big(\int\frac{|\partial_x\psi(n)|^2}{n}\,\mathrm{d}x\Big)^{1/2}\Big(\int n|\partial_x\mathscr{E}|^2\,\mathrm{d}x\Big)^{1/2}\\ &\le \frac{1-\kappa}{2}\Big(\|\partial_x\mathscr{E}\|_{L^2}^2 + \int n\Big|\frac{\partial_x\psi(n)}{n}\Big|^2\,\mathrm{d}x\Big).\end{aligned}$$

The last term can be dominated as follows

$$\int n\Big|\frac{\partial_x\psi(n)}{n}\Big|^2\,\mathrm{d}x \le 2\int n\Big|\frac{\partial_x\psi(n)}{n}+u\Big|^2\,\mathrm{d}x + 2\int n|u|^2\,\mathrm{d}x \tag{5.23}$$

Therefore, we combining these information, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{H}} + \frac{1}{2}\int\frac{\phi}{\mu\nu}\big|\partial_x\Pi\big|^2\,\mathrm{d}x + \Big(1-\frac{\overline{\nu}}{2}\Big)\int\frac{\psi'(n)}{\bar{\phi}^2\phi}|\partial_x n|^2\,\mathrm{d}x\\ \le \frac{1-\kappa}{2}\Big(\|\partial_x\mathscr{E}\|_{L^2} + 2\int n|u|^2\,\mathrm{d}x\Big) + (1-\kappa)\int n\Big|\frac{\partial_x\psi(n)}{n}+u\Big|^2\,\mathrm{d}x.\end{aligned}$$

Applying the Grönwall lemma, we conclude with the following statement.

Proposition 5.1 *Let $N = 1$. We assume that (H1), (H1'), (H2) and (H3) are satisfied with $0 < \underline{\nu} < 2$. Let the initial data fulfill (4.20) together with*

$$\mathscr{M}_1 = \int n_{\mathrm{Init}}\Big|\frac{\partial_x\psi(n_{\mathrm{Init}})}{n_{\mathrm{Init}}} + u_{\mathrm{Init}}\Big|^2\,\mathrm{d}x < \infty. \tag{5.24}$$

Then, for any $0 < T < \infty$, there exists a constant $0 < C_T < \infty$, which depends on $\mathscr{M}_0, \mathscr{M}_1, \overline{\nu}, \kappa$, $\|\partial_x\mathscr{E}\|_{L^2}$ and T, such that the solutions of (1.2)–(1.4) satisfy

$$\begin{aligned}&i)\quad \sup_{0\le t\le T}\int n\Big|\frac{\partial_x\psi(n)}{n}+u\Big|^2\,\mathrm{d}x \le C_T, && ii)\quad \int_0^T\int\frac{1}{2}\int\frac{\phi}{\mu\nu}\big|\partial_x\Pi\big|^2\,\mathrm{d}x\,\mathrm{d}s \le C_T,\\ &iii)\quad \int_0^T\int\frac{\psi'(n)}{\bar{\phi}^2\phi}|\partial_x n|^2\,\mathrm{d}x\,\mathrm{d}s \le C_T, && iv)\quad \sup_{0\le t\le T}\int\Big|\frac{\partial_x\psi(n)}{\sqrt{n}}\Big|^2\,\mathrm{d}x \le C_T.\end{aligned}$$

Estimate iv) is deduced from i) and Proposition 4.2, by using the same argument as in (5.23). From Proposition 5.1 we can establish that n is bounded from below away from vacuum.

Corollary 5.1 *Let the assumptions of Proposition 5.1 be fulfilled. Then, for any $0 < T < \infty$, there exists $0 < m_T < 1$ such that for a.e. $(t, x) \in (0, T) \times \mathbb{R}$, we have $0 < m_T \le n(t, x) \le 1$.*

The proof relies on the application of the following claim.

Lemma 5.1 *Let $u : \mathbb{R} \to [0, 1]$ be such that:*

a) $(1 - u) \in L^1(\mathbb{R})$,

b) There exists a non increasing function $Z : (0, \infty) \to (0, \infty)$ such that $\lim_{s\to 0} Z(s) = +\infty$ and $x \mapsto \partial_x Z(u(x)) \in L^2(\mathbb{R})$.

Then, there exists $\gamma > 0$ such that for a.e. $x \in \mathbb{R}$, we have $u(x) \ge \gamma$.

Proof. From a), we argue by contradiction to show that there exists $0 < \epsilon < 1$, and $R > 0$ such that for any $x \in \mathbb{R}$ we can find a measurable subset $\mathscr{I}_x \subset [x - R, x + R]$ verifying $\mathrm{meas}(\mathscr{I}_x) > 0$ and $u(y) \geq \epsilon$ on $\mathscr{I}_x$. Indeed, assume conversely, for any $\epsilon \in (0, 1)$, $R > 0$ the existence of $x_{\varepsilon,R} \in \mathbb{R}$ such that $u(y) \leq \epsilon$ for a.e. $y \in [x_{\epsilon,R} - R, x_{\epsilon,R} + R]$. Then, we would have

$$2R(1-\epsilon) \leq \int_{x_{\epsilon,R}-R}^{x_{\epsilon,R}+R} \big(1 - u(y)\big)\,\mathrm{d}y \leq \|1 - u\|_{L^1}.$$

Keeping ϵ fixed and letting R run to ∞ yields a contradiction.

