GLOBAL EXISTENCE OF ENTROPY-WEAK SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NON-LINEAR DENSITY DEPENDENT VISCOSITIES

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Abstract. In this paper, we extend considerably the global existence results of entropy-weak solutions related to compressible Navier-Stokes system with density dependent viscosities obtained, independently (using different strategies), by Vasseur-Yu [*Inventiones mathematicae* (2016) and arXiv:1501.06803 (2015)] and by Li-Xin [arXiv:1504.06826 (2015)]. More precisely we are able to consider a physical symmetric viscous stress tensor $\sigma = 2\mu(\rho)\,\mathbb{D}(u) + \big(\lambda(\rho)\mathrm{div}u - P(\rho)\big)\,\mathrm{Id}$ where $\mathbb{D}(u) = [\nabla u + \nabla^T u]/2$ with a shear and bulk viscosities (respectively $\mu(\rho)$ and $\lambda(\rho)$) satisfying the BD relation $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ and a pressure law $P(\rho) = a\rho^\gamma$ (with $a > 0$ a given constant) for any adiabatic constant $\gamma > 1$. The nonlinear shear viscosity $\mu(\rho)$ satisfies some lower and upper bounds for low and high densities (our mathematical result includes the case $\mu(\rho) = \mu\rho^\alpha$ with $2/3 < \alpha < 4$ and $\mu > 0$ constant). This provides an answer to a longstanding mathematical question on compressible Navier-Stokes equations with density dependent viscosities as mentioned for instance by F. Rousset in the Bourbaki 69ème année, 2016–2017, no 1135.

1. Introduction

When a fluid is governed by the barotropic compressible Navier-Stokes equations, the existence of global weak solutions, in the sense of J. Leray (see [32]), in space dimension greater than two remained for a long time without answer, because of the weak control of the divergence of the velocity field which may provide the possibility for the density to vanish (vacuum state) even if initially this is not the case.

There exists a huge literature on this question, in the case of constant shear viscosity μ and constant bulk viscosity λ. Before 1993, many authors such as Hoff [24], Jiang-Zhang [26], Kazhikhov–Shelukhin [29], Serre [44], Veigant–Kazhikhov [45] (to cite just some of them) have obtained partial answers: We can cite, for instance, the works in dimension 1 in 1986 by Serre [44], the one by Hoff [24] in 1987, and the one in the spherical case in 2001 by Jiang-Zhang [26]. The first rigorous approach of this problem in its generality is due in 1993 by P.–L. Lions [35] when the pressure law in terms of the density is given by $P(\rho) = a\rho^\gamma$ where a and γ are two strictly positive constants. He has presented in 1998 a complete theory for $P(\rho) = a\rho^\gamma$ with $\gamma \geq 3d/(d+2)$ (where d is the space dimension) allowing to obtain the result of global existence of weak solutions à la Leray in dimension $d = 2$ and 3 and for general initial data belonging to the energy space. His result has been then extended in 2001 to the case $P(\rho) = a\rho^\gamma$ with $\gamma > d/2$ by

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Feireisl-Novotny-Petzeltova [20] introducing an appropriated method of truncation. Note also in 2014 the paper by Plotnikov-Weigant [42] in dimension 2 for the linear pressure law that means $\gamma = 1$. In 2002, Feireisl [21] has also proved it is possible to consider a pressure $P(\rho)$ law non-monotone on a compact set $[0, \rho_*]$ (with ρ_* constant) and monotone elsewhere. This has been relaxed in 2018 by Bresch-Jabin [13] allowing to consider real non-monotone pressure laws. They have also proved that it is possible to consider some constant anisotropic viscosities. The Lions theory has also been extended recently by Vasseur-Wen-Yu [48] to pressure laws depending on two phases (see also Mastese & *al.* [36], Novotny [40] and Novotny-Pokorny [41]). The method introduced by Bresch-Jabin in [13] has also been recently developped in the bifluid framework by Bresch-Mucha-Zatorska in [15].

When the shear and the bulk viscosities (respectively μ and λ) are assumed to depend on the density ρ, the mathematical framework is completely different. It has been discussed, mathematically, initially in a paper by Bernardi-Pironneau [5] related to viscous shallow-water equations and by P.–L. Lions [35] in his second volume related to mathematics and fluid mechanics. The main ingredient in the constant case which is the compactness in space of the effective flux $F = (2\mu + \lambda)\mathrm{div}u - P(\rho)$ is no longer true for density dependent viscosities. In space dimension greater than one, a real breakthrough has been realized with a series of papers by Bresch-Desjardins [6, 8, 9, 10], (started in 2003 with Lin [11] in the context of Navier-Stokes-Korteweg with linear shear viscosity case) who have identified an information related to the gradient of a function of the density if the viscosities satisfy what is called the Bresch-Desjardins constraint. This information is usually called the BD entropy in the literature with the introduction of the concept of entropy-weak solutions. Using such extra information, they obtained the global existence of entropy-weak solutions in the presence of appropriate drag terms or singular pressure close to vacuum. Concerning the one-dimensional in space case or the spherical case, many important results have been obtained for instance by Burtea-Haspot [16], Ducomet-Necasova-Vasseur [19], Constantin-Drivas-Nguyen-Pasqualottos [18], Guo-Jiu-Xin [22], Haspot [23], Jiang-Xin-Zhang [25], Jiang-Zhang [26], Kanel [30], Li-Li-Xin [33], Mellet-Vasseur [38], Shelukhin [44] without such kind of additional terms. Stability and construction of approximate solutions in space dimension two or three have been investigated during more than fifteen years with a first important stability result without drag terms or singular pressure by Mellet-Vasseur [37]. Several important works for instance by Bresch-Desjardins [6, 8, 9, 10] and Bresch-Desjardins-Lin [11], Bresch-Desjardins-Zatorska [12], Li-Xin [34], Mellet-Vasseur [37], Mucha-Pokorny-Zatorska [39], Vasseur-Yu [46, 47], and Zatorska [49] have also been written trying to find a way to construct approximate solutions. Recently a real breakthrough has been done in two important papers by Li-Xin [34] and Vasseur-Yu [47]: Using two different ways, they got the global existence of entropy-weak solutions for the compressible paper when $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$. Note that in the last paper [34] by Li-Xin, they also consider more general viscosities satisfying the BD relation but with a non-symmetric stress diffusion ($\sigma = \mu(\rho)\nabla u + (\lambda(\rho)\mathrm{div}u - P(\rho))\mathrm{Id}$) and more restrictive conditions on the shear $\mu(\rho)$ viscosity and bulk viscosity $\lambda(\rho)$ and on the pressure law $P(\rho)$ compared to the present paper.

The objective of this current paper is to extend the existence results of global entropy-weak solutions obtained independently (using different strategies) by Vasseur-Yu [47] and

Lin-Xin [34] to answer a longstanding mathematical question on compressible Navier-Stokes equations with density dependent viscosities as mentioned for instance by Rousset [43]. More precisely extending and coupling carefully the two-velocities framework by Bresch-Desjardins-Zatorska [12] with the generalization of the quantum Böhm identity found by Bresch-Couderc-Noble-Vila [7] (proving a generalization of the dissipation inequality used by Jüngel [27] for Navier-Stokes-Quantum system and established by Jüngel-Matthes in [28]) and with the renormalized solutions introduced in Lacroix-Violet and Vasseur [31], we can get global existence of entropy-weak solutions to the following Navier-Stokes equations:

$$\begin{aligned} &\rho_t + \mathrm{div}(\rho u) = 0 \\ &(\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) - 2\mathrm{div}\big(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id}\big) = 0, \end{aligned} \tag{1.1}$$

where

$$\sqrt{\mu(\rho)}\mathbb{S}_\mu = \mu(\rho)\mathbb{D}(u)$$

with data

$$\rho|_{t=0} = \rho_0(x) \geq 0, \qquad \rho u|_{t=0} = m_0(x) = \rho_0 u_0, \tag{1.2}$$

and where $P(\rho) = a\rho^\gamma$ denotes the pressure with the two constants $a > 0$ and $\gamma > 1$, ρ is the density of fluid, u stands for the velocity of fluid, $\mathbb{D}u = [\nabla u + \nabla^T u]/2$ is the strain tensor. As usually, we consider

$$u_0 = \frac{m_0}{\rho_0} \text{ when } \rho_0 \neq 0 \text{ and } u_0 = 0 \text{ elsewhere}, \qquad \frac{|m_0|^2}{\rho_0} = 0 \text{ a.e. on } \{x \in \Omega : \rho_0(x) = 0\}.$$

We remark the following identity

$$2\mathrm{div}\big(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id}\big) = -2\mathrm{div}(\mu(\rho)\mathbb{D}u) - \nabla(\lambda(\rho)\mathrm{div}u).$$

The viscosity coefficients $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ satisfy the Bresch-Desjardins relation introduced in [9]

$$\lambda(\rho) = 2(\rho\mu'(\rho) - \mu(\rho)). \tag{1.3}$$

The relation between the stress tensor $\mathbb{S}_\mu$ and the triple $(\mu(\rho)/\sqrt{\rho}, \sqrt{\rho}u, \sqrt{\rho}v)$ where $v = 2\nabla s(\rho)$ with $s'(\rho) = \mu'(\rho)/\rho$ will be proved in the following way: The matrix $\mathbb{S}_\mu$ is the symetric part of a matrix value function $\mathbb{T}_\mu$ namely

$$\mathbb{S}_\mu = \frac{(\mathbb{T}_\mu + \mathbb{T}_\mu^t)}{2} \tag{1.4}$$

where $\mathbb{T}_\mu$ is defined through

$$\sqrt{\mu(\rho)}\mathbb{T}_\mu = \nabla(\sqrt{\rho}u\,\frac{\mu(\rho)}{\sqrt{\rho}}) - \sqrt{\rho}u \otimes \sqrt{\rho}\nabla s(\rho) \tag{1.5}$$

with

$$s'(\rho) = \mu'(\rho)/\rho, \tag{1.6}$$

and

$$\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\mathrm{Id} = \Big[\mathrm{div}(\frac{\lambda(\rho)}{\mu(\rho)}\sqrt{\rho}u\,\frac{\mu(\rho)}{\sqrt{\rho}}) - \sqrt{\rho}u \cdot \sqrt{\rho}\,\nabla s(\rho)\,\frac{\rho\mu''(\rho)}{\mu'(\rho)}\Big]\mathrm{Id}. \tag{1.7}$$

For the sake of simplicity, we will consider the case of periodic boundary conditions in three dimension in space namely $\Omega = \mathbb{T}^3$. In the whole paper, we assume:

$$\mu \in C^0(\mathbb{R}_+;\, \mathbb{R}_+) \cap C^2(\mathbb{R}_+^*;\, \mathbb{R}), \tag{1.8}$$

where $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+^* = (0,\infty)$. We also assume that there exists two positive numbers α_1, α_2 such that

$$\begin{aligned} &\frac{2}{3} < \alpha_1 < \alpha_2 < 4, \\ &\text{for any } \rho > 0, \qquad 0 < \frac{1}{\alpha_2}\rho\mu'(\rho) \le \mu(\rho) \le \frac{1}{\alpha_1}\rho\mu'(\rho), \end{aligned} \tag{1.9}$$

and there exists a constant $C > 0$ such that

$$\left|\frac{\rho\mu''(\rho)}{\mu'(\rho)}\right| \le C < +\infty. \tag{1.10}$$

Note that if $\mu(\rho)$ and $\lambda(\rho)$ satisfying (1.3) and (1.9), then

$$\lambda(\rho) + 2\mu(\rho)/3 \ge 0$$

and thanks to (1.9)

$$\mu(0) = \lambda(0) = 0.$$

Note that the hypothesis (1.9)–(1.10) allow a shear viscosity of the form $\mu(\rho) = \mu\rho^\alpha$ with $\mu > 0$ a constant where $2/3 < \alpha < 4$ and a bulk viscosity satisfying the BD relation: $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$.

Remark. In [47] and [34] the case $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$ is considered, and in [34] more general cases have been considered but with a non-symmetric viscous term in the three-dimensional in space case, namely $-\text{div}(\mu(\rho)\nabla u) - \nabla(\lambda(\rho)\text{div}u)$. In [34] the viscosities $\mu(\rho)$ and $\lambda(\rho)$ satisfy (1.3) with $\mu(\rho) = \mu\rho^\alpha$ where $\alpha \in [3/4, 2)$ and with the following assumption on the value γ for the pressure $p(\rho) = a\rho^\gamma$:

$$\text{If } \alpha \in [3/4, 1], \qquad \gamma \in (1, 6\alpha - 3)$$

and

$$\text{if } \alpha \in (1, 2), \qquad \gamma \in [2\alpha - 1, 3\alpha - 1].$$

The main result of our paper reads as follows:

Theorem 1.1. *Let $\mu(\rho)$ verify (1.8)–(1.10) and μ and λ verify (1.3). Let us assume the initial data satisfy*

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\rho_0|u_0 + 2\kappa\nabla s(\rho_0)|^2 + \kappa(1-\kappa)\rho_0\frac{|2\nabla s(\rho_0)|^2}{2}\right) dx \\ &\qquad\qquad + \int_\Omega \left(a\frac{\rho_0^\gamma}{\gamma - 1} + \mu(\rho_0)\right) dx \le C < +\infty. \end{aligned} \tag{1.11}$$

with $k \in (0,1)$ given. Let T be given such that $0 < T < +\infty$, then, for any $\gamma > 1$, there exist a renormalized solution to (1.1)-(1.2) as defined in Definition 1.1. Moreover, this renormalized solution with initial data satisfying (1.11) is a weak solution to (1.1)-(1.2) in the sense of Definition 1.2.

Our result may be considered as an improvement of [34] for two reasons: First it takes into account a physical symmetric viscous tensor and secondly, it extends the range of coefficients α and γ. The method is based on the consideration of an approximated system with an extra pressure quantity, appropriate non-linear drag terms and appropriate capillarity terms. This generalizes the Quantum-Navier-Stokes system with quadratic drag terms considered in [46, 47]. First we prove that weak solutions of the approximate solution are renormalized solutions of the system, in the sense of [31]. Then we pass to the limit with respect to r_2, r_1, r_0, r, δ to get renormalized solutions of the compressible Navier-Stokes system. The final step concerns the proof that a renormalized solution of the compressible Navier-Stokes system is a global weak solution of the compressible Navier–Stokes system. Note that, thanks to the technique of renormalized solution introduced in [31], it is not necessary to derive the Mellet-Vasseur type inequality in this paper: This allows us to cover the all range $\gamma > 1$.

First Step. Motivated by the work of [31], the first step is to establish the existence of global κ entropy weak solution to the following approximation

$$\begin{aligned}
&\rho_t + \operatorname{div}(\rho u) = 0\\
&(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \nabla P_\delta(\rho)\\
&\qquad - 2\operatorname{div}\Big(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id}\Big)\\
&\qquad - 2r\operatorname{div}\Big(\sqrt{\mu(\rho)}\mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_r)\mathrm{Id}\Big)\\
&\qquad\qquad + r_0 u + r_1 \frac{\rho}{\mu'(\rho)}|u|^2 u + r_2\rho|u|u = 0
\end{aligned} \tag{1.12}$$

where the barotorpic pressure law and the extra pressure term are respectively

$$P(\rho) = a\rho^\gamma, \qquad P_\delta(\rho) = \delta\rho^{10} \text{ with } \delta > 0. \tag{1.13}$$

The matrix $\mathbb{S}_\mu$ is defined in (1.4) and $\mathbb{T}_\mu$ is given in(1.5)- (1.7). The matrix $\mathbb{S}_r$ is compatible in the following sense:

$$r\sqrt{\mu(\rho)}\mathbb{S}_r = 2r\Big[2\sqrt{\mu(\rho)}\nabla\nabla Z(\rho) - \nabla(\sqrt{\mu(\rho)}\nabla Z(\rho))\Big], \tag{1.14}$$

where

$$Z(\rho) = \int_0^\rho [(\mu(s))^{1/2}\mu'(s)]/s\,ds, \qquad k(\rho) = \int_0^\rho [\lambda(s)\mu'(s)]/\mu(s)^{3/2} ds \tag{1.15}$$

and

$$r\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_r)\mathrm{Id} = r\Big(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} + \frac{1}{2}k(\rho)\Big)\Delta Z(\rho)\mathrm{Id} - \frac{r}{2}\operatorname{div}[k(\rho)\nabla Z(\rho)]\mathrm{Id}. \tag{1.16}$$

Remark. Note that the previous system is the generalization of the quantum viscous Navier-Stokes system considered by Lacroix-Violet and Vasseur in [31] (see also the interesting papers by Antonelli-Spirito [3, 4] and by Carles-Carrapatoso-Hillairet [17]). Indeed if we consider $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$, we can write $\sqrt{\mu(\rho)}\mathbb{S}_r$ as

$$\sqrt{\mu(\rho)}\mathbb{S}_r = 4\sqrt{\rho}\Big[\nabla\nabla\sqrt{\rho} - 4(\nabla\rho^{1/4} \otimes \nabla\rho^{1/4})\Big],$$

using $Z(\rho) = 2\sqrt{\rho}$. The Navier–Stokes equations for quantum fluids was also considered by A. Jüngel in [27].

As the first step generalizing [47], we prove the following result.

Theorem 1.2. *Let $\mu(\rho)$ verifies (1.8)–(1.10) and $\lambda(\rho)$ is given by (1.3). If $r_0 > 0$, then we assume also that $\inf_{s\in[0,+\infty)}\mu'(s) = \epsilon_1 > 0$. Assume that r_1 is small enough compared to r, r_2 is small enough compared to δ, and that the initial values verify*

$$\begin{aligned}\int_\Omega \rho_0 \left(\frac{|u_0 + 2\kappa\nabla s(\rho_0)|^2}{2} + (\kappa(1-\kappa)+r)\frac{|2\nabla s(\rho_0)|^2}{2} \right) dx \\ + \int_\Omega \big(a\frac{\rho_0^\gamma}{\gamma-1} + \mu(\rho_0) + \delta\frac{\rho_0^{10}}{9} + \frac{r_0}{\varepsilon_1}|(\ln\rho_0)_-|\big)\, dx < +\infty,\end{aligned} \tag{1.17}$$

for a fixed $\kappa \in (0,1)$. Then there exists a κ entropy weak solution $(\rho, u, \mathbb{T}_\mu, \mathbb{S}_r)$ to (1.12)–(1.16) satisfying the initial conditions (1.2), in the sense that $(\rho, u, \mathbb{T}_\mu, \mathbb{S}_r)$ satisfies the mass and momentum equations in a weak form, and satisfies the compatibility formula in the sense of definition 1.2. In addition, it verifies the following estimates:

$$\begin{aligned}&\|\sqrt{\rho}\,(u + 2\kappa\nabla s(\rho))\|^2_{L^\infty(0,T;L^2(\Omega))} \le C, && a\|\rho\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} \le C,\\ &\|\mathbb{T}_\mu\|^2_{L^2(0,T;L^2(\Omega))} \le C, && (\kappa(1-\kappa)+r)\|\sqrt{\rho}\nabla s(\rho)\|^2_{L^\infty(0,T;L^2(\Omega))} \le C,\\ &\kappa\|\sqrt{\mu'(\rho)\rho^{\gamma-2}}\nabla\rho\|^2_{L^2(0,T;L^2(\Omega))} \le C,\end{aligned} \tag{1.18}$$

and

$$\begin{aligned}&\delta\|\rho\|^{10}_{L^\infty(0,T;L^{10}(\Omega))} \le C, && \delta\|\sqrt{\mu'(\rho)\rho^8}\nabla\rho\|^2_{L^2(0,T;L^2(\Omega))} \le C,\\ &r_2\|(\frac{\rho}{\mu'(\rho_n)})^{\frac14}u\|^4_{L^4(0,T;L^4(\Omega))} \le C, && r_1\|\rho^{\frac13}|u|\|^3_{L^3(0,T;L^3(\Omega))} \le C,\\ &r_0\|u\|^2_{L^2(0,T;L^2(\Omega))} \le C, && r\|\mathbb{S}_r\|^2_{L^2(0,T;L^2(\Omega))} \le C.\end{aligned} \tag{1.19}$$

Note that the bounds (1.18) provide the following control on the velocity field

$$\|\sqrt{\rho}\,u\|^2_{L^\infty(0,T;L^2(\Omega))} \le C.$$

Moreover let

$$Z(\rho) = \int_0^\rho \frac{\sqrt{\mu(s)}\mu'(s)}{s}\,ds \quad \text{and} \quad Z_1(\rho) = \int_0^\rho \frac{\mu'(s)}{(\mu(s))^{1/4}s^{1/2}}\,ds,$$

we have the extra control

$$r\left[\int_0^T\int_\Omega |\nabla^2 Z(\rho)|^2\,dx\,dt + \int_0^T\int_\Omega |\nabla Z_1(\rho)|^4\,dx\,dt\right] \le C, \tag{1.20}$$

and

$$\begin{aligned}&\|\mu(\rho)\|_{L^\infty(0,T;W^{1,1}(\Omega))} + \|\mu(\rho)u\|_{L^\infty(0,TL^{3/2}(\Omega))\cap L^2(0,T;W^{1,1}(\Omega))} \le C,\\ &\|\partial_t\mu(\rho)\|_{L^\infty(0,T;W^{-1,1}(\Omega))} \le C,\\ &\|Z(\rho)\|_{L^\infty(0,T;L^{1+}(\Omega))} + \|Z_1(\rho)\|_{L^\infty(0,T;L^{1+}(\Omega))} \le C,\end{aligned} \tag{1.21}$$

where $C > 0$ is a constant which depends only on the initial data.

Sketch of proof for Theorem 1.2. To show Theorem 1.2, we need to build the smooth solution to an approximation associated to (1.12). Here, we adapt the ideas developed in [12] to construct this approximation. More precisely, we consider an augmented version of the system which will be more appropriate to construct approximate solutions. Let us explain the idea.

First step: the augmented system. Defining a new velocity field generalizing the one introduced in the BD entropy estimate namely

$$w = u + 2\kappa\nabla s(\rho)$$

and a drift velocity $v = 2\nabla s(\rho)$ and $s(\rho)$ defined in (1.6).

Assuming to have a smooth solution of (1.12) with damping terms, it cavown that (ρ, w, v) satisfies the following system of equations

$$\rho_t + \operatorname{div}(\rho w) - 2\kappa\Delta\mu(\rho) = 0$$

and

$$\begin{aligned}
&(\rho w)_t + \operatorname{div}(\rho u \otimes w) - 2(1-\kappa)\operatorname{div}(\mu(\rho)\mathbb{D}\, w) - 2\kappa\operatorname{div}(\mu(\rho)\mathbf{A}(w))\\
&- (1-\kappa)\nabla(\lambda(\rho)\operatorname{div}(w - \kappa v)) + \nabla\rho^\gamma + \delta\nabla\rho^{10} + 4(1-\kappa)\kappa\operatorname{div}(\mu(\rho)\nabla^2 s(\rho))\\
&= -r_0(w - 2\kappa\nabla s(\rho)) - r_1\rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho))\\
&- r_2\frac{\rho}{\mu'(\rho)}|w - 2\kappa\nabla s(\rho)|^2(w - 2\kappa\nabla s(\rho)) + r\rho\nabla\left(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\, ds)\right),
\end{aligned}$$

and

$$(\rho v)_t + \operatorname{div}(\rho u \otimes v) - 2\kappa\operatorname{div}(\mu(\rho)\nabla v) + 2\operatorname{div}(\mu(\rho)\nabla^t w) + \nabla(\lambda(\rho)\operatorname{div}(w - \kappa v)) = 0,$$

where

$$v = 2\nabla s(\rho), \qquad w = u + \kappa v$$

and

$$K(\rho) = 4(\mu'(\rho))^2/\rho.$$

This is the augmented version for which we will show that there exists global weak solutions, adding an hyperdiffusivity $\varepsilon_2[\Delta^{2s}w - \operatorname{div}((1+|\nabla w|^2)\nabla w)]$ on the equation satisfied by w, and passing to the limit ε_2 goes to zero.

Important remark. Note that recently Bresch-Couderc-Noble-Vila [7] showed the following interesting relation

$$\rho\nabla\left(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\, ds)\right) = \operatorname{div}(F(\rho)\nabla^2\psi(\rho)) + \nabla\left((F'(\rho)\rho - F(\rho))\Delta\psi(\rho)\right),$$

with $F'(\rho) = \sqrt{K(\rho)\rho}$ and $\sqrt{\rho}\psi'(\rho) = \sqrt{K(\rho)}$. Thus choosing

$$F(\rho) = 2\,\mu(\rho) \text{ and therefore } F'(\rho)\rho - F(\rho) = \lambda(\rho),$$

this gives $\psi(\rho) = 2s(\rho)$ and thus

$$\rho\nabla\left(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\, ds)\right) = 2\operatorname{div}\Big(\mu(\rho)\nabla^2\big(2s(\rho)\big)\Big) + \nabla\Big(\lambda(\rho)\Delta\big(2s(\rho)\big)\Big). \tag{1.22}$$

This identity will play a crucial role in the proof. It defines the appropriate capillarity term to consider in the approximate system. Other identities will be used to define the weak solution for the Navier-Stokes-Korteweg system and to pass to the limit in it namely

$$\begin{aligned}2\mu(\rho)\nabla^2(2\mathbf{s}(\rho)) + \lambda(\rho)\Delta(2\mathbf{s}(\rho)) = 4\Big[2\sqrt{\mu(\rho)}\nabla\nabla Z(\rho) - \nabla(\sqrt{\mu(\rho)}\nabla Z(\rho)\Big]\\ + (\frac{2\lambda(\rho)}{\sqrt{\mu(\rho)}} + k(\rho))\Delta Z(\rho)\,\mathrm{Id} - \mathrm{div}[k(\rho)\nabla Z(\rho)]\,\mathrm{Id}.\end{aligned} \tag{1.23}$$

where $Z(\rho) = \int_0^\rho [(\mu(s))^{1/2}\mu'(s)]/s\,ds$ and $k(\rho) = \int_0^\rho \frac{\lambda(s)\mu'(s)}{\mu(s)^{3/2}}ds$.

Note that the case considered in [31, 46, 47] is related $\mu(\rho) = \rho$ and $K(\rho) = 4/\rho$ which corresponds to the quantum Navier-Stokes system. Note that two very interesting papers have been written by Antonelli-Spirito in [1, 2] considering Navier-Stokes-Korteweg systems without such relation between the shear viscosity and the capillary coefficient.

Remark 1.1. The additional pressure $\delta\rho^{10}$ is used in (2.17) thanks to $3\alpha_2 - 2 \leq 10$.

Second Step and main result concerning the compressible Navier-Stokes system. To prove global existence of weak solutions of the compressible Navier-Stokes equations, we follow the strategy introduced in [31, 47]. To do so, first we approximate the viscosity μ by a viscosity μ_{ε_1} such that $\inf_{s\in[0,+\infty)} \mu'_{\varepsilon_1}(s) \geq \varepsilon_1 > 0$. Then we use Theorem 1.2 to construct a κ entropy weak solution to the approximate system (1.12). We then show that this κ entropy weak solution is a renormalized solution of (1.12) in the sense introduced in [31]. More precisely we prove the following theorem:

Theorem 1.3. *Let* $\mu(\rho)$ *verifies* (1.8)–(1.10)*,* $\lambda(\rho)$ *given by* (1.3)*. If* $r_0 > 0$*, then we assume also that* $\inf_{s\in[0,+\infty)}\mu'(s) = \epsilon_1 > 0$*. Assume that* r_1 *is small enough compared to* r *and* r_2 *is small enough compared to* δ*, the initial values verify and*

$$\begin{aligned}\int_\Omega \left(\rho_0\left(\frac{|u_0 + 2\kappa\nabla s(\rho_0)|^2}{2} + (\kappa(1-\kappa)+r)\frac{|2\nabla s(\rho_0)|^2}{2}\right)\right)dx\\ + \int_\Omega\left(a\frac{\rho_0^\gamma}{\gamma-1} + \mu(\rho_0) + \delta\frac{\rho^{10}}{9} + \frac{r_0}{\varepsilon_1}|(\ln\rho_0)_-|\right)dx < +\infty.\end{aligned} \tag{1.24}$$

Then the κ *entropy weak solutions is a renormalized solution of* (1.12) *in the sense of Definition 1.1.*

We then pass to the limit with respect to the parameters r, r_0, r_1, r_2 and δ to recover a renormalized weak solution of the compressible Navier-Stokes equations and prove our main theorem.

Definitions. Following [31] (based on the work in [47]), we will show the existence of renormalized solutions in u. Then, we will show that this renormalized solution is a weak solution. The renormalization provides weak stability of the advection terms $\rho u\otimes u$ together and $\rho u \otimes v$. Let us first define the renormalized solution:

Definition 1.1. Consider $\mu > 0$, $3\lambda + 2\mu > 0$, $r_0 \geq 0$, $r_1 \geq 0$, $r_2 \geq 0$ and $r \geq 0$. We say that $(\sqrt{\rho}, \sqrt{\rho}u)$ is a renormalized weak solution in u, if it verifies (1.18)-(1.21), and for any function $\varphi \in W^{2,\infty}(\mathbb{R}^d)$ with $\varphi(s)s \in L^\infty(\mathbb{R}^d)$, there exists three measures $R_\varphi, \overline{R}^1_\varphi, \overline{R}^2_\varphi \in \mathcal{M}(\mathbb{R}^+ \times \Omega)$, with

$$\|R_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^1_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^2_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} \leq C\|\varphi''\|_{L^\infty(\mathbb{R})},$$

where the constant C depends only on the solution $(\sqrt{\rho}, \sqrt{\rho}u)$, and for any function $\psi \in C_c^\infty(\mathbb{R}^+ \times \Omega)$,

$$\begin{aligned}
&\int_0^T \int_\Omega (\rho\psi_t + \sqrt{\rho}\sqrt{\rho}u \cdot \nabla\psi)\,dx\,dt = 0,\\
&\int_0^T \int_\Omega \big(\rho\varphi(u)\psi_t + \rho\varphi(u) \otimes u : \nabla\psi\big)\,dx\,dt\\
&- \int_0^T \int_\Omega \left(2(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id})\,\varphi'(u)\right) \cdot \nabla\psi\,dxdt\\
&- r\int_0^T \int_\Omega \left(2(\sqrt{\mu(\rho)}\mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_r)\mathrm{Id})\,\varphi'(u)\right) \cdot \nabla\psi\,dxdt\\
&\qquad\qquad\qquad\qquad + F(\rho,u)\,\varphi'(u)\psi\,dx\,dt = \langle R_\varphi, \psi\rangle\,,\\
&\int_0^T \int_\Omega (\mu(\rho)\psi_t + \frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}u \cdot \nabla\psi)\,dxdt - \int_0^T \int_\Omega \frac{\lambda(\rho)}{2\mu(\rho)} Tr(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\psi\,dxdt = 0,
\end{aligned}$$

where $\mathbb{S}_\mu$ is given in (1.4) and $\mathbb{T}_\mu$ is given in (1.7). The matrix $\mathbb{S}_r$ is compatible in (1.14), (1.15), and (1.16).

The vector valued function F is given by

$$\begin{aligned}
F(\rho,u) = &\sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}}\nabla \int_0^\rho \sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\\
&+ \delta\sqrt{\frac{P'_\delta(\rho)\rho}{\mu'(\rho)}}\nabla \int_0^\rho \sqrt{\frac{P'_\delta(s)\mu'(s)}{s}}\,ds - r_0 u - r_1\rho|u|u - \frac{r_2}{\mu'(\rho)}\rho|u|^2u.
\end{aligned} \tag{1.25}$$

For every i, j, k between 1 and d:

$$\sqrt{\mu(\rho)}\varphi_i'(u)[\mathbb{T}_\mu]_{jk} = \partial_j(\mu(\rho)\rho\varphi_i'(u)u_k) - \sqrt{\rho}\;u_k\varphi_i'(u)\sqrt{\rho}\partial_j s(\rho) + \overline{R}^1_\varphi, \tag{1.26}$$

$$r\varphi_i'(u)[\nabla(\sqrt{\mu(\rho)}\nabla Z(\rho))]_{jk} = r\partial_j(\sqrt{\mu(\rho)}\varphi_i'(u)\partial_k Z(\rho)) + \overline{R}^2_\varphi, \tag{1.27}$$

and

$$\|\overline{R}^1_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^2_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|R_\varphi\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} \leq C\|\varphi''\|_{L^\infty}.$$

and for any $\overline{\psi} \in C_c^\infty(\Omega)$:

$$\begin{aligned}
&\lim_{t\to 0} \int_\Omega \rho(t,x)\overline{\psi}(x)\,dx = \int_\Omega \rho_0(x)\overline{\psi}(x)\,dx,\\
&\lim_{t\to 0} \int_\Omega \rho(t,x)u(t,x)\overline{\psi}(x)\,dx = \int_\Omega m_0(x)\overline{\psi}(x)\,dx,\\
&\lim_{t\to 0} \int_\Omega \mu(\rho)(t,x)\overline{\psi}(x)\,dx = \int_\Omega \mu(\rho_0)(x)\overline{\psi}(x)\,dx
\end{aligned}$$

We define a global weak solution of the approximate system or the compressible Navier-Stokes equation (when $r = r_0 = r_1 = r_2 = \delta = 0$) as follows

Definition 1.2. Let $\mathbb{S}_\mu$ the symmetric part of $\mathbb{T}_\mu$ in $L^2((0,T)\times\Omega)$ verifying (1.4)–(1.7) and $\mathbb{S}_r$ the capillary quantity in $L^2((0,T)\times\Omega)$ given by (1.14)–(1.16). Let us denote $P(\rho) = a\rho^\gamma$ and $P_\delta(\rho) = \delta\rho^{10}$. We say that (ρ, u) is a weak solution to (1.12)–(1.15), if it satisfies the *a priori* estimates (1.18)–(1.21) and for any function $\psi \in \mathcal{C}_c^\infty((0,T)\times\Omega)$ verifying

$$
\begin{aligned}
&\int_0^T\int_\Omega (\rho\partial_t\psi + \rho u\cdot\nabla\psi)\,dxdt = 0,\\
&\int_0^T\int_\Omega (\rho u\partial_t\psi + \rho u\otimes u : \nabla\psi)\,dxdt\\
&\qquad -\int_0^T\int_\Omega 2(\sqrt{\mu(\rho)}\mathbb{S}_\mu + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu)\mathrm{Id})\cdot\nabla\psi\,dxdt\\
&\qquad - r\int_0^T\int_\Omega 2(\sqrt{\mu(\rho)}\mathbb{S}_r + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_r)\mathrm{Id})\cdot\nabla\psi\,dxdt\\
&\qquad\qquad + F(\rho,u)\,\psi\,dxdt = 0,\\
&\int_0^\infty\int_\Omega \left(\mu(\rho)\psi_t + \frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}u\cdot\nabla\psi\right)dx\,dt\\
&\qquad\qquad - \int_0^T\int_\Omega \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\psi\,dxdt = 0,
\end{aligned}
\tag{1.28}
$$

with F given through (1.25) and for any $\overline{\psi} \in \mathcal{C}_c^\infty(\Omega)$:

$$
\begin{aligned}
&\lim_{t\to 0}\int_\Omega \rho(t,x)\overline{\psi}(x)\,dx = \int_\Omega \rho_0(x)\overline{\psi}(x)\,dx,\\
&\lim_{t\to 0}\int_\Omega \rho(t,x)u(t,x)\overline{\psi}(x)\,dx = \int_\Omega m_0(x)\overline{\psi}(x)\,dx,\\
&\lim_{t\to 0}\int_\Omega \mu(\rho)(t,x)\overline{\psi}(x)\,dx = \int_\Omega \mu(\rho_0)(x)\overline{\psi}(x)\,dx.
\end{aligned}
$$

Remark. As mentioned in [14], the equation on $\mu(\rho)$ is important: By taking $\psi = \mathrm{div}\varphi$ for all $\varphi \in \mathcal{C}_0^\infty$, we can write the equation satisfied by $\nabla\mu(\rho)$ namely

$$
\begin{aligned}
\partial_t\nabla\mu(\rho) + \mathrm{div}(\nabla\mu(\rho)\otimes u) &= \mathrm{div}(\nabla\mu(\rho)\otimes u) - \nabla\mathrm{div}(\mu(\rho)u)\\
&\qquad - \nabla\Big(\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\Big)\\
&= -\mathrm{div}(\sqrt{\mu(\rho)}^t\mathbb{T}_\mu) - \nabla\Big(\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu)\Big).
\end{aligned}
\tag{1.29}
$$

This will justify in some sense the two-velocities formulation introduced in [12] with the extra velocity linked to $\nabla\mu(\rho)$.