Next we use b) as follows. Let $x \in \mathbb{R}$. For any $y \in \mathscr{I}_x$, we write

$$Z(u(x)) = Z(u(y)) + \int_y^x \partial_x Z(u(\xi))\,\mathrm{d}\xi \leq Z(\varepsilon) + \|\partial_x Z(u)\|_{L^2}\ \sqrt{|x-y|} \leq Z(\varepsilon) + \|\partial_x Z(u)\|_{L^2}\sqrt{R}.$$

This shows that $x \mapsto Z(u(x))$ belongs to $L^\infty(\mathbb{R})$. In turn the behavior of the function Z at the origin implies that u is bounded from below, away from 0. ∎

Proof of Corollary 5.1. We apply directly Lemma 5.1 to $n(t, x)$. To this end we define Z by

$$Z(s) = \int_s^1 \frac{\psi'(\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi = \int_s^1 \frac{\mu(\xi)}{\xi^{3/2}}\,\mathrm{d}\xi$$

so that $n\left|\frac{\partial_x \psi(n)}{n}\right|^2 = |\partial_x Z(n)|^2$. Owing to (H2), we have

$$Z(s) \geq \underline{\mu} \int_s^1 \frac{\mathrm{d}\xi}{\xi^{3/2-r}} \xrightarrow[s\to 0]{} \infty.$$

As a consequence of (4.20), and (5.24), $(1 - n_{\mathrm{Init}}) \in L^1(\mathbb{R})$ with $\partial_x Z(n_{\mathrm{Init}}) \in L^2(\mathbb{R})$, so that n_{Init} is bounded from below by a positive constant. This property propagates for any positive time $0 < T < \infty$. The bound from below m_T depends on $0 < T < \infty$ because the estimate on $\partial_x Z(n)$ in $L^\infty(0, T; L^2(\mathbb{R}))$ in Proposition 5.1 depends on the final time T. ∎

6 The One Dimension Case: Stability of the Solutions

In this Section, we stay in dimension one and we investigate the stability of the solutions of (1.2)–(1.4). We consider (n^k, u^k, p^k, ϕ^k), solutions of the system associated to a sequence of data $(n^k_{\mathrm{Init}}, u^k_{\mathrm{Init}}, p^k_{\mathrm{Init}}, \phi^k_{\mathrm{Init}})$ which verifies (4.20) and (5.24) uniformly with respect to the parameter $k \in \mathbb{N}$. The question we address is two–fold. On the one hand, we wish to deduce compactness properties of the sequence of solutions (n^k, u^k, p^k, ϕ^k), from the estimates established in the previous Section. On the other hand, dealing with subsequences that converge in a certain topology, we wish to determine whether or not the limit defines a solution of (1.2)–(1.4). This analysis leads to the following definition of weak solutions for the problem (1.2)–(1.4).

Definition 6.1 *Let $N = 1$. We say that (n, u, p, ϕ) is a weak solution of* (1.2)–(1.4)*, associated to the initial conditions* (1.5) *which are requested to satisfy* (4.20) *and* (5.24)*, if, for any $0 < T < \infty$,*

- $0 \le n \le 1$*, and* $0 \le \phi = \dfrac{1-n}{\bar\phi} \le \dfrac{1}{\bar\phi}$ *lies in* $H^1 \cap C^0([0,T] \times \mathbb{R})$*,*
- *the estimates in Proposition 4.2 and Proposition 5.1 are satisfied,*
- $\partial_x p$ *belongs to* $H^{-1}((0,T) \times \mathbb{R})$*, with* $\frac{\phi}{\mu\nu}\partial_x p = W = \sqrt{\frac{\phi}{\mu\nu}} Z$*, where* $\mathrm{supp}(W) \subset \{\phi > 0\}$*,* $W \in L^2((0,T) \times \mathbb{R})$*,* $\sqrt{\frac{\mu\nu}{\phi}} W \in L^2((0,T) \times \mathbb{R})$*,*

and for any $\zeta \in C_c^\infty([0,T) \times \mathbb{R})$, we have

$$\int (n\partial_t \zeta + nu\partial_x \zeta)\,\mathrm{d}x\,\mathrm{d}t + \int n_{\mathrm{Init}}(x)\zeta(0,x)\,\mathrm{d}x = 0,$$
$$\int \big(nu\partial_t \zeta + (nu^2 - \mu\partial_x u + \phi)\partial_x \zeta + (\kappa n + \bar\phi\phi)\partial_x \mathscr{E}\zeta\big)\,\mathrm{d}x\,\mathrm{d}t - \langle \partial_x p, \zeta \rangle + \int n_{\mathrm{Init}} u_{\mathrm{Init}}(x)\zeta(0,x)\,\mathrm{d}x = 0,$$
$$u = \frac{\bar\phi^2}{\mu\nu}(\phi\mathscr{E} + \partial_x \phi) + \bar\phi^2 W.$$

The remainder of the paper is devoted to the proof of the following stability statement.