2. The first level of approximation procedure

The goal of this section is to construct a sequence of approximated solutions satisfying the compactness structure to prove Theorem 1.2 namely the existence of weak solutions of the approximation system with capillarity and drag terms. Here we present the first level of approximation procedure.

1. The continuity equation

$$\rho_t + \mathrm{div}(\rho[w]_{\varepsilon_3}) = 2\kappa \mathrm{div}\left([\mu'(\rho)]_{\varepsilon_4}\nabla\rho\right), \tag{2.1}$$

with modified initial data

$$\rho(0,x) = \rho_0 \in C^{2+\nu}(\bar\Omega), \quad 0 < \underline{\rho} \le \rho_0(x) \le \bar\rho.$$

Here ε_3 and ε_4 denote the standard regularizations by mollification with respect to space and time. This is a parabolic equation recalling that in this part $\mathrm{Inf}_{[0,+\infty)}\mu'(s) > 0$. Thus, we can apply the standard theory of parabolic equation to solve it when w is given smooth enough. In fact, the exact same equation was solved in paper [12]. In particular, we are able to get the following bound on the density at this level approximation

$$0 < \underline{\rho} \le \rho(t,x) \le \bar\rho < +\infty. \tag{2.2}$$

2. The momentum equation with drag terms is replaced by its Faedo-Galerkin approximation with the additional regularizing term $\varepsilon_2[\Delta^{2s}w - \mathrm{div}((1+|\nabla w|^2)\nabla w)]$ where $s \ge 2$

$$\begin{aligned}
&\int_\Omega \rho w\cdot\psi\,dx - \int_0^t\int_\Omega \left(\rho([w]_{\varepsilon_3} - 2\kappa\frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho)\otimes w\right) : \nabla\psi\,dx\,dt\\
&+ 2(1-\kappa)\int_0^t\int_\Omega \mu(\rho)\mathbb{D}w : \nabla\psi\,dx\,dt + 2\kappa\int_0^t\int_\Omega \mu(\rho)\mathbf{A}(w) : \nabla\psi\,dx\,dt\\
&+ (1-\kappa)\int_0^t\int_\Omega \lambda(\rho)\mathrm{div}w\mathrm{div}\psi\,dx\,dt - 2\kappa(1-\kappa)\int_0^t\int_\Omega \mu(\rho)\nabla v : \nabla\psi\,dx\,dt\\
&- \kappa(1-\kappa)\int_0^t\int_\Omega \lambda(\rho)\mathrm{div}v\mathrm{div}\psi\,dx\,dt - \int_0^t\int_\Omega \rho^\gamma\mathrm{div}\psi\,dx\,dt - \delta\int_0^t\int_\Omega \rho^{10}\mathrm{div}\psi\,dx\,dt\\
&+ \varepsilon_2\int_0^t\int_\Omega \left(\Delta^s w\cdot\Delta^s\psi + (1+|\nabla w|^2)\nabla w : \nabla\psi\right)\,dx\,dt = -\int_0^t\int_\Omega r_0(w - 2\kappa\nabla s(\rho))\cdot\psi\,dx\,dt\\
&- r_1\int_0^t\int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho))\cdot\psi\,dx\,dt\\
&- r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w - 2\kappa\nabla s(\rho)|^2(w - 2\kappa\nabla s(\rho))\cdot\psi\,dx\,dt\\
&- r\int_0^t\int_\Omega \sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)\mathrm{div}(\rho\psi)\,dx\,dt + \int_\Omega \rho_0 w_0\cdot\psi\,dx
\end{aligned} \tag{2.3}$$

satisfied for any $t > 0$ and any test function $\psi \in C([0,T], X_n)$, where $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$, and $s'(\rho) = \mu'(\rho)/\rho$, and $X_n = \mathrm{span}\{e_i\}_{i=1}^n$ is an orthonormal basis in $W^{1,2}(\Omega)$ with $e_i \in C^\infty(\Omega)$ for any integers $i > 0$.

3. The Faedo-Galerkin approximation for the equation on the drift velocity v reads

$$\begin{aligned}&\int_\Omega \rho v\cdot\phi\,dx-\int_0^t\int_\Omega(\rho([w]_{\varepsilon_3}-2\kappa\frac{[\mu'(\rho)]_{\varepsilon_4}}{\rho}\nabla\rho)\otimes v):\nabla\phi\,dx\,dt\\&+2\kappa\int_0^t\int_\Omega\mu(\rho)\nabla v:\nabla\phi\,dx\,dt+\kappa\int_0^t\int_\Omega\lambda(\rho)\mathrm{div}v\,\mathrm{div}\phi\,dx\,dt\\&-\int_0^t\int_\Omega\lambda(\rho)\mathrm{div}w\mathrm{div}\phi\,dx\,dt+2\int_0^t\int_\Omega\mu(\rho)\nabla^T w:\nabla\phi\,dx\,dt=\int_\Omega\rho_0 v_0\cdot\phi\,dx\end{aligned}\tag{2.4}$$

satisfied for any $t>0$ and any test function $\phi\in C([0,T],Y_n)$, where $Y_n=\mathrm{span}\{b_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^\infty$ is an orthonormal basis in $W^{1,2}(\Omega)$ with $b_i\in C^\infty(\Omega)$ for any integers $i>0$.

The above full approximation is similar to the ones in [12]. We can repeat the same argument as their paper to obtain the local existence of solutions to the Galerkin approximation. In order to extend the local solution to the global one, the uniform bounds are necessary so that the corresponding procedure can be iterated.

2.1. **The energy estimate if the solution is regular enough.** For any fixed $n>0$, choosing test functions $\psi=w$, $\phi=v$ in (2.3) and (2.4), we find that (ρ,w,v) satisfies the following $\kappa-$entropy equality

$$\begin{aligned}&\int_\Omega\left(\rho\left(\frac{|w|^2}{2}+(1-\kappa)\kappa\frac{|v|^2}{2}\right)+\frac{\rho^\gamma}{\gamma-1}+\delta\frac{\rho^{10}}{9}\right)dx+2(1-\kappa)\int_0^t\int_\Omega\mu(\rho)|\mathbb{D}w-\kappa\nabla v|^2\,dx\,dt\\&+(1-\kappa)\int_0^t\int_\Omega\lambda(\rho)(\mathrm{div}w-\kappa\mathrm{div}v)^2\,dx\,dt++2\kappa\int_0^t\int_\Omega\frac{\mu'(\rho)p'(\rho)}{\rho}|\nabla\rho|^2\,dx\,dt\\&+2\kappa\int_0^t\int_\Omega\mu(\rho)|Aw|^2\,dx\,dt+\varepsilon_2\int_0^t\int_\Omega\left(|\Delta^s w|^2+(1+|\nabla w|^2)|\nabla w|^2\right)dx\,dt\\&+r\int_0^t\int_\Omega\sqrt{K(\rho)}\Delta(\int_0^\rho\sqrt{K(s)}\,ds)\mathrm{div}(\rho w)\,dx\,dt+20\kappa\int_0^t\int_\Omega\mu'(\rho)\rho^8|\nabla\rho|^2\,dx\,dt\\&+r_0\int_0^t\int_\Omega(w-2\kappa\nabla s(\rho))\cdot w\,dx\,dt+r_1\int_0^t\int_\Omega\rho|w-2\kappa\nabla s(\rho)|(w-2\kappa\nabla s(\rho))\cdot w\,dx\,dt\\&+r_2\int_0^t\int_\Omega\frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^2(w-2\kappa\nabla s(\rho))\cdot w\,dx\,dt\\&=\int_\Omega\left(\rho_0\left(\frac{|w_0|^2}{2}+(1-\kappa)\kappa\frac{|v_0|^2}{2}\right)+\frac{\rho_0^\gamma}{\gamma-1}+\delta\frac{\rho_0^{10}}{9}\right)dx-\int_0^T\int_\Omega\rho^\gamma\mathrm{div}([w]_{\varepsilon_3}-w)\,dx\,dt\\&-\delta\int_0^T\int_\Omega\rho^{10}\mathrm{div}([w]_{\varepsilon_3}-w)\,dx\,dt,\end{aligned}\tag{2.5}$$

where $s'=\mu'(\rho)/\rho$ and $p(\rho)=\rho^\gamma$. Compared to the calculations made in [12], we have to take care of the capillary term and then to take care of the drag terms showing that they can be controlled using that $\int_{s\in[0,T]}\mu'(s)\geq\varepsilon_1$ for the linear drag, using the extra pressure term $\delta\rho^{10}$ for the quadratic drag term and using the capillary term $r\rho\nabla(\sqrt{K(\rho)}\Delta(\int_0^\rho\sqrt{K(s)}))$ for the cubic drag term. To do so, let us provide some properties on the capillary term and rewrite the terms coming from the drag quantities.

2.1.1. *Some properties on the capillary term.* Using the mass equation, the capillary term in the entropy estimates reads

$$\begin{aligned}\int_\Omega \sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)\,\mathrm{div}(\rho w) &= \frac{r}{2}\frac{d}{dt}\int_\Omega |\nabla \int_0^\rho \sqrt{K(s)}\,ds|^2 \\ + 2\kappa \int_\Omega \sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)\,\Delta\mu(\rho) &= I_1 + I_2.\end{aligned} \tag{2.6}$$

In fact, we write term I_1 as follows

$$\frac{r}{2}\frac{d}{dt}\int_\Omega |\nabla \int_0^\rho \sqrt{K(s)}\,ds|^2 = \frac{r}{2}\frac{d}{dt}\int_\Omega \rho|\nabla s(\rho)|^2\,dx.$$

By (1.22), we have

$$\begin{aligned} I_2 &= \int_\Omega \sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)\,\Delta\mu(\rho) \\ &= -\int_\Omega \rho\nabla\Big(\sqrt{K(\rho)}\Delta(\int_0^\rho \sqrt{K(s)}\,ds)\Big)\cdot\nabla s(\rho) \\ &= \int_\Omega 2\mu(\rho)|2\nabla^2 s(\rho)|^2 + \lambda(\rho)|2\Delta s(\rho)|^2. \end{aligned} \tag{2.7}$$

Control of norms using I_2. Let us first recall that since

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)) > -2\mu(\rho)/3,$$

there exists $\eta > 0$ such that

$$\begin{aligned} &2\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx\,dt + \int_0^T\int_\Omega \lambda(\rho)|\Delta s(\rho)|^2\,dx\,dt \\ &\qquad \geq \eta\Big[2\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx\,dt + \frac{1}{3}\int_0^T\int_\Omega \mu(\rho)|\Delta s(\rho)|^2\,dx\,dt\Big]. \end{aligned}$$

As the second term in the right-hand side is positive, lower bound on the quantity

$$\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx\,dt \tag{2.8}$$

will provide the same lower bound on I_2.

Let us now precise the norms which are controlled by (2.8). To do so, we need to rely on the following lemma on the density. In this lemma, we prove a more general entropy dissipation inequality than the one introduced by Jüngel in [27] and more general than those by Jüngel-Matthes in [28].

Lemma 2.1. *Let $\mu'(\rho)\rho < k\mu(\rho)$ for $2/3 < k < 4$ and*

$$s(\rho) = \int_0^\rho \frac{\mu'(s)}{s}\,ds, \qquad Z(\rho) = \int_0^\rho \frac{\sqrt{\mu(s)}}{s}\mu'(s)\,ds, \qquad Z_1(\rho) = \int_0^\rho \frac{\mu'(s)}{(\mu(s))^{1/4}s^{1/2}}\,ds.$$

i) *Assume $\rho > 0$ and $\rho \in L^2(0,T;H^2(\Omega))$ then there exists $\varepsilon(k) > 0$, such that we have the following estimate*

$$\int_0^T\int_\Omega |\nabla^2 Z(\rho)|^2\,dx\,dt + \varepsilon(k)\int_0^T\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\,dx\,dt \leq \frac{C}{\varepsilon(k)}\int_0^T\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx\,dt,$$

where C is a universal positive constant.
ii) *Consider a sequence of smooth densities $\rho_n > 0$ such that $Z(\rho_n)$ and $Z_1(\rho_n)$ converge strongly in $L^1((0,T)\times\Omega)$ respectively to $Z(\rho)$ and $Z_1(\rho)$ and $\sqrt{\mu(\rho_n)}\nabla^2\mathbf{s}(\rho_n)$ is uniformly bounded in $L^2((0,T)\times\Omega)$. Then*

$$\int_0^T\int_\Omega |\nabla^2 Z(\rho)|^2\,dx\,dt + \varepsilon(k)\int_0^T\int_\Omega |\nabla Z_1(\rho)|^4\,dx\,dt \le C < +\infty$$

Remark 2.1. The case of $Z = 2\sqrt{\rho}$ for the inequality was proved in [27], which is critical to derive the uniform bound on approximated velocity in $L^2(0,T;L^2(\Omega))$ in [46, 47]. The above lemma will play a similar role in this paper.

Proof. Let us first prove the part i). Note that $Z'(\rho) = \frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)$, we get the following calculation:

$$\begin{aligned}\sqrt{\mu(\rho)}\nabla^2 s(\rho) &= \sqrt{\mu(\rho)}\nabla(\frac{\nabla\mu(\rho)}{\rho}) = \sqrt{\mu(\rho)}\nabla\left(\frac{1}{\sqrt{\mu(\rho)}}\nabla Z(\rho)\right)\\ &= \nabla^2 Z(\rho) - \frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\otimes\nabla\sqrt{\mu(\rho)}\\ &= \nabla^2 Z(\rho) - \frac{\rho\nabla Z(\rho)\otimes\nabla Z(\rho)}{2\mu(\rho)^{\frac{3}{2}}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx = \int_\Omega |\nabla^2 Z(\rho)|^2\,dx + \frac{1}{4}\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\,dx\\ - \int_\Omega \frac{\rho}{\mu(\rho)^{\frac{3}{2}}}\nabla^2 Z(\rho) : (\nabla Z(\rho)\otimes\nabla Z(\rho))\,dx.\end{aligned} \tag{2.9}$$

By integration by parts, the cross product term reads as follows

$$\begin{aligned}&-\int_\Omega \frac{\rho}{\mu(\rho)^{\frac{3}{2}}}\nabla^2 Z(\rho) : (\nabla Z(\rho)\otimes\nabla Z(\rho))\,dx\\ &= -\int_\Omega \frac{\rho\sqrt{\mu(\rho)}}{\mu(\rho)}\nabla^2 Z(\rho) : (\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\otimes\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}})\,dx\\ &= \int_\Omega \frac{\rho}{\mu(\rho)}\sqrt{\mu(\rho)}\nabla Z(\rho)\cdot\operatorname{div}(\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}}\otimes\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}})\,dx\\ &\qquad + \int_\Omega \nabla(\frac{\rho}{\sqrt{\mu(\rho)}})\otimes\nabla Z(\rho) : \frac{\nabla Z(\rho)\otimes\nabla Z(\rho)}{\mu(\rho)}\,dx\\ &= I_1 + I_2.\end{aligned} \tag{2.10}$$

To this end, we are able to control I_1 directly,

$$\begin{aligned}|I_1| &\le \varepsilon\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\,dx + \frac{C}{\varepsilon}\int_\Omega \mu(\rho)|\nabla(\frac{\nabla Z(\rho)}{\sqrt{\mu(\rho)}})|^2\,dx\\ &\le \varepsilon\int_\Omega \frac{\rho^2}{\mu(\rho)^3}|\nabla Z(\rho)|^4\,dx + \frac{C}{\varepsilon}\int_\Omega \mu(\rho)|\nabla^2 s(\rho)|^2\,dx,\end{aligned} \tag{2.11}$$

where C is a universal positive constant. We calculate I_2 to have

$$\begin{aligned} I_2 &= \int_\Omega \nabla(\frac{\rho}{\sqrt{\mu(\rho)}}) \otimes \nabla Z(\rho) : \frac{\nabla Z(\rho) \otimes \nabla Z(\rho)}{\mu(\rho)}\, dx \\ &= \int_\Omega \frac{\nabla \rho \otimes \nabla Z(\rho)}{\mu(\rho)^{\frac{3}{2}}} : (\nabla Z(\rho) \otimes \nabla Z(\rho))\, dx \\ &\qquad - \int_\Omega \frac{\rho}{\mu(\rho)^2} \nabla \sqrt{\mu(\rho)} \otimes \nabla Z(\rho) : (\nabla Z(\rho) \otimes \nabla Z(\rho))\, dx \\ &= \int_\Omega \frac{\rho}{\mu(\rho)^2 \mu(\rho)'} |\nabla Z(\rho)|^4\, dx - \frac{1}{2} \int_\Omega \frac{\rho^2}{\mu(\rho)^3} |\nabla Z(\rho)|^4\, dx. \end{aligned} \tag{2.12}$$

Relying on (2.9)-(2.12), we have

$$\begin{aligned} &\int_\Omega |\nabla^2 Z(\rho)|^2\, dx + \int_\Omega \frac{\rho}{\mu(\rho)^2 \mu'(\rho)} |\nabla Z(\rho)|^4\, dx - (\frac{1}{4} + \varepsilon) \int_\Omega \frac{\rho^2}{\mu(\rho)^3} |\nabla Z(\rho)|^4\, dx \\ &\le \frac{C}{\varepsilon} \int_\Omega \mu(\rho) |\nabla^2 s(\rho)|^2\, dx. \end{aligned}$$

Since $k_1 \mu'(s) s \le \mu(s)$, we have

$$\frac{s}{\mu^2(s)\mu'(s)} - (\frac{1}{4} + \varepsilon) \frac{s^2}{\mu(s)^3} \ge (k_1 - \frac{1}{4} - \varepsilon) \frac{s^2}{\mu(s)^3} > \varepsilon \frac{s^2}{\mu(s)^3},$$

where we choose $k_1 > \frac{1}{4}$. This implies

$$\int_\Omega |\nabla^2 Z(\rho)|^2\, dx + \varepsilon \int_\Omega \frac{\rho^2}{\mu(\rho)^3} |\nabla Z(\rho)|^4\, dx \le \frac{C}{\varepsilon} \int_\Omega \mu(\rho) |\nabla^2 s(\rho)|^2\, dx.$$

This ends the proof of part i). Concerning part ii), it suffices to pass to the limit in the inequality proved previously using the lower semi continuity on the left-hand side.

□

2.1.2. *Drag terms control.* We have to discuss three kind of drag terms: Linear drag term, quadratic drag term and finally cubic drag term.

a) *Linear drag terms.* As in previous works [6, 46, 49], we need to choose a linear drag with constant coefficient

$$\begin{aligned} r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot w\, dx\, dt &= r_0 \int_0^t \int_\Omega |w - 2\kappa \nabla s(\rho)|^2\, dx\, dt \\ &+ r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot (2\kappa \nabla s(\rho))\, dx\, dt. \end{aligned} \tag{2.13}$$

The second term on the right side of (2.13) reads

$$\begin{aligned} r_0 \int_0^t \int_\Omega (w - 2\kappa \nabla s(\rho)) \cdot (2\kappa \nabla s(\rho))\, dx\, dt &= r_0 \int_0^t \int_\Omega \rho (w - 2\kappa \nabla s(\rho)) \cdot \frac{2\kappa \nabla s(\rho)}{\rho}\, dx\, dt \\ &= r_0 \int_0^t \int_\Omega \rho (w - 2\kappa \nabla s(\rho)) \cdot 2\kappa \nabla g(\rho)\, dx\, dt \\ &= r_0 \int_0^t \int_\Omega \rho_t g(\rho)\, dx\, dt, \end{aligned}$$

where $g'(\rho) = \frac{s'(\rho)}{\rho} = \frac{\mu'(\rho)}{\rho^2}$ and $g(\rho) = \int_1^\rho \frac{\mu'(r)}{r^2}\,dr$. Letting

$$G(\rho) = \int_1^\rho \int_1^r \frac{\mu'(\zeta)}{\zeta^2}\,d\zeta\,dr,$$

then

$$r_0 \int_\Omega \rho_t g(\rho)\,dx = r_0 \frac{\partial}{\partial_t} \int_\Omega G(\rho)\,dx,$$

which implies

$$r_0 \int_0^t \int_\Omega \rho_t g(\rho)\,dx\,dt = r_0 \int_\Omega G(\rho)\,dx.$$

Meanwhile, since $\lim_{\zeta\to 0} \mu'(\zeta) = \varepsilon_1 > 0$, for any $|\zeta| < \epsilon$ and any small number $\epsilon > 0$, we have $\mu'(\zeta) \geq \frac{\varepsilon_1}{2}$. Thus, we have further estimate on $G(\rho)$ as follows

$$\begin{aligned} G(\rho) = \int_1^\rho \int_1^r \frac{\mu'(\zeta)}{\zeta^2}\,d\zeta\,dr &\geq \frac{\varepsilon_1}{2} \int_1^\rho (1 - \frac{1}{r})\,dr \\ &= \frac{\varepsilon_1}{2}(\rho - 1 - \ln \rho) \\ &\geq -\frac{\varepsilon_1}{4}(\ln \rho)_-, \end{aligned}$$

for any $\rho \leq \epsilon$. Similarly, we can show that

$$G(\rho) \leq 4\varepsilon_1 (\ln \rho)_+$$

for any $\rho \leq \epsilon$. For given number $\epsilon_0 > 0$, if $\rho \geq \epsilon_0$, then we have

$$0 \leq G(\rho) \leq C \int_1^\rho \int_1^r \mu'(\zeta)\,d\zeta\,dr \leq C\mu(\rho)\rho.$$

b) *Quadratic drag term.* We use the same argument as in [12] to handle this term. The quadratic drag term gives

$$\begin{aligned} & r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot w\,dx\,dt \\ & = r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|^3\,dx\,dt \\ & \qquad + r_1 \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho))\,dx\,dt. \end{aligned} \tag{2.14}$$

The second drag term of the right–hand side can be controlled as follows

$$\begin{aligned} & r_1 \left| \int_0^t \int_\Omega \rho|w - 2\kappa\nabla s(\rho)|(w - 2\kappa\nabla s(\rho)) \cdot (2\kappa\nabla s(\rho))\,dx\,dt \right| \\ & \leq r_1 \int_0^t \int_\Omega \mu(\rho)|u||\mathbb{D}u|\,dx\,dt \\ & \leq \frac{1}{2} \int_0^t \int_\Omega \mu(\rho)|\mathbb{D}u|^2\,dx\,dt + \frac{r_1^2}{2} \int_0^t \int_\Omega \mu(\rho)|u|^2\,dx\,dt, \end{aligned} \tag{2.15}$$

and

$$\|\sqrt{\mu(\rho)}|u|\|_{L^2(0,T;L^2(\Omega))} \leq C\|\rho^{\frac{1}{3}}|u|\|_{L^3(0,T;L^3(\Omega))} \|\frac{\sqrt{\mu(\rho)}}{\rho^{\frac{1}{3}}}\|_{L^6(0,T;L^6(\Omega))}.$$

Note that

$$\begin{aligned}\int_0^t\int_\Omega \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt &= \int_0^t\int_{0\le\rho\le 1} \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt + \int_0^t\int_{\rho\ge 1} \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt\\ &\le C\int_0^t\int_{0\le\rho\le 1} \mu(\rho)(\mu'(\rho))^2\,dx\,dt + \int_0^t\int_{\rho\ge 1} \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt \\ &\le C + \int_0^t\int_{\rho\ge 1} \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt.\end{aligned} \tag{2.16}$$

From (1.9), for any $\rho \ge 1$, we have

$$c'\rho^{\alpha_1} \le \mu(\rho) \le c\rho^{\alpha_2},$$

where $2/3 < \alpha_1 \le \alpha_2 < 4$. This yields to

$$\int_0^t\int_{\rho\ge 1} \frac{\mu(\rho)^3}{\rho^2}\,dx\,dt \le c\int_0^t\int_{\rho\ge 1} \rho^{3\,\alpha_2-2}\,dx\,dt \le c\int_0^t\int_\Omega \rho^{10}\,dx \tag{2.17}$$

for any time $t > 0$.

c) *Cubic drag term.* The non-linear cubic drag term gives

$$\begin{aligned} &r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^2(w-2\kappa\nabla s(\rho))\cdot w\,dx\,dt \\ &= r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^4\,dx\,dt \\ &\qquad + r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^2(w-2\kappa\nabla s(\rho))\cdot(2\kappa\nabla s(\rho))\,dx\,dt.\end{aligned} \tag{2.18}$$

The novelty now is to show that we control the second drag term of the right–hand side using the Korteweg-type information on the left-hand side

$$\begin{aligned} &r_2\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^2(w-2\kappa\nabla s(\rho))\cdot(2\kappa\nabla s(\rho))\,dx\,dt \\ &\le r_2\Big(\frac{3}{4}\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|w-2\kappa\nabla s(\rho)|^4 + \frac{(2\kappa)^4}{4}\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|\nabla s(\rho)|^4\Big).\end{aligned} \tag{2.19}$$

Remark that the first term in the right-hand side may be absorbed using the first term in (2.18). Let us now prove that if r_1 small enough, the second term in the right-hand side may be absorbed by the term coming from the capillary quantity in the energy. From Lemma 2.1, we have

$$\int_0^t\int_\Omega \frac{\rho^2}{\mu^3(\rho)}|\nabla Z(\rho)|^4\,dx\,dt = \int_0^t\int_\Omega \frac{1}{\mu(\rho)\rho^2}|\nabla\mu(\rho)|^4\,dx\,dt.$$

It remains to check that

$$\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|\nabla s(\rho)|^4 = \int_0^t\int_\Omega \frac{1}{\mu'(\rho)\rho^3}|\nabla\mu(\rho)|^4\,dx\,dt \le C\int_0^t\int_\Omega \frac{1}{\mu(\rho)\rho^2}|\nabla\mu(\rho)|^4\,dx\,dt.$$

This concludes assuming r_1 small enough compared to r.

2.1.3. *The κ-entropy estimate.* Using the previous calculations, assuming r_2 small enough compared to r, and denoting

$$E[\rho, u+2\kappa\nabla\mathbf{s}(\rho), \nabla\mathbf{s}(\rho)] = \int_\Omega \rho\left(\frac{|u+2\kappa\nabla s(\rho)|^2}{2} + (1-\kappa)\kappa\frac{|\nabla s(\rho)|^2}{2}\right) + \frac{\rho^\gamma}{\gamma-1} + \frac{\delta\rho^{10}}{9} + G(\rho),$$

we get the following κ-entropy estimate

$$\begin{aligned}
&E[\rho, u+2\kappa\nabla\mathbf{s}(\rho), \nabla\mathbf{s}(\rho)](t) + r_0\int_0^t\int_\Omega |u|^2\,dx\,dt \\
&+\frac{r}{2}\int_\Omega |\nabla\int_0^\rho \sqrt{K(s)}\,ds|^2\,dx + 2(1-\kappa)\int_0^t\int_\Omega \mu(\rho)|\mathbb{D}u|^2\,dx\,dt + 20\kappa\int_0^t\int_\Omega \mu'(\rho)\rho^8|\nabla\rho|^2\,dx\,dt \\
&+2(1-\kappa)\int_0^t\int_\Omega (\mu'(\rho)\rho-\mu(\rho))(\mathrm{div}u)^2\,dx\,dt + 2\kappa\int_0^t\int_\Omega \mu(\rho)|A(u+2\kappa\nabla s(\rho))|^2\,dx\,dt \\
&+2\kappa\int_0^t\int_\Omega \frac{\mu'(\rho)p'(\rho)}{\rho}|\nabla\rho|^2\,dx\,dt + r_1\int_0^t\int_\Omega \rho|u|^3\,dx\,dt + \frac{r_2}{4}\int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|u|^4\,dx\,dt \\
&+\kappa r\int_0^t\int_\Omega \mu(\rho)|2\nabla^2 s(\rho)|^2\,dx\,dt + \frac{1}{2}\kappa r\int_0^t\int_\Omega \lambda(\rho)|2\Delta s(\rho)|^2\,dx\,dt \\
&\le \int_\Omega\left(\rho_0\left(\frac{|w_0|^2}{2} + (1-\kappa)\kappa\frac{|v_0|^2}{2}\right) + \frac{\rho_0^\gamma}{\gamma-1} + \frac{\delta\rho_0^{10}}{9} + \frac{r}{2}|\nabla\int_0^{\rho_0}\sqrt{K(s)}\,ds|^2 + G(\rho_0)\right)dx \\
&+C\frac{r_1}{\delta}\int_\Omega E[\rho, u+2\kappa\nabla\mathbf{s}(\rho), \nabla\mathbf{s}(\rho)]dx\,dt.
\end{aligned} \tag{2.20}$$

It suffices now to remark that

$$\begin{aligned}
&\int_0^t\int_\Omega \mu(\rho)|\mathbb{D}u|^2 + \int_0^t\int_\Omega (\mu'(\rho)\rho-\rho)|\mathrm{div}u|^2 \\
&= \int_0^t\int_\Omega \mu(\rho)|\mathbb{D}u - \frac{1}{3}\mathrm{div}u\,\mathrm{Id}|^2\,dxdt + \int_0^t\int_\Omega (\mu'(\rho)\rho-\mu(\rho)+\frac{1}{3}\mu(\rho))|\mathrm{div}u|^2.
\end{aligned}$$

Note that $\alpha_1 > 2/3$, there exists $\varepsilon > 0$ such that

$$\mu'(\rho)\rho - \frac{2}{3}\mu(\rho) > \varepsilon\mu(\rho).$$

Such information and the control of $\sqrt{\mu(\rho)}|A(u)+2\kappa\nabla\mathbf{s}(\rho)|$ in $L^2(0,T;L^2(\Omega))$ allow us, using the Grönwall Lemma and the constraints on the parameters, to get the uniform estimates (1.18)–(1.20).

Now we can show (1.21). First, we have

$$\nabla\mu(\rho) = \frac{\nabla\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho} \in L^\infty(0,T;L^1(\Omega)),$$

due to the mass conservation and the uniform control on $\nabla\mu(\rho)/\sqrt{\rho}$ given in (1.18). Let us now write the equation satisfied by $\mu(\rho)$ namely

$$\partial_t\mu(\rho) + \mathrm{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2}\mathrm{div}u = 0.$$

Recalling that $\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho))$ and the hypothesis on $\mu(\rho)$, we get

$$\frac{d}{dt}\int_\Omega \mu(\rho) \le C\,\big(\int_\Omega |\lambda(\rho)||\mathrm{div}u|^2 + \int_\Omega \mu(\rho)\big),$$

and therefore

$$\mu(\rho) \in L^\infty(0,T;L^1(\Omega)),$$

if $\mu(\rho_0) \in L^1(\Omega)$ due to the fact that $\sqrt{|\lambda(\rho)|}\mathrm{div}u \in L^2(0,T;L^2(\Omega))$. Now, we observe that $\mu(\rho)/\sqrt{\rho}$ is smaller than 1 for $\rho \le 1$ because $\alpha_1 > 2/3$, and smaller than $\mu(\rho)$ for $\rho_n > 1$, then

$$\frac{\mu(\rho)}{\sqrt{\rho}} \in L^\infty(L^1).$$

Meanwhile, thanks to (1.9), we have

$$|\nabla(\mu(\rho)/\sqrt{\rho})| \le \left|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right| + \frac{\mu(\rho)}{2\rho\sqrt{\rho}}|\nabla\rho| \le \left(1+\frac{1}{\alpha_1}\right)\left|\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\right|.$$

By (1.18), $\nabla(\mu(\rho)/\sqrt{\rho})$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and finally $\mu(\rho)/\sqrt{\rho}$ is bounded in $L^\infty(0,T;(L^6(\Omega))$. Thus, we have that

$$\mu(\rho)u = \frac{\mu(\rho)}{\sqrt{\rho}}\sqrt{\rho}u,$$

is uniformly bounded in $L^\infty(0,T;L^{3/2}(\Omega))$. Let us come back to the equation satisfied by $\mu(\rho)$ which reads

$$\partial_t\mu(\rho) + \mathrm{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2}\mathrm{div}u = 0.$$

Recalling that $\lambda(\rho)\mathrm{div}u \in L^\infty(0,T;L^1(\Omega))$, then we get the conclusion on $\partial_t\mu(\rho)$. Let us now to prove that

$$Z(\rho) = \int_0^{\rho_n} \frac{\sqrt{\mu(s)}\mu'(s)}{s}ds \in L^{1+}((0,T)\times\Omega) \text{ uniformly.}$$

Note first that

$$0 \le \frac{\sqrt{\mu(s)}\mu'(s)}{s} \le \alpha_2\frac{\mu(s)^{3/2}}{s^2} \le c_2\alpha_2(s^{3\alpha_1/2-2}1_{s\le 1} + \frac{\mu(s)^{3/2-}}{s^{2-}}1_{s\ge 1}).$$

There exists $\varepsilon > 0$ such that $\alpha_1 > 2/3 + \varepsilon$, thus

$$0 \le \frac{\sqrt{\mu(s)}\mu'(s)}{s} \le c_2\alpha_2(s^{\varepsilon-1}1_{s\le 1} + \frac{\mu(s)^{3/2-}}{s^{2-}}1_{s\ge 1}).$$

Note that $\mu'(s) > 0$ for $s > 0$ and the definition of $Z(\rho)$, we get

$$0 \le Z(\rho) \le C(\rho^\varepsilon + \mu(\rho)^{3/2-})$$

with C independent of n. Thus $Z(\rho) \in L^\infty(0,T;L^{1+}(\Omega))$ uniformly with respect to n. Bound on $Z_1(\rho)$ follows the similar lines.

2.2. **Compactness Lemmas.** In this subsection, we provide general compactness lemmas which will be used several times in this paper.

Some uniform compactness.

Lemma 2.2. *Assume we have a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ satisfying the estimates in Theorem 1.2, uniformly with respect to n. Then, there exists a function $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ such that, up to a subsequence,*

$$\mu(\rho_n) \to \mu(\rho) \text{ in } \mathcal{C}([0,T];L^{3/2}(\Omega) \text{ weak}),$$

and

$$\rho_n \to \rho \text{ a.e. in } (0,T)\times\Omega.$$

Moreover

$$\rho_n \to \rho \text{ in } L^{(4\gamma/3)^+}((0,T)\times\Omega),$$

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \rightharpoonup \sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}}\nabla\Big(\int_0^{\rho}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \text{ in } L^1((0,T)\times\Omega)$$

and

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \in L^{1+}((0,T)\times\Omega).$$

If $\delta_n>0$ is such that $\delta_n\to\delta\geq 0$, then

$$\delta_n\rho_n^{10} \to \delta\rho^{10} \quad \text{in } L^{\frac{4}{3}}((0,T)\times\Omega).$$

Proof. From the estimate on $\mu(\rho_n)$ and Aubin-Lions lemma, up to a subsequence, we have

$$\mu(\rho_n) \to \mu(\rho) \text{ in } \mathcal{C}([0,T];L^{3/2}(\Omega) \text{ weak})$$

and therefore using that $\mu'(s)>0$ on $(0,+\infty)$ with $\mu(0)=0$, we get the conclusion on ρ_n. Let us now recall that

$$\frac{\alpha_1}{\rho_n} \leq \frac{\mu'(\rho_n)}{\mu(\rho)} \leq \frac{\alpha_2}{\rho_n} \tag{2.21}$$

and therefore

$$c_1\rho_n^{\alpha_2} \leq \mu(\rho_n) \leq c_2\rho_n^{\alpha_1} \qquad \text{for } \rho_n\leq 1,$$

and

$$c_1\rho_n^{\alpha_1} \leq \mu(\rho_n) \leq c_2\rho_n^{\alpha_2} \qquad \text{for } \rho\geq 1.$$

with c_1 and c_2 independent on n. Note that

$$\sqrt{\frac{p'(\rho_n)\mu'(\rho_n)}{\rho_n}}\nabla\rho_n \in L^\infty(0,T;L^2(\Omega)) \text{ uniformly.} \tag{2.22}$$