Theorem 6.1 *Let the assumptions of Proposition 5.1 be fulfilled and consider a sequence of solutions (n^k, u^k, p^k, ϕ^k). Then we can assume that n^k (resp. ϕ^k) converges uniformly on compact sets to n (resp. ϕ), u^k converges strongly in $L^2_{\mathrm{loc}}((0,T) \times \mathbb{R})$ to u, and $\partial_x p^k$ converges to $\partial_x p$ weakly-$\star$ in $H^{-1}((0,T) \times \mathbb{R})$. The limit is a weak solution of* (1.2)–(1.4)*, where the product $\phi\partial_x p$ belongs to $L^2((0,T) \times \mathbb{R})$, and $\sqrt{\phi}\partial_x p \in L^2((0,T) \times \mathbb{R})$.*

In what follows, we will make use of the following classical compactness statement (referred to as the Aubin-Simon lemma).

Lemma 6.1 (Cor. 4 [62]) *Let X, B, Y be Banach spaces such that $X \subset B \subset Y$, the embedding $X \subset B$ being compact. Let $0 < T < \infty$ and $1 \le p, q \le \infty$. Let $\mathscr{B}$ be a bounded set in $L^p(0,T;X)$ such that $\{\partial_t f,\ f \in \mathscr{B}\}$ is bounded in $L^q(0,T;Y)$.*

- *If p is finite and $q \ge 1$, then $\mathscr{B}$ is relatively compact in $L^p(0,T;B)$.*
- *If $p = +\infty$ and $q > 1$, then $\mathscr{B}$ is relatively compact in $C^0([0,T];B)$.*

We consider $0 < T < \infty$, fixed once for all. From the previous Section, we know that

- $0 \le n^k \le 1$, and $0 \le \phi^k = \dfrac{1-n^k}{\bar\phi} \le \dfrac{1}{\bar\phi}$,
- ϕ^k, $\phi^k|\ln(\phi^k)|$ and $\phi^k\mathscr{E}$ are bounded in $L^\infty(0,\infty; L^1(\mathbb{R}))$,

- $\sqrt{n^k}u^k$ is bounded in $L^\infty(0,\infty;L^2(\mathbb{R}))$,
- $\sqrt{\mu^k}\partial_x u^k$ is bounded in $L^2((0,\infty)\times\mathbb{R})$,
- We set $\Pi^k = \ln(\phi^k) + \bar\phi(p^k + \mathscr{E})$; then $\sqrt{\frac{\phi^k}{\mu^k\nu^k}}\partial_x\Pi^k$ is bounded in $L^2((0,\infty)\times\mathbb{R})$.

Furthermore, with Proposition 5.1 and Corollary 5.1, we have established that

$$\text{there exists } m_T \text{, which does not depend on } k \text{, such that } 0 < m_T \le n^k(t,x) \le 1.$$

Coming back to Proposition 5.1 we deduce the following uniform estimates

- u^k and $\partial_x n^k$ are bounded in $L^\infty(0,T;L^2(\mathbb{R}))$,
- $\partial_x u^k$ is bounded in $L^2((0,T)\times\mathbb{R})$.

Finally, the dissipated term in Proposition 5.1-iii) also tells us that

$$\sqrt{\phi^k} \text{ is bounded in } L^2(0,T;H^1(\mathbb{R}))$$

because $\frac{\psi'(n)}{\phi}|\partial_x n|^2 = 4\bar\phi^2\frac{\mu(n)}{n}|\partial_x\sqrt{\phi}|^2$ and $\mu(n)/n$ is bounded from below, by (H2) and Corollary 5.1. As a consequence of the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we note that

$$u^k \text{ is bounded in } L^2(0,T;L^\infty(\mathbb{R})) \text{ and, thus, in } L^4((0,T)\times\mathbb{R}).$$

The following observation is very specific to the one-dimension framework, and it will be crucial to the analysis.

Lemma 6.2 *We have* $u^k = \frac{\bar\phi\phi^k}{\mu^k\nu^k}\partial_x\Pi^k$.

Proof. The condition (1.4) in dimension one means that $u^k(t,x) - \frac{\bar\phi\phi^k}{\mu^k\nu^k}\partial_x\Pi^k(t,x) = \Upsilon(t)$ for a certain function $\Upsilon : [0,\infty) \to \mathbb{R}$ which only depends on the time variable. However, the left hand side belongs to $L^\infty(0,T;L^2(\mathbb{R}))$, which thus forces $\Upsilon(t) = 0$. ∎

We turn to the compactness analysis.