Let us prove that there exists ε such that

$$I_0 = \int_0^T\int_\Omega \rho_n^{\frac{4\gamma}{3}+\varepsilon} < C$$

with C independent on n and the parameters. We first remark that it suffices to look at it when $\rho_n \geq 1$ and to remark there exists ε such that $\varepsilon \leq (\gamma-1)/3$. Let us take such parameter then

$$\int_0^T \int_\Omega \rho_n^{\frac{4\gamma}{3}+\varepsilon} 1_{\rho\geq 1} \leq \int_0^T \int_\Omega \rho_n^{\frac{2\gamma}{3}+\gamma-\frac{1}{3}} 1_{\rho\geq 1} \leq \int_0^T \int_\Omega \rho_n^{\frac{2\gamma}{3}+\gamma+\alpha_1-1} 1_{\rho\geq 1}$$

recalling that $\alpha_1 > 2/3$. Following [34], it remains to prove that

$$I_1 = \int_0^T \int_\Omega \big[\rho_n^{[5\gamma+3(\alpha_1-1)]/3}\, 1_{\rho\geq 1}\big] < +\infty$$

uniformly. Denoting

$$I_2 = \int_0^T \int_\Omega \big[\rho^{[5\gamma+3(\alpha_2-1)]/3}\, 1_{\rho\leq 1}\big]$$

and using the bounds on $\mu(\rho_n)$ in terms of power functions in ρ, which are different if $\rho_n \geq 1$ or $\rho_n \leq 1$, we can write:

$$I_1 \leq I_1 + I_2 \leq C_a \int_0^T \int_\Omega \rho_n^{2\gamma/3} P'(\rho_n)\, \mu(\rho_n) \leq C_a \int_0^T \|\rho_n^\gamma\|_{L^1(\Omega)}^{2/3} \|P'(\rho_n)\mu(\rho_n)\|_{L^3(\Omega)}$$

where C does not depend on n. Using the Poincaré-Wirtinger inequality, one obtains that

$$\begin{aligned}\|P'(\rho_n)\mu(\rho_n)\|_{L^3(\Omega)} &= \|\sqrt{P'(\rho_n)\mu(\rho_n)}\|_{L^6(\Omega)}^2 \\ &\leq \|\sqrt{P'(\rho_n)\mu(\rho_n)}\|_{L^1(\Omega)} + \|\nabla\big[\sqrt{P'(\rho_n)\mu(\rho_n)}\big]\|_{L^2(\Omega)}^2.\end{aligned}$$

Let us now check that the two terms are uniformly bounded in time. First we caculate

$$\nabla\big[\sqrt{P'(\rho_n)\mu(\rho_n)}\big] = \frac{P''(\rho_n)\mu(\rho_n) + P'(\rho_n)\mu'(\rho_n)}{\sqrt{P'(\rho_n)\mu(\rho_n)}} \nabla\rho_n$$

and using (2.21), we can check that

$$\frac{P''(\rho_n)\mu(\rho_n) + P'(\rho_n)\mu'(\rho_n)}{\sqrt{P'(\rho_n)\mu(\rho_n)}} \leq \sqrt{\frac{P'(\rho_n)\mu'(\rho_n)}{\rho_n}}.$$

Therefore, using (2.22), uniformly with respect to n, we get

$$\sup_{t\in[0,T]} \|\nabla\big[\sqrt{P'(\rho_n)\mu(\rho_n)}\big]\|_{L^2(\Omega)}^2 < +\infty.$$

Let us now check that uniformly with respect to n

$$\sup_{t\in[0,T]} \|\sqrt{P'(\rho_n)\mu(\rho_n)}\|_{L^1(\Omega)} < +\infty. \tag{2.23}$$

Using the bounds on $\mu(\rho_n)$, we have

$$\int_\Omega \sqrt{P'(\rho_n)\mu(\rho_n)} \leq C \int_\Omega \Big[\rho_n^{(\gamma-1+\alpha_1)/2} 1_{\rho_n\leq 1} + \rho_n^{(\gamma-1+\alpha_2)/2} 1_{\rho_n\geq 1}\Big]$$

with C independent on n. Recalling that $\alpha_1 \geq 2/3$ and $\alpha_2 < 4$, we can check that

$$\int_\Omega \sqrt{P'(\rho_n)\mu(\rho_n)} \leq C \int_\Omega \Big[\rho_n^{\gamma/3} + \rho_n^{\frac{\gamma}{2}} \rho_n^{\frac{3}{2}}\Big],$$

and therefore using that $\rho_n^\gamma \in L^\infty(0,T;L^1(\Omega))$ and $\rho_n \in L^\infty(0,T;L^{10}(\Omega))$, we get (2.23). This ends the proof of the convergence of ρ_n to ρ in $L^{(4\gamma/3)^+}((0,T)\times\Omega$.

Let us now focus on the convergence of

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big). \tag{2.24}$$

First let us recall that

$$\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \in L^\infty(0,T;L^2(\Omega)) \text{ uniformly.}$$

Let us now prove that

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}} \in L^{2+}((0,T)\times\Omega). \tag{2.25}$$

Recall first that $\alpha_1 > \frac{2}{3}$, we just have to consider $\rho_n \geq 1$. We write

$$\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}1_{\rho_n\geq 1} \leq C\rho_n^{\gamma-\alpha_1+1}1_{\rho_n\geq 1} \leq C\rho_n^{\gamma+1/3}1_{\rho_n\geq 1} \leq C\rho_n^{\frac{4\gamma}{3}}1_{\rho_n\geq 1}.$$

We can use the fact that $\rho_n^{(4\gamma/3)^+} \in L^1((0,T)\times\Omega)$ uniformly to conclude on (2.25). Thanks to

$$\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}} \to \sqrt{\frac{P'(\rho)\rho}{\mu'(\rho)}} \text{ in } L^2((0,T)\times\Omega)$$

and

$$\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \to \nabla\Big(\int_0^{\rho}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big) \text{ weakly in } L^2((0,T)\times\Omega),$$

we have the weak convergence of (2.24) in $L^1((0,T)\times\Omega)$.

We now investigate limits on u independent of the parameters. We need to differentiate the case with hyper-viscosity $\varepsilon_2 > 0$, from the case without. In the case with hyper-viscosity, the estimate depends on ε_1 because of the drag force r_1, while the estimate in the case $\varepsilon_2 = 0$ is independent of all the other parameters. This is why we will consider the limit ε_2 converges to 0 first.

Lemma 2.3. *Assume that $\varepsilon_1 > 0$ is fixed. Then, there exists a constant $C > 0$ depending on ε_1 and C_{in}, but independent of all the other parameters (as long as they are bounded), such that for any initial values $(\rho_0, \sqrt{\rho_0}u_0)$ verifying (1.24) for $C_{in} > 0$ we have*

$$\|\partial_t(\rho u)\|_{L^{1+}(0,T;W^{-s,2}(\Omega))} \leq C,$$
$$\|\nabla(\rho u)\|_{L^2(0,T;L^1(\Omega))} \leq C.$$

Assume now that $\varepsilon_2 = 0$. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function, positive for $\rho > 0$, such that

$$\Phi(\rho) + |\Phi'(\rho)| \leq Ce^{-\frac{1}{\rho}}, \qquad \text{for } \rho \leq 1,$$
$$\Phi(\rho) + |\Phi'(\rho)| \leq Ce^{-\rho}, \qquad \text{for } \rho \geq 2.$$

Assume that the initial values $(\rho_0, \sqrt{\rho_0}u_0)$ verify (1.24) for a fixed $C_{in} > 0$. Then, there exists a constant $C > 0$ independent of $\varepsilon_1, r_0, r_1, r_2, \delta$ (as long as they are bounded), such that

$$\|\partial_t [\Phi(\rho)u]\|_{L^{1+}(0,T;W^{-2,1}(\Omega))} \leq C,$$
$$\|\nabla [\Phi(\rho)u]\|_{L^2(0,T;L^1(\Omega))} \leq C.$$

Proof. We split the proof into the two cases.

Case 1: Assume that $\varepsilon_1 > 0$. From the equation on ρu and the *a priori* estimates, we find directly that

$$\|\partial_t(\rho u)\|_{L^{1+}(0,T;W^{-s,2}(\Omega))} \leq C + r_1^{1/4} \frac{\|\rho\|^{1/4}_{L^1((0,T)\times\Omega)}}{\|\mu'(\rho)\|_{L^\infty((0,T)\times\Omega)}} \left(r_1 \int_0^T \int_\Omega \rho|u|^4 \, dx \, dt \right)^{3/4} \leq C(1+1/\varepsilon_1).$$

We have $\mu(\rho) \geq \varepsilon_1 \rho$, and from (1.18), we have the *a priori* estimate

$$\|\nabla\sqrt{\rho}\|^2_{L^\infty(0,T;L^2(\Omega))} \leq \frac{C}{\varepsilon_1}.$$

Hence

$$\begin{aligned}\|\nabla(\rho u)\|_{L^2(0,T;L^1(\Omega))} \quad &\leq \left\| \frac{\rho}{\sqrt{\mu}(\rho)} \right\|_{L^\infty(0,T;L^2(\Omega))} \|\sqrt{\mu}(\rho)\nabla u\|_{L^2(0,T;L^2(\Omega)))} \\ &+2\|\nabla\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))}\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C.\end{aligned}$$

Case 2: Assume now that $\varepsilon_2 = 0$. Multiplying the equation on (ρu) by $\Phi(\rho)/\rho$, we get, as for the renormalization, that

$$\|\partial_t [\Phi(\rho)u]\|_{L^{1+}(0,T;W^{-2,1}(\Omega))} \leq C.$$

Note that

$$\begin{aligned}\|\nabla [\Phi(\rho)u]\|_{L^2(0,T;L^1(\Omega))} &\leq \left\| \frac{\Phi(\rho)}{\sqrt{\mu}(\rho)} \right\|_{L^\infty} \|\sqrt{\mu}(\rho)\nabla u\|_{L^2(L^2)} \\ &+2\left\|\frac{\Phi'(\rho)}{\mu'(\rho)}\right\|_{L^\infty((0,T)\times\Omega)}\|\mu'(\rho)\nabla\sqrt{\rho}\|_{L^\infty(0,T;L^2(\Omega))}\|\sqrt{\rho}u\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C.\end{aligned}$$

□

Lemma 2.4. *Assume either that $\varepsilon_{2,n} = 0$, or $\varepsilon_{1,n} = \varepsilon_1 > 0$. Let $(\rho_n, \sqrt{\rho_n}u_n)$ be a sequence of solutions for a family of bounded parameters with uniformly bounded initial values verifying (1.24) with a fixed C_{in}. Assume that there exists $\alpha > 0$, and a smooth function $h : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ such that ρ_n^α is uniformly bounded in $L^p((0,T)\times\Omega)$ and $h(\rho_n, u_n)$ is uniformly bounded in $L^q((0,T)\times\Omega)$, with*

$$\frac{1}{p} + \frac{1}{q} < 1.$$

Then, up to a subsequence, ρ_n converges to a function ρ strongly in L^1, $\sqrt{\rho_n}u_n$ converges weakly to a function q in L^2. We define $u = q/\sqrt{\rho}$ whenever $\rho \neq 0$, and $u = 0$ on the vacuum where $\rho = 0$. Then $\rho_n^\alpha h(\rho_n, u_n)$ converges strongly in L^1 to $\rho^\alpha h(\rho, u)$.

Proof. Thanks to the uniform bound on the kinetic energy $\int \rho_n|u_n|^2$, and to Lemma 2.2, up to a subsequence, ρ_n converges strongly in $L^1((0,T)\times\Omega)$ to a function ρ, and $\sqrt{\rho_n}u_n$ converges weakly in $L^2((0,T)\times\Omega)$ to a function q.

We want to show that, up to a subsequence, $u_n\mathbf{1}_{\{\rho>0\}}$ converges almost every where to $u\mathbf{1}_{\{\rho>0\}}$. We consider the two cases. First, if $\varepsilon_{1,n}=\varepsilon_1>0$, then from Lemma 2.3 and the Aubin-Lions Lemma, $\rho_n u_n$ converges strongly in $C^0(0,T;L^1(\Omega))$ to $\sqrt{\rho}q=\rho u$. Up to a subsequence, both ρ_n and $\rho_n u_n$ converges almost everywhere to, respectively, ρ and ρu. For almost every $(t,x)\in\{\rho>0\}$, for n big enough, $\rho_n(t,x)>0$, so $u_n=\rho_n u_n/\rho_n$ at this point converges u. If $\varepsilon_{2,n}=0$ we use the second part of Lemma 2.3 and thanks to the Aubin-Lions Lemma, $\Phi(\rho_n)u_n$ converges strongly in $C^0(0,T;L^1(\Omega))$ to $\Phi(\rho)u$. We still have, up to a subsequence, both ρ_n and $\Phi(\rho_n)u_n$ converging almost everywhere to, respectively, ρ and $\phi(\rho)u$ (we used the fact that $\Phi(r)/\sqrt{r}=0$ at $r=0$). Since $\Phi(r)\neq 0$ for $r\neq 0$, for almost every $(t,x)\in\{\rho>0\}$, for n big enough, $\Phi(\rho_n)(t,x)>0$, so $u_n=\Phi(\rho_n)u_n/\Phi(\rho_n)$ at this point converges u.

Note that
$$\rho_n^\alpha h(\rho_n,u_n)=\rho_n^\alpha h(\rho_n,u_n)\mathbf{1}_{\{\rho>0\}}+\rho_n^\alpha h(\rho_n,u_n)\mathbf{1}_{\{\rho=0\}}.$$
The first term converges almost everywhere to $\rho^\alpha h(\rho,u)\mathbf{1}_{\{\rho>0\}}$, and therefore to $\rho^\alpha h(\rho,u)$ in L^1 by the Lebesgue's theorem. The second part can be estimated as follows
$$\|\rho_n^\alpha h(\rho_n,u_n)\mathbf{1}_{\{\rho=0\}}\|_{L^1}\leq \|h(\rho_n,u_n)\|_{L^q}\|\rho_n^\alpha\mathbf{1}_{\{\rho=0\}}\|_{L^{p-\varepsilon}}.$$
But $\rho_n^\alpha\mathbf{1}_{\{\rho=0\}}$ converges almost everywhere to 0, by the Lebesgue's theorem, the last term converges to 0. □

Some compactness when the parameters are fixed. For any positive fixed δ, r_0, r_1, r_2 and r, to recover a weak solution to (1.12), we only need to handle the compactness of the terms
$$r\rho_n\nabla\left(\sqrt{K(\rho_n)}\Delta(\int_0^{\rho_n}\sqrt{K(s)}\,ds)\right)$$
and
$$\frac{\rho_n}{\mu'(\rho_n)}|u_n|^2u_n.$$
Indeed due to the term $r_0\rho_n|u_n|u_n$ and the fact that $\inf_{s\in[0,+\infty)}\mu'(s)>\varepsilon_1>0$, one obtains the compactness for all other terms in the same way as in [12, 37].

Capillarity term. To pass to the limits in
$$r\rho_n\nabla\left(\sqrt{K(\rho_n)}\Delta(\int_0^{\rho_n}\sqrt{K(s)}\,ds)\right),$$
we use the identity
$$\begin{aligned}\rho\nabla&\left(\sqrt{K(\rho_n)}\Delta(\int_0^{\rho_n}\sqrt{K(s)}\,ds)\right)\\&=4\Big[2\mathrm{div}(\sqrt{\mu(\rho_n)}\nabla\nabla Z(\rho_n))-\Delta(\sqrt{\mu(\rho_n)}\nabla Z(\rho_n)\Big]\\&\quad+\Big[\nabla\big[(\frac{2\lambda(\rho_n)}{\sqrt{\mu(\rho_n)}}+k(\rho_n))\Delta Z(\rho_n)\big]-\nabla\mathrm{div}[k(\rho_n)\nabla Z(\rho_n)]\Big]\end{aligned}\tag{2.26}$$

where $Z(\rho_n) = \int_0^{\rho_n} [(\mu(s))^{1/2}\mu'(s)]/s\,ds$ and $k(\rho_n) = \int_0^{\rho_n} \frac{\lambda(s)\mu'(s)}{\mu(s)^{3/2}}ds$. It allows us to rewrite the weak form coming for the capillarity term as follows

$$\begin{aligned}
&\int_0^t \int_\Omega \sqrt{K(\rho_n)}\Delta(\int_0^{\rho_n} \sqrt{K(s)}\,ds)\mathrm{div}(\rho_n\psi)\,dx\,dt \\
&= 4\int_0^t \int_\Omega \big(2\sqrt{\mu(\rho_n)}\nabla\nabla Z(\rho_n) : \nabla\psi + \sqrt{\mu(\rho_n)}\nabla Z(\rho_n)\cdot\Delta\psi\big) \\
&\qquad + \int_0^t \int_\Omega \big(\frac{2\lambda(\rho_n)}{\sqrt{\mu(\rho_n)}} + k(\rho_n)\big)\Delta Z(\rho_n)\,\mathrm{div}\psi + k(\rho_n)\nabla Z(\rho_n).\nabla\mathrm{div}\psi\big) \\
&= A_1 + A_2.
\end{aligned}$$

In fact, with Lemma 2.2 at hand, we are able to have compactness of A_1 and A_2 easily. Concerning A_1, we know that

$$\sqrt{\mu(\rho_n)} \to \sqrt{\mu(\rho)} \text{ in } L^p((0,T); L^q(\Omega)) \text{ for all } p < +\infty \text{ and } q < 3.$$

Note that $\nabla\nabla Z(\rho_n)$ is uniformly bounded in $L^2(0,T; L^2(\Omega))$, we have $\nabla Z(\rho_n)$ is uniformly bounded in $L^2(0,T; L^6(\Omega))$, because $\int_\Omega \nabla Z(\rho_n) = 0$ due to the periodic condition. Thus we have following weak convergence

$$\int_\Omega \sqrt{\mu(\rho_n)}\nabla Z(\rho_n)\cdot\Delta\psi\,dx \to \int_\Omega \sqrt{\mu}\nabla Z\cdot\Delta\psi\,dx,$$

and

$$\int_\Omega \sqrt{\mu(\rho_n)}\nabla\nabla Z(\rho_n)\nabla\psi\,dx \to \int_\Omega \sqrt{\mu}\nabla\nabla Z : \nabla\psi\,dx,$$

thanks to Lemma 2.2. We conclude that $Z = Z(\rho)$, thanks to the bound on $Z(\rho_n)$ and the strong convergence on ρ_n. Thus using the compactness on ρ_n, the passage to the limit in A_1 is done. Concerning A_2, we just have to look at the coefficients

$$k(\rho_n) = \int_0^{\rho_n} \lambda(s)\mu'(s)/\mu(s)^{3/2}\,ds, \qquad j(\rho_n) = 2\lambda(\rho_n)/\sqrt{\mu(\rho_n)}.$$

Recalling the assumptions on $\mu(s)$ and the relation $\lambda(s) = 2(\mu'(s)s - \mu(s))$, we have

$$2(\alpha_1 - 1)\mu(s) \le \lambda(s) \le 2(\alpha_2 - 1)\mu(s),$$

and

$$\frac{\alpha_1}{\sqrt{\mu(s)}s} \le \frac{\mu'(s)}{\mu(s)^{3/2}} \le \frac{\alpha_2}{\sqrt{\mu(s)}s}.$$

This means that the coefficients $k(\rho_n)$ and $j(\rho_n)$ are comparable to $\sqrt{\mu(\rho_n)}$. Using the compactness of the density ρ_n and the informations on $\mu(\rho_n)$ given in Corollary 2.2, we conclude the compactness of A_2 doing as for A_1.

Cubic non-linear drag term. We will use Lemma 2.4 to show the compactness of

$$\frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n.$$

More precisely, we write

$$\frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n = \rho_n^{\frac{1}{6}}\sqrt{\frac{\rho_n}{\mu'(\rho_n)}}|u_n|^2 \rho_n^{\frac{1}{3}}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}} = \rho_n^{1/6} h(\rho_n, |u_n|), \tag{2.27}$$

By Lemma 2.2, there exists $\varepsilon > 0$ such that $\rho_n^{\frac{1}{6}}$ is uniformly bounded in $L^\infty(0,T;L^{6\gamma+\varepsilon}(\Omega))$ and $\rho_n \to \rho$ a.e., so

$$\rho_n^{\frac{1}{6}} \to \rho^{\frac{1}{6}} \quad \text{in } L^{6\gamma+\varepsilon}((0,T)\times\Omega)). \tag{2.28}$$

Note that $\sqrt{\frac{\rho_n}{\mu'(\rho_n)}}|u_n|^2$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$, and $\inf_{s\in[0,+\infty)}\mu'(s) \geq \varepsilon_1 > 0$, $\rho_n^{\frac{1}{3}}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}}$ is uniformly bounded in $L^3(0,T;L^3(\Omega))$, thus

$$h(\rho_n,|u_n|) = \sqrt{\frac{\rho_n}{\mu'(\rho_n)}}|u_n|^2\rho_n^{\frac{1}{3}}|u_n|\frac{1}{\sqrt{\mu'(\rho_n)}} \in L^{\frac{6}{5}}(0,T;L^{\frac{6}{5}}(\Omega)) \text{ uniformly.} \tag{2.29}$$

By Lemma 2.4 and (2.27)–(2.29), we deduce that

$$\int_0^t\int_\Omega \frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n\,dx\,dt \to \int_0^t\int_\Omega \frac{\rho}{\mu'(\rho)}|u|^2 u\,dx\,dt. \square$$

Relying on the compactness stated in this section and the compactness in [37], we are able to follow the argument in [12] to show Theorem 1.2. Thanks to term $r_0\rho_n|u_n|u_n$, we have

$$\int_0^T\int_\Omega r_0\rho_n|u_n|^4\,dx\,dt \leq C.$$

This gives us that

$$\sqrt{\rho_n}u_n \to \sqrt{\rho}u \;\text{ strongly in } L^2(0,T;L^2(\Omega)).$$

With above compactness of this section, we are able to pass to the limits for recovering a weak solution. In fact, to recover a weak solution to (1.12), we have to pass to the limits as the order of $\varepsilon_4 \to 0$, $n \to \infty$, $\varepsilon_3 \to 0$ and $\varepsilon \to 0$ respectively. In particular, when passing to the limit ε_3 tends to zero, we also need to handle the identification of v with $2\nabla s(\rho)$. Following the same argument in [12], one shows that v and $2\nabla s(\rho)$ satisfy the same moment equation. By the regularity and compactness of solutions, we can show the uniqueness of solutions. By the uniqueness, we have $v = 2\nabla s(\rho)$. This ends the proof of Theorem 1.2.

3. From weak solutions to renormalized solutions to the approximation

This section is dedicated to show that a weak solution is a renormalized solution for our last level of approximation namely to show Theorem 1.3. First, we introduce a new function

$$[f(t,x)]_\varepsilon = f*\eta_\varepsilon(t,x), \text{ for any } \; t>\varepsilon, \qquad \text{and } \; [f(t,x)]_\varepsilon^x = f*\eta_\varepsilon(x)$$

where

$$\eta_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+1}}\eta(\frac{t}{\varepsilon},\frac{x}{\varepsilon}), \qquad \text{and } \eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(\frac{x}{\varepsilon}),$$

with η a smooth nonnegative even function compactly supported in the space time ball of radius 1, and with integral equal to 1. In this section, we will rely on the following two lemmas to proceed our ideas. Let ∂ be a partial derivative in one direction (space or time) in these two lemmas. The first one is the commutator lemma of DiPerna and Lions, see [35].

Lemma 3.1. *Let $f \in W^{1,p}(\mathbb{R}^N \times \mathbb{R}^+)$, $g \in L^q(\mathbb{R}^N \times \mathbb{R}^+)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have*

$$\|[\partial(fg)]_\varepsilon - \partial(f([g]_\varepsilon))\|_{L^r(\mathbb{R}^N \times \mathbb{R}^+)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^N \times \mathbb{R}^+)}\|g\|_{L^q(\mathbb{R}^N \times \mathbb{R}^+)}$$

for some $C \geq 0$ independent of ε, f and g, r is determined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$[\partial(fg)]_\varepsilon - \partial(f([g]_\varepsilon)) \to 0 \quad in \ L^r(\mathbb{R}^N \times \mathbb{R}^+)$$

as $\varepsilon \to 0$ if $r < \infty$. Moreover, in the same way if $f \in W^{1,p}(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have

$$\|[\partial(fg)]^x_\varepsilon - \partial(f([g]^x_\varepsilon))\|_{L^r(\mathbb{R}^N)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}$$

for some $C \geq 0$ independent of ε, f and g, r is determined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$[\partial(fg)]^x_\varepsilon - \partial(f([g]^x_\varepsilon)) \to 0 \quad in \ L^r(\mathbb{R}^N)$$

as $\varepsilon \to 0$ if $r < \infty$.

We also need another very standard lemma as follows.

Lemma 3.2. *If $f \in L^p(\Omega \times \mathbb{R}^+)$ and $g \in L^q(\Omega \times \mathbb{R}^+)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $H \in W^{1,\infty}(\mathbb{R})$, then*

$$\begin{aligned}
&\int_0^T \int_\Omega [f]_\varepsilon g \, dx \, dt = \int_0^T \int_\Omega f[g]_\varepsilon \, dx \, dt, \\
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega [f]_\varepsilon g \, dx \, dt = \int_0^T \int_\Omega fg \, dx \, dt, \\
&\partial[f]_\varepsilon = [\partial f]_\varepsilon, \\
&\lim_{\varepsilon \to 0} \|H([f]_\varepsilon) - H(f)\|_{L^s_{loc}}(\Omega \times \mathbb{R}^+) = 0, \quad \text{for any } 1 \leq s < \infty.
\end{aligned}$$

We define a nonnegative cut-off functions ϕ_m for any fixed positive m as follows.

$$\phi_m(y) \begin{cases} = 0, & \text{if } 0 \leq y \leq \frac{1}{2m}, \\ = 2my - 1, & \text{if } \frac{1}{2m} \leq y \leq \frac{1}{m}, \\ = 1, & \text{if } \frac{1}{m} \leq y \leq m, \\ = 2 - \frac{y}{m}, & \text{if } m \leq y \leq 2m, \\ = 0, & \text{if } y \geq 2m. \end{cases} \tag{3.1}$$

It enables to define an approximated velocity for the density bounded away from zero and bounded away from infinity. It is crucial to process our procedure, since the gradient approximated velocity is bounded in $L^2((0,T) \times \Omega)$. In particular, we introduce $u_m = u\phi_m(\rho)$ for any fixed $m > 0$. Thus, we can show ∇u_m is bounded in $L^2(0,T; L^2(\Omega))$ due to (3.1). In fact,

$$\begin{aligned}
\nabla u_m &= \phi'_m(\rho) u \otimes \nabla\rho + \phi_m(\rho)\frac{1}{\sqrt{\mu(\rho)}}\mathbb{T}_\mu \\
&= \big(\phi'_m(\rho)\frac{(\mu(\rho)\rho)^{1/4}}{(\mu'(\rho))^{\frac{3}{4}}}\big)\big((\frac{\rho}{\mu'(\rho)})^{\frac{1}{4}}u\big) \otimes \big(\frac{\mu'(\rho)}{\rho^{\frac{1}{2}}\mu(\rho)^{\frac{1}{4}}}\nabla\rho\big) + \phi_m(\rho)\frac{1}{\sqrt{\mu(\rho)}}\mathbb{T}_\mu.
\end{aligned}$$

Similarly to [31], thanks to the cut-off function (3.1) and for m fixed, $\phi_m'(\rho)(\mu(\rho)\rho)^{\frac{1}{4}}/(\mu'(\rho))^{\frac{3}{4}}$ and $\phi_m(\rho)/\sqrt{\mu(\rho)}$ are bounded. Then ∇u_m is bounded in $L^2((0,T)\times\Omega)$ using the estimates with $r>0$ and $r_2>0$, and hence for $\varphi\in W^{2,+\infty}(\mathbb{R})$, we get $\nabla\varphi'((u_m)_j)$ is bounded in $L^2((0,T)\times\Omega)$ for $j=1,2,3$.

The following estimates are necessary. We state them in the lemma as follows.

Lemma 3.3. *There exists a constant $C>0$ depending only on the fixed solution $(\sqrt{\rho},\sqrt{\rho}u)$, and C_m depending also on m such that*

$$
\begin{aligned}
&\|\rho\|_{L^\infty(0,T;L^{10}(\Omega))}+\|\rho u\|_{L^3(0,T;L^{\frac{5}{2}}(\Omega))}+\|\rho|u|^2\|_{L^2(0,T;L^{\frac{10}{7}}(\Omega))}\\
&+\|\sqrt{\mu}\big(|\mathbb{S}_\mu|+r|\mathbb{S}_r|\big)\|_{L^2(0,T;L^{\frac{10}{7}}(\Omega))}+\|\frac{\lambda(\rho)}{\mu(\rho)}\|_{L^\infty((0,T)\times\Omega)}\\
&+\|\sqrt{\frac{P'(\rho_n)\rho_n}{\mu'(\rho_n)}}\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P'(s)\mu'(s)}{s}}\,ds\Big)\|_{L^{1+}((0,T)\times\Omega)}\\
&+\|\sqrt{\frac{P_\delta'(\rho_n)\rho_n}{\mu'(\rho_n)}}\nabla\Big(\int_0^{\rho_n}\sqrt{\frac{P_\delta'(s)\mu'(s)}{s}}\,ds\Big)\|_{L^{1+}((0,T)\times\Omega)}+\|r_0u\|_{L^2((0,T)\times\Omega)}\le C,
\end{aligned}
$$

and

$$
\|\nabla\phi_m(\rho)\|_{L^4((0,T)\times\Omega}+\|\partial_t\phi_m(\rho)\|_{L^2((0,T\times\Omega))}\le C_m.
$$

Proof. By (1.19), we have $\rho\in L^\infty(0,T;L^{10}(\Omega))$. Now we have $\nabla\sqrt{\rho}\in L^\infty(0,T;L^2(\Omega))$ because $\mu'(s)\ge\varepsilon_1$ and $\mu'(\rho)\nabla\rho/\sqrt{\rho}\in L^\infty((0,T);L^2(\Omega))$. Note that

$$
\rho u=\rho^{\frac{2}{3}}\rho^{\frac{1}{3}}u,
$$

$\rho^{\frac{2}{3}}\in L^\infty(0,T;L^{15}(\Omega))$ and $\rho^{\frac{1}{3}}u\in L^3(0,T;L^3(\Omega))$, ρu is bounded in $L^3(0,T;L^{\frac{5}{2}}(\Omega))$.

By (1.19), we have $(\frac{\rho}{\mu'(\rho)})^{1/2}|u|^2\in L^2((0,T)\times\Omega)$. Note that

$$
\rho|u|^2=(\rho\mu'(\rho))^{1/2}(\frac{\rho}{\mu'(\rho)})^{1/2}|u|^2,
$$

it is bounded in $L^2(0,T;L^{\frac{10}{7}}(\Omega))$, where we used facts that $\mu(\rho)\in L^\infty(0,T;L^{5/2}(\Omega))$ (recalling that for $\rho\ge1$ we have $\mu(\rho)\le c\rho^4$ and $\rho\in L^\infty(0,T;L^{10}(\Omega))$) and $\mu'(\rho)\rho\le\alpha_2\mu(\rho)$.

Similarly, we get $\sqrt{\mu}(|\mathbb{S}_\mu|+r|\mathbb{S}_r|)\in L^2(0,T;L^{10/7}(\Omega))$ by (1.18). The $L^\infty((0,T)\times\Omega)$ bound for $\lambda(\rho)/\mu(\rho)$ may be obtained easily due to (1.3) and (1.9).

Concerning the estimates related to the pressures, we just have to look at the proof in Lemma 2.2. Note that

$$
\nabla\phi_m(\rho)=\phi_m'(\rho)\nabla\rho=\phi_m'(\rho)\frac{\rho^{1/2}\mu(\rho)^{1/4}}{\mu'(\rho)}[\frac{\mu'(\rho)}{\rho^{1/2}\mu(\rho)^{1/4}}\nabla\rho]
$$

by (1.20), we conclude that $\nabla\phi_m(\rho)$ is bounded in $L^4((0,T)\times\Omega)$. It suffices to recall that thanks to the cut-off function ϕ_m, we have $\phi_m'(\rho)\rho^{1/2}\mu(\rho)^{1/4}/\mu'(\rho)$ bounded in $L^\infty((0,T)\times$

$\Omega)$. Similarly, we write

$$\begin{aligned}\partial_t\phi_m(\rho) &= \phi_m'(\rho)\partial_t\rho = -\phi_m'(\rho)\mathrm{div}(\rho u)\\ &= -\phi_m'(\rho)\frac{\rho}{\sqrt{\mu}}\mathrm{Tr}(\mathbb{T}_\mu) - \big(\phi_m'(\rho)\frac{(\mu(\rho)\rho)^{\frac14}}{(\mu'(\rho))^{\frac34}}\big)\big(\frac{\rho^{\frac14}}{(\mu'(\rho))^{\frac14}}u\big)\cdot\big(\frac{\mu'(\rho)}{\rho^{1/2}\mu(\rho)^{1/4}}\nabla\rho\big),\end{aligned}$$

which provides $\partial_t\phi_m(\rho)$ bounded in $L^2(0,T;L^2(\Omega))$ thanks to (1.18), (1.19) and (1.20). and using the cut-off function property to bound the extra quantiies in $L^\infty((0,T)\times\Omega)$ as previously.

□

Lemma 3.4. *The κ-entropic weak solution constructed in Theorem 1.2 is a renormalized solution, in particular, we have*

$$\begin{aligned}&\int_0^T\int_\Omega\big(\rho\varphi(u)\psi_t + (\rho\varphi(u)\otimes u)\nabla\psi\big)\\ &-\int_0^T\int_\Omega \nabla\psi\varphi'(u)\big[2\big(\sqrt{\mu(\rho)}(\mathbb{S}_\mu + r\,\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + r\sqrt{\mu(\rho)}\mathbb{S}_r)\mathrm{Id}\big]\\ &-\int_0^T\int_\Omega \psi\varphi''(u)\mathbb{T}_\mu\big[2\big((\mathbb{S}_\mu + r\,\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\mathbb{S}_\mu + r\mathbb{S}_r)\mathrm{Id}\big]\\ &+\int_0^T\int_\Omega \psi\varphi'(u)F(\rho,u)\big)\,dx\,dt = 0,\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}&\sqrt{\mu(\rho)}\varphi_i'(u)[\mathbb{T}_\mu]_{jk} = \partial_j(\mu\varphi_i'(u)u_k) - \sqrt{\rho}u_k\varphi_i'(u)\frac{\nabla\mu}{\sqrt{\rho}} + \bar{R}^1_\varphi,\\ &\sqrt{\mu(\rho)}\varphi_i'(u)[\mathbb{S}_r]_{jk} = 2\sqrt{\mu(\rho)}\varphi_i'(u)\partial_j\partial_k Z(\rho) - 2\partial_j(\sqrt{\mu(\rho)}\partial_k Z(\rho)\varphi_i'(u)) + \bar{R}^2_\varphi\\ &\frac{\lambda(\rho)}{2\mu(\rho)}\varphi_i'(u)\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{T}_\mu) = \mathrm{div}\big(\frac{\lambda(\rho)}{\mu(\rho)}\sqrt{\rho}u\frac{\mu(\rho)}{\sqrt{\rho}}\varphi'(u)\big)\\ &\qquad\qquad - \sqrt{\rho}u\cdot\sqrt{\rho}\nabla s(\rho)\frac{\rho\mu''(\rho)}{\mu(\rho)}\varphi'(u) + \bar{R}^3_\varphi\\ &\frac{\lambda(\rho)}{\mu(\rho)}\varphi'(u)\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_r) = \varphi_i'(u)\big(\frac{\lambda(\rho)}{\sqrt{\mu(\rho)}} + \frac12 k(\rho)\big)\Delta Z(\rho)\\ &\qquad\qquad - \frac12\mathrm{div}(k(\rho)\varphi_i'(u)\nabla Z(\rho)) + \bar{R}^4_\varphi\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\bar{R}^1_\varphi &= \varphi_i''(u)\mathbb{T}_\mu\sqrt{\mu(\rho)}u\\ \bar{R}^2_\varphi &= 2\varphi_i''(u)\mathbb{T}_\mu\nabla Z(\rho)\\ \bar{R}^3_\varphi &= -\varphi_i''(u)\mathbb{T}_\mu\cdot\sqrt{\mu(\rho)}u\frac{\lambda(\rho)}{\mu(\rho)}\\ \bar{R}^4_\varphi &= \frac{k(\rho)}{2\sqrt{\mu(\rho)}}\varphi_i''(u)\mathbb{T}_\mu\cdot\nabla Z(\rho)\end{aligned} \tag{3.4}$$

Proof. We choose a function $\Big[\phi_m'([\rho]_\varepsilon)\psi\Big]_\varepsilon$ as a test function for the continuity equation with $\psi \in C_c^\infty((0,T)\times\Omega)$. Using Lemma 3.2, we have

$$\begin{aligned}0 &= \int_0^T \int_\Omega \big(\partial_t\Big[\phi_m'([\rho]_\varepsilon)\psi\Big]_\varepsilon \rho + \rho u\cdot\nabla\Big[\phi_m'([\rho]_\varepsilon)\psi\Big]_\varepsilon\big)\,dx\,dt\\ &= -\int_0^T\int_\Omega \big(\phi_m'([\rho]_\varepsilon)\psi\,\partial_t[\rho]_\varepsilon + \mathrm{div}([\rho u]_\varepsilon)\phi_m'([\rho]_\varepsilon)\psi\big)\,dx\,dt\\ &= \int_0^T\int_\Omega \left(\psi_t\phi_m([\rho]_\varepsilon) - \psi\phi_m'([\rho]_\varepsilon)\big[\frac{\rho}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}_\mu) + 2\sqrt{\rho}u\cdot\nabla\sqrt{\rho}\big]_\varepsilon\right)\,dx\,dt.\end{aligned} \tag{3.5}$$

Using Lemma 3.3 and Lemma 3.2, and passing into the limit as ε goes to zero, from (3.5), we get:

$$\begin{aligned}0 &= \int_0^T\int_\Omega \big(\psi_t\phi_m(\rho) - \psi\phi_m'(\rho)[\frac{\rho}{\sqrt{\mu}}\mathrm{Tr}(\mathbb{T}_\mu) + 2\sqrt{\rho}u\cdot\nabla\sqrt{\rho}]\big)\,dx\,dt\\ &= \int_0^T\int_\Omega \big(\psi_t\phi_m(\rho) - \psi\big[\phi_m'(\rho)\frac{\rho}{\sqrt{\mu}}\mathrm{Tr}(\mathbb{T}_\mu) + u\cdot\nabla\phi_m(\rho)\big]\big)\,dx\,dt,\end{aligned} \tag{3.6}$$

thanks to $\psi\nabla\phi_m(\rho)\in L^4((0,T)\times\Omega)$, $u\in L^2((0,T)\times\Omega)$, and ψ compactly supported.