Lemma 6.3 *The sequence* $\left(n^k\right)_{k\in\mathbb{N}}$ *is relatively compact in* $C^0([0,T]\times[-R,+R])$ *for any* $0 < R < \infty$.

Proof. By (1.2), we have $\partial_t n^k = -\partial_x(n^k u^k) = -u^k\partial_x n^k - n^k\partial_x u^k$, where the right hand side is bounded (at least) in $L^2((0,T)\times\mathbb{R})$. For any $0 < R < \infty$, $H^1((-R,R)) \subset L^2((-R,R))$ embeds compactly in $C^0([-R,+R])$ and the statement follows from Lemma 6.1. Note also that combining the estimates on ϕ^k tells us that ϕ^k is bounded in $H^1((0,T)\times\mathbb{R})$. ∎

Up to now the compactness arguments are quite standard. A difficulty is related to a lack of estimate on the pressure p^k and on $\partial_t(n^k u^k)$ or $\partial_t u^k$. This difficulty already appears in the models investigated in [45]. However, in [45] the pressure is defined in domains where the fluid becomes compressible by a simple local law, p^k being given as a (non negative) function of the density n^k. Hence, it is not clear how to adapt the techniques elaborated in [45] to the non local definition of the pressure we are dealing with.

Let us consider the evolution of $\phi^k u^k$:

$$\begin{aligned}\partial_t(\phi^k u^k) + \partial_x(\phi^k u^k u^k) &= \phi^k(\partial_t u^k + u^k\partial_x u^k) + u^k(\partial_t\phi^k + \partial_x(\phi^k u^k))\\ &= \frac{\phi^k}{n^k}\partial_x(-p^k - \phi^k + \mu^k\partial_x u^k) - \frac{\phi^k}{n^k}(\kappa n^k + \bar\phi\phi^k)\partial_x\mathscr{E} + \frac{1}{\bar\phi}u^k\partial_x u^k.\end{aligned}$$

We rewrite the viscous term as follows

$$\frac{\phi^k}{n^k}\partial_x(\mu^k\partial_x u^k) = \partial_x\Big(\frac{\phi^k}{n^k}\mu^k\partial_x u^k\Big) - \partial_x\Big(\frac{\phi^k}{n^k}\Big)\mu^k\partial_x u^k$$

so that it appears of a sum of a term bounded in $L^2(0,T;H^{-1}(\mathbb{R}))$ and a term bounded in $L^2(0,T;L^1(\mathbb{R}))$. For the pressure term, we get

$$\frac{\phi^k}{n^k}\partial_x p^k = \frac{\phi^k}{\bar\phi n^k}\partial_x\Pi^k - \frac{\phi^k}{n^k}\partial_x\mathscr{E} - \frac{1}{\bar\phi n^k}\partial_x\phi^k = \frac{\sqrt{\mu^k\nu^k\phi^k}}{n^k}\ \sqrt{\frac{\phi^k}{\mu^k\nu^k}}\partial_x\Pi^k - \frac{\phi^k}{n^k}\partial_x\mathscr{E} + \frac{1}{\bar\phi^2 n^k}\partial_x n^k.$$

Hence this quantity is bounded in $L^2((0,T)\times\mathbb{R})$. Finally $\frac{\phi^k}{n^k}(\kappa n^k + \bar\phi\phi^k)\partial_x\mathscr{E}$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$, $\frac{\phi^k}{n^k}\partial_x\phi^k = -\frac{\phi^k}{\bar\phi n^k}\partial_x n^k$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$, $u^k\partial_x u^k = \frac{1}{2}\partial_x|u^k|^2$ is bounded in functional spaces like $L^2(0,T;L^1(\mathbb{R}))$, $L^2(0,T;H^{-1}(\mathbb{R}))$ or $L^1(0,T;L^2(\mathbb{R}))$, while $\partial_x(\phi^k u^k u^k)$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}))$. Therefore, we can apply Lemma 6.1 as follows: $\phi^k u^k$ is bounded in $L^2(0,T;H^1(\mathbb{R}))$, because $\phi^k u^k$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}))$ and $\partial_x(\phi^k u^k) = \phi^k\partial_x u^k + u^k\partial_x\phi^k$ is bounded in $L^2((0,T)\times\mathbb{R})$, with moreover $\partial_t(\phi^k u^k)$ bounded in $L^2(0,T;H^{-s}(\mathbb{R}))$ for a sufficiently large exponent $s>0$. Since for any $0<R<\infty$, $H^1((-R,R))$ embeds compactly to $L^2((-R,R))$, we conclude with the following claim.

Lemma 6.4 *The sequence $\big(\phi^k u^k\big)_{k\in\mathbb{N}}$ is relatively compact in $L^2((0,T)\times(-R,+R))$ for any $0<R<\infty$.*

Looking similarly at the estimates on each term but the pressure in (1.3), we deduce that

Lemma 6.5 *The sequence $\big(\partial_x p^k\big)_{k\in\mathbb{N}}$ is bounded in $H^{-1}((0,T)\times\mathbb{R})$, with $\big(\sqrt{\phi^k}\partial_x p^k\big)_{k\in\mathbb{N}}$ bounded in $L^2((0,T)\times\mathbb{R})$.*

Proof. The second assertion comes from the expression of Π^k, once we have remarked that $\partial_x\sqrt{\phi^k}$ is bounded in $L^2((0,T)\times\mathbb{R})$, by virtue of Proposition 5.1. ∎

Owing to the compactness properties we have established, possibly at the price of extracting subsequences, we can suppose that

- n^k converges to n uniformly on $[0,T]\times[-R,R]$ for any $0<R<\infty$, and $\phi^k=(1-n^k)/\bar\phi$ converges to $\phi=(1-n)/\bar\phi$ uniformly on $[0,T]\times[-R,R]$,

- u^k converges to u weakly in $L^2((0,T)\times\mathbb{R})$,

- p^k admits a limit p in $\mathscr{D}'((0,T)\times\mathbb{R})$, with $\partial_x p^k$ converging to $\partial_x p$ weakly$-\star$ in $H^{-1}((0,T)\times\mathbb{R})$.

Furthermore, let us set $q^k=\phi^k u^k$; we can assume that q^k converges to q strongly in $L^2((0,T)\times[-R,+R])$ for any $0<R<\infty$ and a. e.

Lemma 6.6 *We can also assume that ϕ^k converges to ϕ pointwise, strongly in $L^p((0,T)\times\mathbb{R})$, for any $1\le p<\infty$, and it is dominated by an integrable function.*

Proof. With the estimates in Proposition 4.2, we can apply the Dunford-Pettis theorem; it tells us that $\big(\phi^k\big)_{k\in\mathbb{N}}$ is relatively compact for the weak topology in $L^1((0,T)\times\mathbb{R})$. Then, extracting subsequences if necessary, ϕ^k converges to ϕ both pointwise (owing to Lemma 6.3) and weakly in $L^1((0,T)\times\mathbb{R})$. Combining Dunford-Pettis and Egoroff theorem, it implies that ϕ^k converges to ϕ strongly in $L^1((0,T)\times\mathbb{R})$. Since $0\le\phi^k\le 1/\bar\phi$, by interpolation the convergence also holds in $L^p((0,T)\times\mathbb{R})$, for any $1\le p<\infty$. Applying the partial reciprocal of the Lebesgue theorem, we can assume that ϕ^k is dominated. For a detailed presentation of the arguments we refer the reader to [33, Sections 7.3.2, 7.3.3 & 7.6 and Lemma 3.31]. ∎

Lemma 6.7 *Up to a subsequence u^k converges to $u=\frac{q}{\phi}\mathbf{1}_{\phi>0}$ a.e. $(0,T)\times\mathbb{R}$, strongly in $L^2_{\text{loc}}((0,T)\times\mathbb{R})$, and weakly in $L^4((0,T)\times\mathbb{R})$. In particular $u(t,x)$ vanishes when $\phi(t,x)=0$.*

Proof. We split $u^k=u^k\mathbf{1}_{\phi>0}+u^k\mathbf{1}_{\phi=0}$, where Lemma 6.2 yields

$$\begin{aligned}
\int_{\{\phi=0\}}|u^k|\,\mathrm{d}x\,\mathrm{d}t &= \int_{\{\phi=0\}}\Big|\frac{\bar\phi\phi^k}{\mu^k\nu^k}\partial_x\Pi^k\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{\{\phi=0\}}\frac{\bar\phi}{\sqrt{\mu^k\nu^k}}\big|\sqrt{\phi^k}-\sqrt{\phi}\big|\,\Big|\sqrt{\frac{\phi^k}{\mu^k\nu^k}}\partial_x\Pi^k\Big|\,\mathrm{d}x\,\mathrm{d}t\\
&\le C_T\left(\int\big|\sqrt{\phi^k}-\sqrt{\phi}\big|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}
\end{aligned}$$

where $C_T>0$ is a constant which does not depend on k, by using the Cauchy-Schwarz inequality and the available estimates. By virtue of Lemma 6.6, we conclude that $u^k\mathbf{1}_{\phi=0}$ tends to 0 in $L^1((0,T)\times\mathbb{R})$. Next, for any fixed (t,x) such that $\phi(t,x)>0$, since ϕ^k converges to ϕ pointwise, we can find $K(t,x)\in\mathbb{N}$ such that $\phi^k(t,x)>0$ when $k\ge K(t,x)$. Thus, for $k\ge K(t,x)$ it makes sense to write $u^k\mathbf{1}_{\phi>0}(t,x)=\frac{q^k(t,x)}{\phi^k(t,x)}\mathbf{1}_{\phi>0}(t,x)$. Therefore, $u^k\mathbf{1}_{\phi>0}(t,x)$ converges to $\frac{q}{\phi}\mathbf{1}_{\phi>0}(t,x)$ a.e. Since u^k is bounded in $L^4((0,T\times\mathbb{R}))$, we conclude that it converges strongly in $L^p((0,T)\times(-R,R))$ for any $0<R<\infty$ and $1\le p<4$, and its limit is $u=\frac{q}{\phi}\mathbf{1}_{\phi>0}$. ∎