Similarly, we can choose $[\psi\phi_m(\rho)]_\varepsilon$ as a test function for the momentum equation. In particular, we have the following lemma.

Lemma 3.5.

$$\int_0^T\int_\Omega [\psi\phi_m(\rho)]_\varepsilon\big(\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u)\big)\,dx\,dt$$

tends to

$$-\int_0^T\int_\Omega \psi_t\rho u_m + \nabla\psi\cdot(\rho u\otimes u_m + \psi(\partial_t\phi_m(\rho) + u\cdot\nabla\phi_m(\rho))\rho u\,dx\,dt$$

as $\varepsilon\to 0$.

Proof. By Lemma 3.1, we can show that

$$\int_0^T\int_\Omega [\psi\phi_m(\rho)]_\varepsilon\partial_t(\rho u)\,dx\,dt \to -\int_0^T\int_\Omega \partial_t\psi\rho u_m + \psi\partial_t\phi_m(\rho)\rho u\,dx\,dt.$$

For the second term, we have

$$\begin{aligned}&\int_0^T\int_\Omega \big[\psi\phi_m(\rho)\big]_\varepsilon \mathrm{div}(\rho u\otimes u)\,dx\,dt = \int_0^T\int_\Omega \psi\phi_m(\rho)\big[\mathrm{div}(\rho u\otimes u)\big]_\varepsilon\,dx\,dt\\ &= \Big(\int_0^T\int_\Omega \psi\phi_m(\rho)\big[\mathrm{div}(\rho u\otimes u)\big]_\varepsilon\,dx\,dt - \int_0^T\int_\Omega \psi\phi_m(\rho)\big[\mathrm{div}(\rho u\otimes u)\big]_\varepsilon^x\,dx\,dt\Big)\\ &+ \int_0^T\int_\Omega \psi\phi_m(\rho)\big[\mathrm{div}(\rho u\otimes u)\big]_\varepsilon^x\,dx\,dt\\ &= R_1 + R_2,\end{aligned}$$

where $[f(t,x)]_\varepsilon = f(t,x) * \eta_\varepsilon(t,x)$ and $[f(t,x)]_\varepsilon^x = f * \eta_\varepsilon(x)$ with $\varepsilon > 0$ a small enough number. We write R_1 in the following way

$$\begin{aligned} R_1 &= \int_0^T \int_\Omega \psi \phi_m(\rho) \big[\operatorname{div}(\rho u \otimes u)\big]_\varepsilon \,dx\,dt - \int_0^T \int_\Omega \psi \phi_m(\rho) \big[\operatorname{div}(\rho u \otimes u)\big]_\varepsilon^x \,dx\,dt \\ &= \int_0^T \int_\Omega \psi \nabla \phi_m(\rho) : \big[\rho u \otimes u\big]_\varepsilon \,dx\,dt - \int_0^T \int_\Omega \psi \nabla \phi_m(\rho) : \big[\rho u \otimes u\big]_\varepsilon^x \,dx\,dt. \end{aligned}$$

Thanks to Lemma 3.3, $\rho|u|^2 \in L^2(0,T;L^{10/7}(\Omega))$ and $\psi\nabla\phi_m(\rho) \in L^4((0,T)\times\Omega)$, we conclude that $R_1 \to 0$ as $\varepsilon \to 0$. Meanwhile, we can apply Lemma 3.1 to R_2 directly, thus

$$\begin{aligned} &\int_0^T \int_\Omega \psi \phi_m(\rho) \big[\operatorname{div}(\rho u \otimes u)\big]_\varepsilon^x \,dx\,dt \\ &= \Big(\int_0^T \int_\Omega \psi \phi_m(\rho) \big[\operatorname{div}(\rho u \otimes u)\big]_\varepsilon^x \,dx\,dt - \int_0^T \int_\Omega \psi \phi_m(\rho) \operatorname{div}(\rho u \otimes [u]_\varepsilon^x) \,dx\,dt \Big) \\ &+ \int_0^T \int_\Omega \psi \phi_m(\rho) \operatorname{div}(\rho u \otimes [u]_\varepsilon^x) \,dx\,dt \\ &= R_{21} + R_{22}. \end{aligned}$$

By Lemma 3.1, we have $R_{21} \to 0$ as $\varepsilon \to 0$. The term R_{22} will be calculated in the following way,

$$\begin{aligned} &\int_0^T \int_\Omega \psi \phi_m(\rho) \operatorname{div}(\rho u \otimes [u]_\varepsilon^x) \,dx\,dt \\ &= \int_0^T \int_\Omega \psi \phi_m(\rho) \operatorname{div}(\rho u) [u]_\varepsilon^x \,dx\,dt + \int_0^T \int_\Omega \psi \phi_m(\rho) \rho u \cdot \nabla [u]_\varepsilon^x \,dx\,dt \\ &= \int_0^T \int_\Omega \psi \operatorname{div}(\rho u) [u_m]_\varepsilon^x \,dx\,dt + \int_0^T \int_\Omega \psi \rho u \nabla (\phi_m(\rho) [u]_\varepsilon^x) \,dx\,dt - \\ &\int_0^T \int_\Omega \psi [u]_\varepsilon^x \cdot \nabla \phi_m(\rho) \rho u \,dx\,dt \\ &= -\int_0^T \int_\Omega \nabla \psi \rho u \otimes [u_m]_\varepsilon^x \,dx\,dt - \int_0^T \int_\Omega \psi \cdot [u]_\varepsilon^x \nabla \phi_m(\rho) \rho u \,dx\,dt, \end{aligned}$$

which tends to

$$-\int_0^T \int_\Omega \nabla \psi \rho u \otimes u_m \,dx\,dt - \int_0^T \int_\Omega \psi \cdot u \nabla \phi_m(\rho) \rho u \,dx\,dt,$$

as $\varepsilon \to 0$.

□

For the other terms in the momentum equation, we can follow the same way as above method for (3.6) to have

$$
\begin{aligned}
&\int_0^T\int_\Omega \big(\psi_t\rho u_m + \nabla\psi\cdot(\rho u\otimes u_m - 2\phi_m(\rho)(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + r\mathbb{S}_r)\mathrm{Id}))\\
&+\int_0^T\int_\Omega \psi(\partial_t\phi_m(\rho) + u\cdot\nabla\phi_m(\rho))\rho u\\
&-\int_0^T\int_\Omega 2\psi(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + r\mathbb{S}_r)\mathrm{Id})\nabla\phi_m(\rho) + \psi\phi_m(\rho)F(\rho,u)\big)\,dx\,dt\\
&=0.
\end{aligned}
$$

Thanks to (3.6), we have

$$
\begin{aligned}
&\int_0^T\int_\Omega \big(\psi_t\rho u_m + \nabla\psi\cdot(\rho u\otimes u_m - 2\phi_m(\rho)(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}(\mathbb{S}_\mu + r\mathbb{S}_r))\mathrm{Id})\\
&-\int_0^T\int_\Omega \psi\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}_\mu)\rho u - \psi\phi_m(\rho)F(\rho,u)\\
&-\int_0^T\int_\Omega 2\psi(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}(\mathbb{S}_\mu + r\mathbb{S}_r))\mathrm{Id})\nabla\phi_m(\rho)\big)\,dx\,dt = 0.
\end{aligned}
\tag{3.7}
$$

The goal of this subsection is to derive the formulation of renormalized solution following the idea in [31]. We choose the function $\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon$ as a test function in (3.7). As the same argument of Lemma 3.5, we can show that

$$
\begin{aligned}
&\int_0^T\int_\Omega \big(\partial_t\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon\rho u_m + \nabla\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon : (\rho u\otimes u_m)\big)\,dx\,dt\\
&\to \int_0^T\int_\Omega \big(\rho\varphi(u_m)\psi_t + \rho u\otimes\varphi(u_m)\nabla\psi\big)\,dx\,dt,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T\int_\Omega \nabla\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon\big(-2\phi_m(\rho)(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + r\mathbb{S}_r))\mathrm{Id}\big)\\
&+\big[\psi\varphi'([u_m]_\varepsilon)\big]_\varepsilon\big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}_\mu)\rho u\\
&-2(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}(\mathbb{S}_\mu + r\mathbb{S}_r)\mathrm{Id}))\nabla\phi_m(\rho) + \phi_m(\rho)F(\rho,u)\big)\,dx\,dt\\
&\to\int_0^T\int_\Omega \nabla(\psi\varphi'(u_m))\ \big(-2\phi_m(\rho)(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}(\mathbb{S}_\mu + r\mathbb{S}_r))\mathrm{Id})\big)\\
&+\psi\varphi'(u_m)\big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}_\mu)\rho u\\
&-2(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu + r\mathbb{S}_r))\nabla\phi_m(\rho) + \phi_m(\rho)F(\rho,u)\big)\,dx\,dt
\end{aligned}
$$

as ε goes to zero. Putting these two limits together, we have

$$\begin{aligned}
&\int_0^T\int_\Omega \big(\rho\varphi(u_m)\psi_t+\rho u\otimes\varphi(u_m)\nabla\psi\big)\\
&+\nabla\psi\varphi'(u_m)\big(-2\phi_m(\rho)(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r)+\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu+r\mathbb{S}_r))\big)\\
&+\psi\varphi''(u_m)\nabla u_m\big(-\phi_m(\rho)2(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r)+\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu+r\mathbb{S}_r))\big) \qquad (3.8)\\
&+\psi\varphi'(u_m)\big(-\phi_m'(\rho)\frac{\rho}{\sqrt{\mu(\rho)}}\mathrm{Tr}(\mathbb{T}_\mu)\rho u-2(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r)\\
&+\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}\mathbb{S}_\mu+r\mathbb{S}_r))\nabla\phi_m(\rho)+\phi_m(\rho)F(\rho,u)\big)\,dx\,dt=0.
\end{aligned}$$

Now we should pass to the limit in (3.8) as m goes to infinity. To this end, we should keep the following convergences in mind:

$$\begin{aligned}
&\phi_m(\rho)\ \text{ converges to } 1, \quad \text{for almost every}(t,x)\in\mathbb{R}^+\times\Omega,\\
&u_m \text{ converges to } u, \quad \text{for almost every}(t,x)\in\mathbb{R}^+\times\Omega, \qquad (3.9)\\
&|\rho\phi_m'(\rho)|\le 2, \quad \text{and converges to 0 for almost every}(t,x)\in\mathbb{R}^+\times\Omega.
\end{aligned}$$

We can find that

$$\begin{aligned}
&\sqrt{\mu(\rho)}\nabla u_m=\sqrt{\mu(\rho)}\nabla(\phi_m(\rho)u)=\phi_m(\rho)\sqrt{\mu(\rho)}\nabla u+\phi_m'(\rho)\sqrt{\mu(\rho)}u\cdot\nabla\rho\\
&=\frac{\phi_m(\rho)}{\sqrt{\mu(\rho)}}\big(\nabla(\mu(\rho)u)-\sqrt{\rho}u\cdot\frac{\nabla\mu(\rho)}{\sqrt{\rho}}\big)+\frac{\sqrt{\rho}}{\mu(\rho)^{\frac34}}\big(\frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)\nabla\rho\big)\big(\frac{\rho^{\frac14}}{(\mu'(\rho))^{\frac14}}u\big)\big(\phi_m'(\rho)\frac{\mu(\rho)^{\frac34}\rho^{\frac14}}{(\mu'(\rho))^{\frac34}}\big)\\
&=\phi_m(\rho)\mathbb{T}_\mu+\frac{\sqrt{\rho}}{\mu(\rho)^{\frac34}}\big(\frac{\sqrt{\mu(\rho)}}{\rho}\mu'(\rho)\nabla\rho\big)\big(\frac{\rho^{\frac14}}{(\mu'(\rho))^{\frac14}}u\big)\big(\phi_m'(\rho)\frac{\mu(\rho)^{\frac34}\rho^{\frac14}}{(\mu'(\rho))^{\frac34}}\big)\\
&=A_{1m}+A_{2m}.
\end{aligned}$$

Note that

$$|\phi_m'(\rho)\frac{\mu(\rho)^{\frac34}\rho^{\frac14}}{(\mu'(\rho))^{\frac34}}|\le C|\phi_m'(\rho)\rho|,$$

thus $\phi_m'(\rho)\mu(\rho)^{\frac34}\rho^{\frac14}/(\mu(\rho)')^{\frac34}$ converges to zero for almost every (t,x). Thus, the Dominated convergence theorem yields that A_{2m} converges to zero as $m\to\infty$. Meanwhile, the Dominated convergence theorem also gives us A_{1m} converges to $\mathbb{T}_\mu$ in $L^2_{t,x}$. Hence, with (3.9) at hand, letting $m\to\infty$ in (3.8), one obtains that

$$\begin{aligned}
&\int_0^T\int_\Omega\big(\rho\varphi(u)\psi_t+\rho u\otimes\varphi(u)\nabla\psi\big)-2\nabla\psi\varphi'(u)\big((\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r)\\
&+\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}(\sqrt{\mu(\rho)}(\mathbb{S}_\mu+r\mathbb{S}_r))\mathrm{Id}\big)-2\psi\varphi''(u)\mathbb{T}_\mu((\mathbb{S}_\mu+r\mathbb{S}_r)\\
&+\frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}((\mathbb{S}_\mu+r\mathbb{S}_r)\mathrm{Id})+\psi\varphi'(u)F(\rho,u)\big)\,dx\,dt=0.
\end{aligned}$$

From now, we denote $R_\varphi = 2\psi\varphi''(u)\mathbb{T}_\mu((\mathbb{S}_\mu + r\mathbb{S}_r) + \frac{\lambda(\rho)}{2\mu(\rho)}\mathrm{Tr}((\mathbb{S}_\mu + r\mathbb{S}_r)\mathrm{Id})$. This ends the proof of Theorem 1.3.

□

4. RENORMALIZED SOLUTIONS AND WEAK SOLUTIONS

The main goal of this section is the proof of Theorem 1.1 that obtains the existence of renormalized solutions of the Navier-Stokes equations without the additional terms, thus the existence of weak solutions of the Navier-Stokes equations.

4.1. **Renormalized solutions.** In this subsection, we will show the existence of renormalized solutions. To this end, we need the following lemma of stability.

Lemma 4.1. *For any fixed $\alpha_1 < \alpha_2$ as in (1.9) and consider sequences δ_n, r_{0n}, r_{1n} and r_{2n}, such that $r_{i,n} \to r_i \geq 0$ with $i = 0, 1, 2$ and then $\delta_n \to \delta \geq 0$. Consider a family of $\mu_n : \mathbb{R}^+ \to \mathbb{R}^+$ verifying (1.9) and (1.10) for the fixed α_1 and α_2 such that*

$$\mu_n \to \mu \quad \text{in } C^0(\mathbb{R}^+).$$

Then, if (ρ_n, u_n) verifies (1.18)-(1.21), up to a subsequence, still denoted n, the following convergences hold.
1. The sequence ρ_n convergences strongly to ρ in $C^0(0,T;L^p(\Omega))$ for any $1 \leq p < \gamma$.
2. The sequence $\mu_n(\rho_n)\,u_n$ converges to $\mu(\rho)u$ in $L^\infty(0,T;L^p(\Omega)$ for $p \in [1, 3/2)$.
3. The sequence $(\mathbb{T}_\mu)_n$ convergences to $\mathbb{T}_\mu$ weakly in $L^2(0,T;L^2(\Omega))$.
4. For every function $H \in W^{2,\infty}(\overline{\mathbb{R}^d})$ and $0 < \alpha < 2\gamma/\gamma+1$, we have that $\rho_n^\alpha H(u_n)$ convergences to $\rho^\alpha H(u)$ strongly in $L^p(0,T;\Omega)$ for $1 \leq p < \frac{2\gamma}{(\gamma+1)\alpha}$. In particular, $\sqrt{\mu(\rho_n)}H(u_n)$ convergences to $\sqrt{\mu(\rho)}H(u)$ strongly in $L^\infty(0,T;L^2(\Omega))$.

Proof. Using (1.21), the Aubin-Lions lemma gives us, up to a subsequence,

$$\mu_n(\rho_n) \to \tilde{\mu} \quad \text{in } \; C^0(0,T;L^q(\Omega))$$

for any $q < \frac{3}{2}$. But

$$\sup|\mu_n - \mu| \to 0$$

as $n \to \infty$. Thus, we have

$$\mu_n(\rho_n) \to \tilde{\mu}(t,x) \quad \text{in } \; C^0([0,T];L^q(\Omega)), \tag{4.1}$$

so up to a subsequence,

$$\mu(\rho_n) \to \tilde{\mu}(t,x) \;\text{ a. e.}$$

Note that μ is increasing function, so it is invertible, and μ^{-1} is continuous. This implies that $\rho_n \to \rho$ a.e. with $\mu(\rho) = \tilde{\mu}(t,x)$. Together with (4.1) and ρ_n is uniformly bounded in $L^\infty(0,T;L^\gamma(\Omega))$, thus we get part 1.