Having at hand these results, we can pass to the limit $k \to \infty$ in both the mass conservation and the momentum balance. We obtain

$$\partial_t n + \partial_x(nu) = 0,$$
$$\partial_t(nu) + \partial_x(nuu) + \partial_x p - \partial_x(\mu \partial_x u) = -(\kappa n + \bar{\phi}\phi)\partial_x \mathscr{E} - \nabla_x \phi.$$

We indeed remind the reader that the pressure p^k admits a limit p at least in the sense of $\mathscr{D}'((0,T)\times \mathbb{R})$. It remains to justify that the constraint

$$\partial_x u^k = \partial_x \left(\frac{\bar{\phi}\phi^k}{\mu^k \nu^k} \partial_x \Pi^k \right) = \partial_x \left(\frac{\bar{\phi}}{\mu^k \nu^k} \partial_x \phi^k + \frac{\bar{\phi}^2}{\mu^k \nu^k} \phi^k \partial_x \mathscr{E} \right) + \partial_x \left(\frac{\bar{\phi}^2}{\mu^k \nu^k} \phi^k \partial_x p^k \right)$$

also passes to the limit. Only the last term poses a challenge. We proceed in two steps: firstly, we identify the limit by means of a defect measure, secondly, we justify that the defect term vanishes.

Let us set $T^k = \partial_x p^k$ and $\Gamma^k = \frac{\phi^k}{\mu^k \nu^k}$. We know that T^k is bounded in $H^{-1}((0,T)\times\mathbb{R})$ and $\sqrt{\Gamma^k} T^k$ is bounded in $L^2((0,T)\times\mathbb{R})$. We can thus suppose that

$$T^k \text{ converges weakly}-\star \text{ in } H^{-1}((0,T)\times\mathbb{R}) \text{ to } T = \partial_x p,$$

while Γ^k converges uniformly on compact sets to $\Gamma = \frac{\phi}{\mu\nu}$. We also know that Γ^k converges weakly in $H^1((0,T)\times\mathbb{R})$ to Γ. In particular, since for any $\zeta \in C_c^\infty((0,T)\times\mathbb{R})$, the mapping $\Gamma \mapsto \zeta\Gamma$ is continuous on $H^1((0,T)\times\mathbb{R})$, the product ΓT is well defined in $\mathscr{D}'((0,T)\times\mathbb{R})$. Since $\Gamma^k T^k$ is bounded in $L^2((0,T)\times\mathbb{R})$, we shall assume that

$$\Gamma^k T^k \text{ converges weakly in } L^2((0,T)\times\mathbb{R}) \text{ to } W.$$

We are going to write $\Gamma^k T^k$ in two different ways, which will allow us to identify the limit and the properties of ΓT. We start by splitting

$$\Gamma^k T^k = \Gamma^k T^k \mathbf{1}_{\phi>0} + \big(\sqrt{\Gamma^k} - \sqrt{\Gamma}\big)\sqrt{\Gamma^k} T^k \mathbf{1}_{\phi=0}.$$

Reproducing the arguments used in the proof of Lemma 6.7, we show that the last term in the right hand side tends to 0 in $L^1((0,T)\times\mathbb{R})$. Letting k go to ∞, we deduce that $W = W\mathbf{1}_{\phi>0}$: the limit $W \in L^2((0,T)\times\mathbb{R})$ is supported in $\{\phi > 0\}$. Next, we write

$$\Gamma^k T^k = (\Gamma^k - \Gamma)T^k + \Gamma T^k,$$

where $\Gamma T^k \rightharpoonup \Gamma T$ in $\mathscr{D}'((0,T)\times\mathbb{R})$. Let us denote by S the distribution defined by

$$\langle S, \zeta \rangle_{\mathscr{D}', C_c^\infty} = \lim_{k\to\infty} \int (\Gamma^k - \Gamma) T^k \zeta \, \mathrm{d}x \, \mathrm{d}t = \int W\zeta \, \mathrm{d}x \, \mathrm{d}t - \langle T, \Gamma\zeta \rangle_{H^{-1}, H_0^1}.$$

Since $(\Gamma^k - \Gamma)T^k \mathbf{1}_{\phi=0} = (\sqrt{\Gamma^k} - \sqrt{\Gamma})\sqrt{\Gamma^k} T^k \mathbf{1}_{\phi=0}$ tends to 0 in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R})$, we actually have

$$\langle S, \zeta \rangle_{\mathscr{D}', C_c^\infty} = \lim_{k\to\infty} \int (\Gamma^k - \Gamma) T^k \mathbf{1}_{\phi>0} \zeta \, \mathrm{d}x \, \mathrm{d}t.$$