Note that

$$\nabla\frac{\mu(\rho_n)}{\sqrt{\rho_n}} = \frac{\sqrt{\rho_n}\nabla\mu(\rho_n)}{\rho_n} - \frac{\mu(\rho_n)\nabla\rho_n}{2\rho\sqrt{\rho_n}},$$

thus

$$\left|\nabla\frac{\mu(\rho_n)}{\sqrt{\rho_n}}\right| \leq C\,|\sqrt{\rho_n}|\left|\frac{\nabla\mu(\rho_n)}{\sqrt{\rho_n}}\right|,$$

so $\nabla \frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$, thanks to (1.18). Using (1.21), we have $\frac{\mu(\rho_n)}{\sqrt{\rho_n}}$ is bounded in $L^\infty(0,T;W^{1,2}(\Omega))$, thus it is uniformly bounded in $L^\infty(0,T;L^6(\Omega))$.

On the other hand, $\sqrt{\rho_n}u_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. From Lemma 2.4, we have

$$\mu(\rho_n)u_n = \frac{\mu(\rho_n)}{\sqrt{\rho_n}}\sqrt{\rho_n}u_n \to \mu(\rho)u \quad \text{in} \quad L^\infty(0,T;L^q(\Omega))$$

for any $1 \leq q < \frac{3}{2}$. Since $(\mathbb{T}_\mu)_n$ is bounded in $L^2(0,T;L^2(\Omega))$, and so, up to a sequence, convergences weakly in $L^2(0,T;L^2(\Omega))$ to a function $\mathbb{T}_\mu$. Using Lemma 2.4, this gives part 4. $\square$

With Lemma 4.1, we are able to recover the renormalized solutions of Navier-Stokes equations without any additional term by letting $n \to \infty$ in (3). We state this result in the following Lemma. In this lemma, we fix μ such that $\varepsilon_1 > 0$.

Lemma 4.2. *For any fixed $\varepsilon_1 > 0$, there exists a renormalized solution $(\sqrt{\rho}, \sqrt{\rho}u)$ to the initial value problem* (1.1)-(1.2).

Proof. We can use Lemma 4.1 to pass to the limits for the extra terms. We will have to follow this order: let r_2 goes to zero, then r_1 tends to zero, after that r_0, δ, r go to zero together.

– If $r_2 = r_2(n) \to 0$, we just write

$$r_2 \frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n = r_2^{\frac{1}{4}} \big(\frac{\rho_n}{\mu'(\rho_n)}\big)^{\frac{1}{4}} \big(\frac{\rho_n}{\mu'(\rho_n)}\big)^{\frac{3}{4}}|u_n|^2 u_n,$$

and $\mu'(\rho_n) \geq \varepsilon_1 > 0$, so $\big(\frac{\rho_n}{\mu'(\rho_n)}\big)^{\frac{1}{4}} \leq C|\rho_n|^{\frac{1}{4}}$, thus,

$$r_2 \frac{\rho_n}{\mu'(\rho_n)}|u_n|^2 u_n \to 0 \text{ in } L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}(\Omega)).$$

– For $r_1 = r(n) \to 0$,

$$|r_1 \rho_n |u_n| u_n| \leq r^{\frac{1}{3}} \rho_n^{\frac{1}{3}} r^{\frac{2}{3}} \rho_n^{\frac{2}{3}} |u_n|^2,$$

which convergences to zero in $L^{\frac{3}{2}}(0,T;L^{\frac{9}{7}}(\Omega))$ using the drag term control in the energy and the information on the pressure law $P(\rho) = a\rho^\gamma$.

– For $r_0 = r_0(n) \to 0$, it is easy to conclude that

$$r_0 u_n \to 0 \text{ in } L^2((0,T) \times \Omega).$$

– We now consider the limit $r \to 0$ of the term

$$r\rho_n \nabla \left(\sqrt{K(\rho_n)} \Delta \big(\int_0^{\rho_n} \sqrt{K(s)}\, ds \big) \right).$$

Note the following identity

$$\rho_n \nabla \left(\sqrt{K(\rho_n)} \Delta \big(\int_0^{\rho_n} \sqrt{K(s)}\, ds \big) \right) = 2\mathrm{div}\Big(\mu(\rho_n)\nabla^2\big(2s(\rho_n)\big)\Big) + \nabla\Big(\lambda(\rho_n)\Delta\big(2s(\rho_n)\big)\Big),$$

we only need to focus on $\operatorname{div}\Big(\mu(\rho_n)\nabla^2\big(2s(\rho_n)\big)\Big)$ since the same argument holds for the other term. Since

$$\begin{aligned}
r\int_\Omega &\operatorname{div}\Big(\mu(\rho_n)\nabla^2\big(2s(\rho_n)\big)\Big)\psi\,dx\\
&= r\int_\Omega \frac{\rho_n}{\mu_n}\nabla Z(\rho_n)\otimes\nabla Z(\rho_n)\nabla\psi\,dx + r\int_\Omega \mu_n\nabla s(\rho_n)\Delta\psi\,dx\\
&= r\int_\Omega \frac{\rho_n}{\mu_n}\nabla Z(\rho_n)\otimes\nabla Z(\rho_n)\nabla\psi\,dx + r\int_\Omega \sqrt{\mu_n}\nabla Z(\rho_n)\Delta\psi\,dx,
\end{aligned}$$

the first term can be controlled as

$$\big|r\int_\Omega \sqrt{\mu_n}\nabla Z(\rho_n)\Delta\psi\,dx\big| \le Cr^{\frac12}\|\sqrt{\mu(\rho_n)}\|_{L^2(0,T;L^2(\Omega))}\|\sqrt{r}\nabla Z(\rho_n)\|_{L^2(0,T;L^2(\Omega))}\to 0,$$

thanks to (1.20) and (1.21); and the second term as

$$\begin{aligned}
&\big|\int_\Omega \frac{\rho_n}{\mu_n}\nabla Z(\rho_n)\otimes\nabla Z(\rho_n)\nabla\psi\,dx\big| \le \sqrt{r}\sqrt{r}\int_\Omega \sqrt{\mu(\rho_n)}\frac{\rho_n}{\mu(\rho_n)^{\frac32}}|\nabla Z(\rho_n)|^2|\nabla\psi|\,dx\\
&\le C\|\sqrt{r}\frac{\rho_n}{\mu(\rho_n)^{\frac32}}|\nabla Z(\rho_n)|^2\|_{L^2(0,T;L^2(\Omega))}\|\sqrt{\mu(\rho_n)}\|_{L^2(0,T;L^2(\Omega))}r^{\frac12}\to 0.
\end{aligned}$$

– Concerning the quantity $\delta\rho^{10}$, thanks to $\mu'_{\varepsilon_1}(\rho)\ge\varepsilon_1>0$, $\sqrt{\delta}|\nabla\rho^5|$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. This gives us that $\delta^{\frac{1}{30}}\rho$ is uniformly bounded in $L^{10}(0,T;L^{30}(\Omega))$. Thus, we have

$$\left|\int_0^T\int_\Omega \delta\rho^{10}\nabla\psi\,dx\,dt\right| \le C(\psi)\delta^{\frac23}\|\delta^{\frac13}\rho^{10}\|_{L^1(0,T;L^3(\Omega))}\to 0$$

as $\delta\to 0$.

With Lemma 4.1 at hand, we are ready to recover the renormalized solutions to (1.1)-(1.2). By part 1 and part 2 of Lemma 4.1, we are able to pass to the limits on the continuity equation. Thanks to part 4 of Lemma 4.1,

$$\sqrt{\mu(\rho_n)}\varphi'(u_n)\to\sqrt{\mu(\rho)}\varphi'(u)\quad\text{in}\quad L^\infty(0,T;L^2(\Omega)).$$

With the help of Lemma 2.2, we can pass to the limit on pressure, thus we can recover the renormalized solutions.

□

4.2. **Recover weak solutions from renormalized solutions.** In this part, we can recover the weak solutions from the renormalized solutions constructed in Lemma 4.2. Now we show that Lemma 4.2 is valid without the condition $\varepsilon_1>0$. For such a μ, we construct a sequence μ_n converging to μ in $C^0(\mathbb{R}^+)$ and such that $\varepsilon_{1n}=\inf\mu'_n>0$. Lemma 4.1 shows that, up to a subsequence,

$$\rho_n\to\rho\quad\text{in}\;\; C^0(0,T;L^p(\Omega))$$

and

$$\rho_n u_n\to\rho u\quad\text{in}\; L^\infty(0,T;L^{\frac{p+1}{2p}}(\Omega))$$

for any $1\le p<\gamma$, where $(\rho,\sqrt{\rho}u)$ is a renormalized solution to (1.1).

Now, we want to show that this renormalized solution is also a weak solution in the sense of Definition 1.2. To this end, we introduce a non-negative smooth function $\Phi : \mathbb{R} \to \mathbb{R}$ such that it has a compact support and $\Phi(s) = 1$ for any $-1 \leq s \leq 1$. Let $\tilde{\Phi}(s) = \int_0^s \Phi(r)\,dr$, we define

$$\varphi_n(y) = n\tilde{\Phi}(\frac{y_1}{n})\Phi(\frac{y_2}{n})....\Phi(\frac{y_N}{n})$$

for any $y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$.

Note that φ_n is bounded in $W^{2,\infty}(\mathbb{R}^N)$ for any fixed $n > 0$, $\varphi_n(y)$ converges everywhere to y_1 as n goes to infinity, φ_n' is uniformly bounded in n and converges everywhere to unit vector $(1, 0, ...0)$, and

$$\|\varphi_n''\|_{L^\infty} \leq \frac{C}{n} \to 0$$

as n goes to infinity. This allows us to control the measures in Definition 1.1 as follows

$$\|R_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^1_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} + \|\overline{R}^2_{\varphi_n}\|_{\mathcal{M}(\mathbb{R}^+\times\Omega)} \leq C\|\varphi_n''\|_{L^\infty(\mathbb{R})} \to 0$$

as n goes to infinity. Using this function φ_n in the equation of Definition 1.1, the Lebesgue's Theorem gives us the equation on ρu_1 in Definition 1.2 by passing limits as n goes to infinity. In this way, we are able to get full vector equation on ρu by permuting the directions. Applying the Lebesgue's dominated convergence Theorem, one obtains (1.4) by passing to limit in (1.26) with $i = 1$ and the function φ_n. Thus, we have shown that the renormalized solution is also a weak solution.

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References

[1] P. Antonelli, S. Spirito. Global existence of weak solutions to the Navier-Stokes-Korteweg equations. ArXiv:1903.02441 (2019).

[2] P. Antonelli, S. Spirito. On the compactness of weak solutions to the Navier-Stokes-Korteweg equations for capillary fluids. ArXiv:1808.03495 (2018).

[3] P. Antonelli, S. Spirito. Global Existence of Finite Energy Weak Solutions of Quantum Navier-Stokes Equations. *Archive of Rational Mechanics and Analysis,* 225 (2017), no. 3, 1161–1199.

[4] P. Antonelli, S. Spirito. On the compactness of finite energy weak solutions to the Quantum Navier-Stokes equations. *J. of Hyperbolic Differential Equations,* 15 (2018), no. 1, 133–147.

[5] C. Bernardi, O. Pironneau. On the shallow water equations at low Reynolds number. *Comm. Partial Differential Equations* 16 (1991), no. 1, 59–104.

[6] D. Bresch, B. Desjardins. Existence of global weak solutions for 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.,* 238 (2003), no.1-3, 211–223.

[7] D. Bresch, F. Couderc, P. Noble, J.-P. Vila. A generalization of the quantum Bohm identity: Hyperbolic CFL condition for Euler–Korteweg equations. *C.R. Acad. Sciences Paris* Volume 354, Issue 1, 39–43, (2016).

[8] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models. *J. Math. Pures Appl.* (9) 86 (2006), no. 4, 362–368.
[9] D. Bresch, B. Desjardins. Quelques modèles diffusifs capillaires de type Korteweg. *C. R. Acad. Sci. Paris,* section mécanique, **332**, no. 11, 881–886, (2004).
[10] D. Bresch, B. Desjardins. Weak solutions via the total energy formulation and their quantitative properties - density dependent viscosities. In: Y. Giga, A. Novotný (éds.) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids. Springer, Berlin (2017).
[11] D. Bresch, B. Desjardins, Chi-Kun Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations* 28 (2003), no. 3-4, 843–868.
[12] D. Bresch, B. Desjardins, E. Zatorska. Two-velocity hydrodynamics in Fluid Mechanics, Part II. Existence of global κ-entropy solutions to compressible Navier-Stokes system with degenerate viscosities. *J. Math. Pures Appl.* Volume 104, Issue 4, 801–836 (2015).
[13] D. Bresch, P.-E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math.* (2) 188 (2018), no. 2, 577-684.
[14] D. Bresch, I. Lacroix-Violet, M. Gisclon. On Navier-Stokes-Korteweg and Euler-Korteweg systems: Application to quantum fluids models. To appear in *Arch. Rational Mech. Anal.* (2019).
[15] D. Bresch, P. Mucha, E. Zatorska. Finite-energy solutions for compressible two-fluid Stokes system. *Arch. Rational Mech. Anal.,* 232, Issue 2, (2019), 987–1029.
[16] C. Burtea, B. Haspot. New effective pressure and existence of global strong solution for compressible Navier-Stokes equations with general viscosity coefficient in one dimension. arXiv:1902.02043 (2019).
[17] R. Carles, K. Carrapatoso, M. Hillairet. Rigidity results in generalized isothermal fluids. *Annales Henri Lebesgue,* 1, (2018), 47–85.
[18] P. Constantin, T. Drivas, H.Q. Nguyen, F. Pasqualotto. Compressible fluids and active potentials. ArXiv:1803.04492.
[19] B. Ducomet, S. Necasova, A. Vasseur, On spherically symmetric motions of a viscous compressible barotropic and self-graviting gas. *J. Math. Fluid Mech.* 13 (2011), no. 2, 191–211.
[20] E. Feireisl, A. Novotný, H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.* **3** (2001), 358–392.
[21] E. Feireisl. Compressible Navier–Stokes Equations with a Non-Monotone Pressure Law. *J. Diff. Eqs* 183, no 1, 97–108, (2002).
[22] Z. Guo, Q. Jiu, Z. Xin. Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients. *SIAM J. Math. Anal.* 39 (2008), no. 5, 1402–1427.
[23] B. Haspot. Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D. *Mathematische Nachrichten,* 291 (14-15), 2188–2203, (2018).
[24] D. Hoff. Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data. *Trans. Amer. Math. Soc.* 303 (1987), no. 1, 169–181.
[25] S. Jiang, Z. Xin, P. Zhang. Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. *Methods Appl. Anal.* 12 (2005), no. 3, 239–251.
[26] S. Jiang, P. Zhang. On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations. *Comm. Math. Phys.* 215 (2001), no. 3, 559–581.
[27] A. Jüngel. Global weak solutions to compressible Navier-Stokes equations for quantum fluids. *SIAM J. Math. Anal.* 42 (2010), no. 3, 1025–1045.
[28] A. Jüngel, D. Matthes. The Derrida–Lebowitz-Speer-Spohn equations: Existence, uniqueness, and Decay rates of the solutions. *SIAM J. Math. Anal.,* 39(6), (2008), 1996–2015.
[29] A.V. Kazhikhov, V.V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.* 41 (1977), no. 2, 273–282.; translated from *Prikl. Mat. Meh.*41 (1977), no. 2, 282–291(Russian).
[30] J. I. Kanel. A model system of equations for the one-dimensional motion of a gas. *Differ. Uravn.* 4 (1968), 721–734 (in Russian).
[31] I. Lacroix-Violet, A. Vasseur. Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit. *J. Math. Pures Appl.* (9) 114 (2018), 191–210.
[32] J. Leray. Sur le mouvement d'un fluide visqueux remplissant l'espace, *Acta Math.* 63 (1934), 193–248.

[33] H.L. Li, J. Li, Z.P. Xin. Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations. *Comm. Math. Phys.,* 281, 401–444 (2008).

[34] J. Li, Z.P. Xin. Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities. arXiv:1504.06826 (2015).

[35] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2. Compressible models.* Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[36] D. Maltese, M. Michalek, P. Mucha, A. Novotny, M. Pokorny, E. Zatorska. Existence of weak solutions for compressible Navier-Stokes with entropy transport. *J. Differential Equations,* 261, No. 8, 4448–4485 (2016)

[37] A. Mellet, A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations* 32 (2007), no. 1-3, 431–452.

[38] A. Mellet, A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. .*SIAM J. Math. Anal.* 39 (2007/08), no. 4, 1344–1365.

[39] P.B. Mucha, M. Pokorny, E. Zatorska. Approximate solutions to a model of two-component reactive flow. *Discrete Contin. Dyn. Syst. Ser.* S, 7, No. 5 , 1079–1099 (2014).

[40] A. Novotny. Weak solutions for a bi-fluid model of a mixture ot two compressible non interacting fluids. Submitted (2018).

[41] A. Novotny, M. Pokorny. Weak solutions for some compressible multi-component fluid models. Submitted (2018).

[42] P.I. Plotnikov, W. Weigant. Isothermal Navier-Stokes equations and Radon transform. *SIAM J. Math. Anal.* 47 (2015), no. 1, 626–653.

[43] F. Rousset. Solutions faibles de l'équation de Navier-Stokes des fluides compressible [d'après A. Vasseur et C. Yu]. Séminaire Bourbaki, 69ème année, 2016–2017, no 1135.

[44] D. Serre. Solutions faibles globales des équations de Navier-Stokes pour un fluide compressible. *C. R. Acad. Sci. Paris.* I Math. 303 (1986), no. 13, 639–642.

[45] V. A. Vaigant, A. V. Kazhikhov. On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid. (Russian). *Sibirsk. Mat. Zh.* 36 (1995), no. 6, 1283-1316, ii; translation in *Siberian Math. J.* 36 (1995), no.6, 1108–1141.

[46] A. Vasseur, C. Yu. Global weak solutions to compressible quantum Navier-Stokes equations with damping. *SIAM J. Math. Anal.* 48 (2016), no. 2, 1489–1511.

[47] A. Vasseur, C. Yu. Existence of Global Weak Solutions for 3D Degenerate Compressible Navier-Stokes Equations. *Inventiones mathematicae* (2016), 1–40.

[48] A. Vasseur, H. Wen, C. Yu. Global weak solution to the viscous two-phase model with finite energy. *To appear in J. Math Pures Appl.* (2018).

[49] E. Zatorska. On the flow of chemically reacting gaseous mixture. *J. Diff. Equations.* 253 (2012) 3471–3500.

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