Nevertheless we are going to establish that S is supported in the set $\{\phi = 0\}$. Indeed, since ϕ is continuous, $\{\phi > 0\}$ is an open set. We can thus deal with trial functions $\zeta \in C_c^\infty((0,T)\times\mathbb{R})$, compactly supported in $\{\phi > 0\}$. Since Γ is continuous and positive on the compact set $\mathrm{supp}(\zeta)$, there exists $\alpha > 0$ such that $\Gamma(t,x) \geq \alpha > 0$ for any $(t,x) \in \mathrm{supp}(\zeta)$. Furthermore, since Γ^k converges uniformly to Γ on compact sets, there exists $K \in \mathbb{N}$ large enough so that $\Gamma^k(t,x) \geq \alpha/2 > 0$ on $\mathrm{supp}(\zeta)$ for any $k \geq K$. For such a test function, it follows that

$$\left|\int (\Gamma^k - \Gamma)T^k \zeta \,\mathrm{d}x\,\mathrm{d}t\right| = \left|\int \frac{\Gamma^k - \Gamma}{\sqrt{\Gamma^k}} \sqrt{\Gamma^k} T^k \zeta \,\mathrm{d}x\,\mathrm{d}t\right| \leq C_T \sqrt{\frac{2}{\alpha}}\, \|\zeta\|_{L^2}\, \|\Gamma^k - \Gamma\|_{L^\infty(\mathrm{supp}(\zeta))} \xrightarrow[k\to\infty]{} 0.$$

Lemma 6.8 *The distribution* ΓT *can be written* $W + S$*, where* $W \in L^2((0,T)\times\mathbb{R})$*, with* $\mathrm{supp}(W) \subset \{\phi > 0\}$*, and* $\mathrm{supp}(S) \subset \{\phi = 0\}$*. Furthermore, the limit of* $\Gamma^k T^k$ *is given by the "regular part"* W *only.*

We are going to prove that S vanishes. The argument relies on the identification of S as a defect measure. The conclusion follows by discussing the support of this measure.

Lemma 6.9 *There exists a signed measure* $\mathfrak{m}$ *such that* $S = \sqrt{\Gamma}\mathfrak{m}$*.*

Proof. We rewrite the momentum equation as

$$-\partial_t(n^k u^k) + \partial_x \mathscr{F}^k + \mathscr{U}^k = \partial_x p^k$$

with the following shorthand notation

$$\mathscr{F}^k = \mu^k \partial_x u^k - \phi^k - n^k |u^k|^2, \qquad \mathscr{U}^k = -(\kappa n^k + \bar{\phi}\phi^k)\partial_x \mathscr{E}.$$

From the previous statements, we know that

$$\begin{aligned}
&n^k u^k \to nu \text{ strongly in } L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}),\\
&\mathscr{F}^k \rightharpoonup \mathscr{F} = \mu\partial_x u - \phi - n|u|^2 \text{ weakly in } L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}),\\
&\mathscr{U}^k \to \mathscr{U} = -(\kappa n + \bar{\phi}\phi)\partial_x \mathscr{E} \text{ strongly in } L^2_{\mathrm{loc}}((0,T)\times\mathbb{R}).
\end{aligned}$$

Therefore, multiplying by $\Gamma \in H^1((0,T)\times\mathbb{R})$ and letting k run to ∞, we obtain, at least in the sense of distributions,

$$\begin{aligned}
\Gamma\partial_x p &= W + S = \lim_{k\to\infty}(\Gamma\partial_x p^k)\\
&= \lim_{k\to\infty}\Big\{ -\partial_t(n^k u^k \Gamma) + n^k u^k \partial_t \Gamma + \partial_x(\mathscr{F}^k \Gamma) - \mathscr{F}^k \partial_x \Gamma + \mathscr{U}^k \Gamma \Big\}\\
&= -\partial_t(nu\Gamma) + nu\partial_t\Gamma + \partial_x(\mathscr{F}\Gamma) - \mathscr{F}\partial_x\Gamma + \mathscr{U}\Gamma.
\end{aligned} \tag{6.25}$$

Similarly, we multiply the momentum equation by Γ^k and we let k go to ∞. We obtain

$$\begin{aligned}
\lim_{k\to\infty}\Gamma^k \partial_x p^k &= W\\
&= \lim_{k\to\infty}\Big\{ -\partial_t(n^k u^k \Gamma^k) + n^k u^k \partial_t \Gamma^k + \partial_x(\mathscr{F}^k \Gamma^k) - \mathscr{F}^k \partial_x \Gamma^k + \mathscr{U}^k \Gamma^k \Big\}.
\end{aligned}$$

Several terms appear as product of sequences that converge strongly and sequences that converge weakly in L^2_{loc}, so that we can write

$$\lim_{k\to\infty}(\Gamma^k\partial_x p^k) = W = -\partial_t(nu\Gamma) + nu\partial_t\Gamma + \partial_x(\mathscr{F}\Gamma) + \mathscr{U}\Gamma - \lim_{k\to\infty}(\mathscr{F}^k\partial_x\Gamma^k).$$

The last term can be recast as

$$\mathscr{F}^k\partial_x\Gamma^k = 2\sqrt{\Gamma^k}\ \mathscr{F}^k\partial_x\sqrt{\Gamma^k},$$

where Γ^k converges uniformly on compact sets to the continuous function Γ and $\mathscr{F}^k\partial_x\sqrt{\Gamma^k}$ is bounded in $L^1((0,T)\times\mathbb{R})$. Therefore, we can assume that there exists a signed measure $\mathfrak{m}$ on $(0,T)\times\mathbb{R}$ such that

$$\lim_{k\to\infty}\mathscr{F}^k\partial_x\sqrt{\Gamma^k} = \mathscr{F}\partial_x\sqrt{\Gamma} + \frac{1}{2}\mathfrak{m}.$$

Accordingly, we are led to

$$\begin{aligned}\lim_{k\to\infty}\Gamma^k\partial_x p^k &= W = -\partial_t(nu\Gamma) + nu\partial_t\Gamma + \partial_x(\mathscr{F}\Gamma) + \mathscr{U}\Gamma - \sqrt{\Gamma}(2\mathscr{F}\partial_x\sqrt{\Gamma} + \mathfrak{m})\\ &= -\partial_t(nu\Gamma) + nu\partial_t\Gamma + \partial_x(\mathscr{F}\Gamma) + \mathscr{U}\Gamma - \mathscr{F}\partial_x\Gamma - \sqrt{\Gamma}\mathfrak{m}.\end{aligned}$$

We conclude by comparing to (6.25). ∎

Corollary 6.1 *The defect measure S vanishes.*

Proof. We associate to the measures S and $\mathfrak{m}$ their total variation $|\mathfrak{m}|$ and $|S|$, which are positive and finite measures on $(0,T)\times\mathbb{R}$, see [55, Th. 6.2 & 6.4]. In particular, by the Radon-Nikodym theorem, there exists a function h such that for any $(t,x)\in(0,T)\times\mathbb{R}$, we have $|h(t,x)|=1$ and $\mathfrak{m}=h|\mathfrak{m}|$, $S=\sqrt{\Gamma}\mathfrak{m}=h\sqrt{\Gamma}|\mathfrak{m}|=h|S|$, and $|S|=\sqrt{\Gamma}|\mathfrak{m}|$, see [55, Th. 6.12]. Let $\epsilon>0$. We denote $\mathscr{O}_\epsilon=\{(t,x)\in(0,T)\times\mathbb{R},\ \phi(t,x)<\epsilon\}$, which is an open set in $(0,T)\times\mathbb{R}$. Since $|\mathfrak{m}|$ is a finite measure, there exists $C>0$ such that, for any $\psi\in C^\infty_c(\mathscr{O}_\epsilon)$, we have

$$|\langle|S|,\psi\rangle| = |\langle|\mathfrak{m}|,\sqrt{\Gamma}\psi\rangle| \le C\sqrt{\epsilon}\|\psi\|_\infty.$$

It means that $|S|(\mathscr{O}_\epsilon)\le C\sqrt{\epsilon}$. Since $\{\phi=0\}=\bigcap_{\epsilon>0}\mathscr{O}_\epsilon$, we conclude that $\{\phi=0\}$ is $|S|-$negligible. However, we already know that the support of S is included into $\{\phi=0\}$; it follows that $S=0$. Indeed, for any trial function $\psi\in C^\infty_c((0,T)\times\mathbb{R})$, we have

$$\langle S,\psi\rangle = \int\psi h\,\mathrm{d}|S| = \int_{\{\phi=0\}}\psi h\,\mathrm{d}|S| = 0.$$

∎

Eventually, we conclude with the following observation. Since $\sqrt{\Gamma^k}\partial_x p^k$ is bounded in $L^2((0,T)\times\mathbb{R})$, we can assume that it converges weakly to some $Z\in L^2((0,T)\times\mathbb{R})$. However $\sqrt{\Gamma^k}$ converges strongly to $\sqrt{\Gamma}$ in $L^2((0,T)\times\mathbb{R})$ so that we can write

$$\Gamma^k\partial_x p^k = \sqrt{\Gamma^k}\sqrt{\Gamma^k}\partial_x p^k \rightharpoonup W = \sqrt{\Gamma}Z,\quad Z\in L^2((0,T)\times\mathbb{R}).$$

This concludes the proof of Theorem 6.1.

This result is a first step towards a proof of the existence of weak solutions. It remains to construct a sequence of approximated problems, that admit weak solutions satisfying uniformly the estimates. This step is really challenging because the estimated quantities are highly non linear and lack of robustness with respect to the structure of the system. In this direction, we refer the reader to [14] for the construction of approximated solutions for compressible Navier-Stokes equations and shallow-water system: additional terms needs to be included in the equations and not all of them can be removed in the asymptotic process.

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